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ON A STUDY OF BEST PROXIMITY POINTS FOR R-PROXIMAL CONTRACTIONS IN GAUGE SPACES

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In this paper, we prove the existence and uniqueness of best proximity points for non-self mappings satisfying the R-proximal contraction condition of the first or the second kind, in the setting of complete gauge spaces. The class of R-functions is a consistent one, and includes many other important classes, such as simulation functions or manageable functions. Moreover, a direct connection could be established between the classes of Geraghty functions and R-functions, respectively. Based on these connections, some classical best proximity points outcomes are recovered as corollaries.

Keywords: approximately compact set, simulation function, Geraghty's proximal contractions, R-function, R-proximal contraction of the first kind and second kind, best proximity point.

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1. Introduction

Fixed point theory is an extremely important tool in the field of nonlinear analysis, which provides necessary and sufficient conditions for the existence of solutions to nonlinear equations of the form $Tx = x$, where T is a self mapping. When T is not a self mapping, this equation might not have a solution. That is the reason why researchers became interested in finding approximate solutions to such equations. Best proximity point theory focuses on strategies to find approximate solutions to the nonlinear equation $x = Tx$, where T is a non-self mapping. Several authors explored best proximity point theorems for different types of proximal contractions. In [3], Basha defined the notion of a proximal contraction with respect to a non-self mapping and proved some best proximity point theorems. This notion was further generalized by Basha and Shahzad [4]. In [6], there are established some best proximity point results by the use of generalized weak contractions with discontinuous control functions. In [11], some coincidence point results are proved, by means of $(\mathcal{Z}, g)$-contractions. In [15], various types of Geraghty proximal contractions are used in order to study best proximity properties. In [16], Nashine *et al.* introduced the notions of rational proximal contraction of the first and of the second type, and stated several existence and uniqueness results, while in [17] Shatanawi and Pitea studied best proximity points and best proximity coupled points in a complete metric space with (P)-property. In [13], an algorithmic approach for proximal points was proposed. Recently, several researchers have discussed the existence of best proximity points for mappings satisfying a proximal contraction involving some auxiliary functions. In [14], the notion of $\mathcal{Z}$-contraction with respect to

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a simulation function was introduced. Also, some auxiliary functions were given by Hierro and Shahzad [12].

The structure of gauge spaces has been extensively used, due to the fact that it allows generalizations of some classic results, which can be obtain, therefore, as particular cases. In [10], Frigon proved fixed point results for generalized contractions on gauge spaces. In [1], some homotopy invariant results are presented in this setting, by means of generalized contractive mappings. Fixed point theorems are stated in [5], with respect to mappings which fulfill generalized weakly contractive conditions on ordered gauge spaces. In [7], fixed point theorems are proved in this framework, with regard to $\alpha - \psi_\lambda$-contractions. In [8], this setting is used to develop results which are applied to solve a second order nonlinear initial value problem. This context was also chosen in [9], in order to develop some fixed point results. Later on, these results were extended by several authors [2].

The purpose of this paper is to prove the existence and uniqueness of best proximity points for non-self mappings satisfying proximal contraction type conditions, based on an auxiliary function over the structure of gauges spaces.

2. **Preliminaries**

We begin by giving some definitions to illustrate and characterize the gauge spaces.

Definition 2.1 ([8]). *Let X be a nonempty set. A function $d\colon X \times X \to [0,\infty)$ is called a pseudo metric on X if for each $x,y,z \in X$, the following axioms hold:*

(i) $d(x,x) = 0$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

Definition 2.2 ([8]). *Let X be a nonempty set endowed with the pseudo metric d. The d-ball of radius $\epsilon > 0$ centered at $x \in X$ is the set*

$$B(x,d,\epsilon) = \{y \in X : d(x,y) < \epsilon\}.$$

In the following, $\mathfrak{A}$ is a family of indices. Among the classes of pseudo metrics, we are interested in the next one, which possesses properties adequate to developing our theory.

Definition 2.3 ([8]). *A family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of pseudo metrics on a nonempty set X is said to be separating if for each pair $(x,y) \in X \times X$, with $x \neq y$, there exists $d_v \in \mathfrak{F}$, with $d_v(x,y) \neq 0$.*

Definition 2.4 ([8]). *Let X be a nonempty set and $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ be a family of pseudo metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having as subbasis the family of balls*

$$\mathfrak{B}(\mathfrak{F}) = \{B(x,d_v,\epsilon) : x \in X, d_v \in \mathfrak{F}, \text{ and } \epsilon > 0\}$$

is called the topology induced by the family $\mathfrak{F}$ of pseudo metrics. The pair $(X,\mathfrak{T}(\mathfrak{F}))$ is called a gauge space. If we consider $\mathfrak{F}$ as being separating, note that $(X,\mathfrak{T}(\mathfrak{F}))$ is Hausdorff.

Some of the tools we will use in the sequel are presented in the next definition.

Definition 2.5 ([8]). *Let $(X,\mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of pseudo metrics on X. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(i) the sequence $\{x_n\}$ converges to x, if for each $v \in \mathfrak{A}$ and $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $d_v(x_n,x) < \epsilon$ for all $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$.

(ii) the sequence $\{x_n\}$ is a Cauchy sequence, if for each $v \in \mathfrak{A}$ and $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $d_v(x_n,x_m) < \epsilon$ for all $n,m \geq N_0$.

(iii) $(X,\mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence is convergent.

(iv) a subset of X is said to be closed if it contains the limit of each convergent sequence included in it.

Mongkolkeha *et. al.* [15] introduced two types of Geraghty proximal contractions and proved best proximity point theorems in connection with them. In order to define these notions, we need the following one.

Definition 2.6. *A function* $\Phi\colon [0,\infty) \to [0,1)$ *is called a Geraghty function if for each* $\{t_n\} \subset [0,\infty)$*, the relation* $\Phi(t_n) \to 1$ *necessarily implies* $t_n \to 0$*.*

We are now in a position to recollect the notions introduced by Mongkolkeha *et. al.* [15].

Definition 2.7 ([15])**.** *Let* (X, d) *be a metric space and* A, B *be nonempty subsets of* X*. A mapping* $T\colon A \to B$ *is called a Geraghty proximal contraction of the first kind if there exists a Geraghty function* $\phi\colon [0,\infty) \to [0,1)$*, such that* $d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2)$ *implies that*

$$d(u_1, u_2) \le \phi(d(x_1, x_2))d(x_1, x_2), \text{ for all } u_1, u_2, x_1, x_2 \in A.$$

Definition 2.8 ([15])**.** *Let* (X, d) *be a metric space, and* A, B *be nonempty subsets of* X*. A mapping* $T\colon A \to B$ *is called a Geraghty proximal contraction of the second kind if there exists a Geraghty function* $\phi\colon [0,\infty) \to [0,1)$*, such that* $d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2)$ *implies that*

$$d(Tu_1, Tu_2) \le \phi(d(Tx_1, Tx_2))d(Tx_1, Tx_2), \text{ for all } u_1, u_2, x_1, x_2 \in A.$$

Khojasteh [14] introduced an implicit type function, known as simulation function. Later on, this notion was modified by Hierro [11].

Definition 2.9. [14] *A simulation function is a mapping* $\zeta\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ *satisfying the following conditions:*

(ζ_1) $\zeta(0, 0) = 0$*;*

(ζ_2) $\zeta(t, s) < s - t$*, for all* $t, s > 0$*;*

(ζ_3) *if* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

The third condition is symmetric in both arguments of ζ. However, in practice, they have different meaning and play different roles. Hierro *et al.* [11] slightly modified the above definition in order to highlight this difference, and also to enlarge the family of simulation functions. For this purpose they replaced condition (ζ_3) of the above definition by the following one:

(ζ_{3a}) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Remark 2.1 ([11])**.** *Note that every simulation function in the sense of Khojasteh* [14] *is also a simulation function in the sense of Hierro et al.* [11]*, but the converse is not true.*

Hierro and Shahzad [12] used the idea of simulation function in order to define the following family of functions.

Definition 2.10 ([12])**.** *Let* $D \subseteq \mathbb{R}$ *be a nonempty subset. A mapping* $\varrho\colon D \times D \to \mathbb{R}$ *is known as an R-function if it satisfies the following two conditions:*

(ϱ_1) *If* $\{a_n\} \subset (0,\infty) \cap D$ *is a sequence such that* $\varrho(a_{n+1}, a_n) > 0$ *for all* $n \in \mathbb{N}$*, then* $a_n \to 0$*.*

(ϱ_2) *If* $\{a_n\}, \{b_n\} \subset (0,\infty) \cap D$ *are two sequences converging to the same limit* $L \ge 0$ *and verifying* $L < a_n$ *and* $\varrho(a_n, b_n) > 0$ *for all* $n \in \mathbb{N}$*, then* $L = 0$*.*

We denote by R_D the family of R-functions whose domain is $D \times D$. In some cases, for given R-functions $\varrho\colon D \times D \to \mathbb{R}$, we will also consider the following property:

(ϱ_3) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap D$ are two sequences such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Proposition 2.1 ([12])**.** *If $\varrho(t,s) \le s - t$, for all $t,s \in (0,\infty) \cap D$, then ($\varrho_3$) holds.*

Lemma 2.1 ([12])**.** *Every simulation function is an R-function which also fulfills (ϱ_3).*

The following examples defines an R-function which is not a simulation function, proving that this notion is more general.

Example 2.1 ([12])**.** *Let $\varrho\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$,*

$$\varrho(t,s) = \begin{cases} \frac{1}{2}s - t, & \text{if } t < s; \\ 0, & \text{if } t \ge s. \end{cases}$$

Then ϱ is an R-function on $[0,\infty)$ that also satisfies condition (ϱ_3). But ϱ is not a simulation function.

In addition, a straightforward connection between Geraghty functions and R-functions is provided below.

Lemma 2.2 ([12])**.** *If $\phi\colon [0,\infty) \to [0,1)$ is a Geraghty function, then $\acute{\varrho}_\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$,*

$$\acute{\varrho}_\phi(t,s) = \phi(s)s - t, \text{ for all } t,s \in [0,\infty),$$

is an R-function on $[0,\infty)$ satisfying condition (ϱ_3).

The following proposition points out an interesting property of R-functions, which is going to be helpful in our development.

Proposition 2.2 ([12])**.** *If $\varrho \in R_D$, then $\varrho(a,a) \le 0$, for all $a \in (0,\infty) \cap D$.*

3. **Main Results**

In this section, we will introduce proximal contraction conditions by using a family of R-functions and prove some results that ensure the existence of best proximity points of such mappings.

Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of psuedo metrics, where $\mathfrak{A}$ is a family of indices. Then *range of* $\mathfrak{F}$ is defined as

$$ran(\mathfrak{F}) = \{d_v(x,y) : x,y \in X \text{ and } v \in \mathfrak{A}\} \subseteq [0,\infty).$$

A sequence $\{x_n\}$ is called *asymptotically regular* on $(X, \mathfrak{T}(\mathfrak{F}))$ if $d_v(x_n, x_{n+1}) \to 0$ as $n \to \infty$, for each $v \in \mathfrak{A}$.

Let A, B be nonempty subsets of a gauge space $(X, \mathfrak{T}(\mathfrak{F}))$ induced by the family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of psuedo metrics. Then, let us define

$$d_v(A,B) = \inf\{d_v(a,b) : a \in A, b \in B\}$$

$$A_0 = \{a \in A : d_v(a,b) = d_v(A,B) \text{ for each } v \in \mathfrak{A}, \text{ for some } b \in B\}$$

$$B_0 = \{b \in B : d_v(a,b) = d_v(A,B) \text{ for each } v \in \mathfrak{A}, \text{ for some } a \in A\}.$$

The set B is said to be *approximately compact* with respect to A, if for some $x \in A$, every sequence $\{y_n\}$ of B, satisfying the condition $d_v(x, y_n) \to d_v(x, B)$ as $n \to \infty$, for all $v \in \mathfrak{A}$, has a convergent subsequence.

Subsequently, we consider that A and B are nonempty subsets of a gauge space $(X, \mathfrak{T}(\mathfrak{F}))$ induced by the family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of psuedo metrics. A point $x \in A$ is said to be a best proximity point of $T\colon A \to B$ if $d_v(x, Tx) = d_v(A,B)$ for each $v \in \mathfrak{A}$. Also note that for a mapping $T\colon A \to B$ such that A_0 is nonempty and $T(A_0) \subseteq B_0$, there exists a sequence $\{x_n\} \subseteq A_0$ based on $x_0 \in A_0$ with $d_v(x_{n+1}, Tx_n) = d_v(A,B)$, for each $v \in \mathfrak{A}$. Such a sequence is called a *proximal Picard sequence.*

In the following definition, we introduce the notion of R-proximal contractions of the first kind, on the structure of gauge spaces.

Definition 3.1. *A mapping $T\colon A \to B$ is said to be an R-proximal contraction of the first kind if there exists an R-function $\varrho \in R_D$ such that $ran(\mathfrak{F}) \subseteq D$ and $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ implies that*

$$\varrho(d_v(u_1, u_2), d_v(x_1, x_2)) > 0, \text{ for each } v \in \mathfrak{A}, \tag{1}$$

where $u_1, u_2, x_1, x_2 \in A$, provided that at least one of the elements $d_v(u_1, u_2)$, $d_v(x_1, x_2)$ is not null.

Inspired by this definition, by replacing the R-functions with simulation functions, we could similarly define the concept of $\mathcal{Z}$-proximal contraction.

Definition 3.2. *A mapping $T\colon A \to B$ is said to be an $\mathcal{Z}$-proximal contraction of the first kind if there exists a simulation function function $\zeta\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ implies that*

$$\zeta(d_v(u_1, u_2), d_v(x_1, x_2)) > 0 \text{ for each } v \in \mathfrak{A},$$

where $u_1, u_2, x_1, x_2 \in A$, provided that at least one of the elements $d_v(u_1, u_2)$, $d_v(x_1, x_2)$ is not null.

Remark 3.1. *Based on Lemma 2.1, one could notice that each $\mathcal{Z}$-proximal contraction of the first kind is also an R-proximal contraction of the first kind.*

A similar statement could be phrased for Geraghty proximal contractions, although their definition is slightly different.

Lemma 3.1. *Each Geraghty proximal contraction of the first kind with respect to all the pseudo metrics of the family $\mathfrak{F}$ is an R-proximal contraction of the first kind.*

Proof. Let $T\colon A \to B$ be a Geraghty proximal contraction of the first kind. Then there exists a Geraghty function $\phi\colon [0, \infty) \to [0, 1)$, such that $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ implies that

$$d_v(u_1, u_2) \le \phi(d_v(x_1, x_2))d_v(x_1, x_2), \text{ for all } u_1, u_2, x_1, x_2 \in A \text{ and } v \in \mathfrak{A}.$$

We shall prove next that the above inequality becomes strict when at least one of the elements $d_v(u_1, u_2)$, $d_v(x_1, x_2)$ is not null. Indeed, assuming that $d_v(u_1, u_2) \ne 0$ it leads to the conclusion that $d_v(x_1, x_2)$ is not zero also. Hence, we should have in mind the condition $d_v(x_1, x_2) \ne 0$.

Let $\varphi\colon [0, \infty) \to [0, 1)$ be the function defined by $\varphi(t) = \dfrac{1 + \phi(t)}{2}$. This is also a Geraghty function and $\phi(t) < \varphi(t)$, $\forall t \in [0, \infty)$. Therefore, if $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ and $d_v(x_1, x_2) \ne 0$, then

$$d_v(u_1, u_2) \le \phi(d_v(x_1, x_2))d_v(x_1, x_2) < \varphi(d_v(x_1, x_2))d_v(x_1, x_2).$$

On the other side, Lemma 2.2 guarantees that $\acute{\varrho}_\varphi\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\acute{\varrho}_\varphi(t, s) = \varphi(s)s - t, \text{ for all } t, s \in [0, \infty),$$

is an R-function on $[0, \infty)$ satisfying condition (ϱ_3). Hence,

$$\acute{\varrho}_\varphi(d_v(u_1, u_2), d_v(x_1, x_2)) > 0, \text{ for each } v \in \mathfrak{A},$$

where $u_1, u_2, x_1, x_2 \in A$, $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$, provided that $d_v(x_1, x_2)$ is not null. □

Theorem 3.1. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X, such that B is approximately compact with respect to A, and A_0 is nonempty. Let $T\colon A \to B$ be an R-proximal contraction of the first kind with respect to $\varrho \in R_D$, such that $T(A_0) \subseteq B_0$. Further, assume that at least one of the following conditions hold:*

(a) T is continuous;

(b) The function ϱ satisfies condition (ϱ_3);

(c) $\varrho(t,s) \leq s - t$ for all $t, s \in (0,\infty) \cap D$.

Then there exists a unique element $x \in A$ such that $d_v(x, Tx) = d_v(A, B)$, for each $v \in \mathfrak{A}$.

Proof. Let $x_0 \in A_0$ be an arbitrary point and $\{x_n\}$ be a proximal Picard sequence of T based on x_0; that is, $d_v(x_{n+1}, Tx_n) = d_v(A, B)$, for all $n \in \mathbb{N}$ and $v \in \mathfrak{A}$. For a given index $v \in \mathfrak{A}$, consider the sequence $a_n^v = d_v(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Let us assume first that $a_{n_0}^v = 0$ for some positive integer n_0. Then

$$\begin{aligned} d_v(A,B) &\leq d_v(x_{n_0}, Tx_{n_0}) \\ &\leq d_v(x_{n_0}, x_{n_0+1}) + d_v(x_{n_0+1}, Tx_{n_0}) \\ &= a_{n_0}^v + d_v(A,B) \\ &= d_v(A,B). \end{aligned}$$

Hence x_{n_0} is a best proximity point for T.

Assume now that $\{a_n^v\} \subseteq (0,\infty)$, for all $n \in \mathbb{N}$. Since T is an R-proximal contraction of the first kind with respect to $\varrho \in R_D$, we get

$$\varrho(a_{n+1}^v, a_n^v) = \varrho(d_v(x_{n+1}, x_{n+2}), d_v(x_n, x_{n+1})) > 0$$

Applying condition (ϱ_1), we obtain, as $n \to \infty$,

$$d_v(x_n, x_{n+1}) = a_n^v \to 0.$$

This shows that $\{x_n\}$ is an asymptotically regular sequence.

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exist $L > 0$, some $v \in \mathfrak{A}$, and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $k \leq n(k) \leq m(k)$ and $d_v(x_{n(k)-1}, x_{m(k)-1}) \leq L < d_v(x_{n(k)}, x_{m(k)})$, for all $k \in \mathbb{N}$. Using the fact that $\{x_n\}$ is asymptotically regular, as well as the triangle inequality, one finds $\lim_{k\to\infty} d_v(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d_v(x_{n(k)-1}, x_{m(k)-1}) = L$. Then we have $a_k = d_v(x_{n(k)}, x_{m(k)}) \to L$, $b_k = d_v(x_{n(k)-1}, x_{m(k)-1}) \to L$, $L < d_v(x_{n(k)}, x_{m(k)}) = a_k$, and $\varrho(a_k, b_k) = \varrho(d_v(x_{n(k)}, x_{m(k)}), d_v(x_{n(k)-1}, x_{m(k)-1})) > 0$ for all $k \in \mathbb{N}$. So, condition (ϱ_2) guarantees that $L = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in A. Since A is a closed subset of the complete space X, there exists $x^* \in A$ such that $x_n \to^{\mathfrak{F}} x^*$.

Now, we show that x^* is a best proximity point of T. Three cases are to be studied, as follows.

Case (a): T is continuous. Then, $d_v(x^*, Tx^*) = \lim_{n\to\infty} d_v(x_{n+1}, Tx_n) = d_v(A, B)$ for each $v \in \mathfrak{A}$. Hence, x^* is a best proximity point of T.

Case (b): ϱ satisfies condition (ϱ_3). In this case, we use the assumption that B is approximately compact with respect to A. By using the triangular inequality, we get

$$\begin{aligned} d_v(x^*, B) &\leq d_v(x^*, Tx_n) \\ &\leq d_v(x^*, x_{n+1}) + d_v(x_{n+1}, Tx_n) \\ &= d_v(x^*, x_{n+1}) + d_v(A,B) \\ &\leq d_v(x^*, x_{n+1}) + d_v(x^*, B), \text{ for all } v \in \mathfrak{A}. \end{aligned}$$

Therefore, $d_v(x^*, Tx_n) \to d_v(x^*, B)$, for each $v \in \mathfrak{A}$. Now by using the fact that B is approximately compact with respect to A, the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to an element y in B. It follows that, for each $v \in \mathfrak{A}$, $d_v(x^*, y) = \lim_{k\to\infty} d_v(x_{n_k+1}, Tx_{n_k}) = d_v(A,B)$, and hence x^* is an element of A_0. As $T(A_0) \subseteq B_0$, there exists $u \in A$ such that $d_v(u, Tx^*) = d_v(A,B)$ for each $v \in \mathfrak{A}$. Furthermore, for $v \in \mathfrak{A}$, consider $a_n^v = d_v(u, x_{n+1})$ and $b_n^v = d_v(x^*, x_n)$. Then we conclude that $b_n^v \to 0$ and by the definition of an R-proximal contraction of the first kind, we get $\varrho(a_n^v, b_n^v) = \varrho(d_v(u, x_{n+1}), d_v(x^*, x_n)) > 0$ for all $n \in \mathbb{N}$. Hence, by condition (ϱ_3), we get $a_n^v \to 0$. Thus, we conclude that $x^* = u$, because the space is separating. Hence, we have $d_v(x^*, Tx^*) = d_v(u, Tx^*) = d_v(A,B)$, for each $v \in \mathfrak{A}$.

Case (c): $\varrho(t,s) \le s - t$ for all $t, s \in (0,\infty) \cap D$. Then, by Proposition 2.1, we are again in Case (b).

Hence, in each case, x^* is a best proximity of T. Finally, we show that x^* is the unique best proximity point of T. Suppose that u^* is another best proximity point of T, such that $x^* \ne u^*$. By (1), we have $\varrho(d_v(x^*, u^*), d_v(x^*, u^*)) > 0$ for an index $v \in \mathfrak{A}$ (since $\mathfrak{F}$ is separating). But this inequality contradicts Proposition 2.2. So, the best proximity point of T is unique. □

Example 3.1. *On $X = \mathbb{R}^n$ consider the pseudo metrics defined by*

$$d_m((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) = \sum_{i=1}^{m} |x_i - y_i|$$

where $m \in \{1, 2, \cdots, n\}$. Take $A = \{(0, x_1, \cdots, x_{n-1}) : x_1, \cdots, x_{n-1} \in \mathbb{R}\}$ and $B = \{(1, x_1, \cdots, x_{n-1}) : x_1, \cdots, x_{n-1} \in \mathbb{R}\}$. Define $T : A \to B$ by $T((0, x_1, \cdots, x_{n-1})) = \left(1, \frac{x_1}{2}, \frac{x_2}{2^2}, \cdots, \frac{x_{n-1}}{2^{n-1}}\right)$ and $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\varrho(t,s) = s - t$. For the n-tuples $x = (0, x_1, x_2, \cdots, x_{n-1})$ and $y = (0, y_1, y_2, \cdots, y_{n-1})$ in A, the equalities $d_m(u, Tx) = d_m(v, Ty) = d_m(A,B)$ lead to

$$d_m\Big(u, \Big(1, \frac{x_1}{2}, \frac{x_2}{2^2}, \cdots, \frac{x_{n-1}}{2^{n-1}}\Big)\Big) = d_m\Big(v, \Big(1, \frac{y_1}{2}, \frac{y_2}{2^2}, \cdots, \frac{y_{n-1}}{2^{n-1}}\Big)\Big) = 1,$$

for all the indices $m \in \{1, 2, \cdots, n\}$. These produce the unique solutions $u = \big(0, \frac{x_1}{2}, \frac{x_2}{2^2}, \cdots, \frac{x_{n-1}}{2^{n-1}}\big)$ and $v = \big(0, \frac{y_1}{2}, \frac{y_2}{2^2}, \cdots, \frac{y_{n-1}}{2^{n-1}}\big) \in A$. On the other side,

$$\varrho(d_m(u,v), d_m(x,y)) = \varrho\Big(d_m\Big(\Big(0, \frac{x_1}{2}, \frac{x_2}{2^2}, \cdots, \frac{x_{n-1}}{2^{n-1}}\Big), \Big(0, \frac{y_1}{2}, \frac{y_2}{2^2}, \cdots, \frac{y_{n-1}}{2^{n-1}}\Big)\Big), \\ d_m((0, x_1, x_2, \cdots, x_{n-1}), (0, y_1, y_2, \cdots, y_{n-1}))\Big) > 0$$

for each $m \in \{1, 2, \cdots, n\}$, whenever at least one of the above $d_m(\cdot,\cdot)$ is nonzero. The rest of the conditions in the Theorem 3.1 are also fulfilled. Thus, we have an unique $x \in A$ such that $d_m(x, Tx) = d_m(A,B)$, for each $m \in \{1, 2, \cdots, n\}$.

We introduce now our first consequence of this theorem, by means of the notion of $\mathcal{Z}$-proximal contractions. Having in mind this notion exposed in Definition 3.2, the following result becomes an immediate consequence of Theorem 3.1 and Remark 3.1.

Corollary 3.1. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that B is approximately compact with respect to A, and A_0 is nonempty. Let $T\colon A \to B$ be a $\mathcal{Z}$-proximal contraction of the first kind such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

Similarly, Theorem 3.1 and Lemma 3.1 lead to a best proximity point outcome for Geraghty proximal contractions as follows.

Corollary 3.2. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that B is approximately compact with respect to A, and A_0 is nonempty. Let $T\colon A \to B$ be a*

Geraghty proximal contraction of the first kind satisfying $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.

In the following definition, we introduce the notion of R-proximal contraction of the second kind.

Definition 3.3. *A mapping $T\colon A \to B$ is said to be an R-proximal contraction of the second kind if there exists an R-function $\varrho \in R_D$ such that $ran(\mathfrak{F}) \subseteq D$ and $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ implies that*

$$\varrho(d_v(Tu_1, Tu_2), d_v(Tx_1, Tx_2)) > 0, \text{ for each } v \in \mathfrak{A}, \tag{2}$$

where $u_1, u_2, x_1, x_2 \in A$, provided that at least one of the elements $d_v(Tu_1, Tu_2)$, $d_v(Tx_1, Tx_2)$ is not null.

Same as before, we could adequately define the concept of $\mathcal{Z}$-proximal contraction of the second kind as follows.

Definition 3.4. *A mapping $T\colon A \to B$ is said to be an $\mathcal{Z}$-proximal contraction of the second kind if there exists a simulation function function $\zeta\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ implies that*

$$\zeta(d_v(Tu_1, Tu_2), d_v(Tx_1, Tx_2)) > 0 \text{ for each } v \in \mathfrak{A},$$

where $u_1, u_2, x_1, x_2 \in A$, provided that at least one of the elements $d_v(Tu_1, Tu_2)$, $d_v(Tx_1, Tx_2)$ is not null.

Remark 3.2. *Obviously, each $\mathcal{Z}$-proximal contraction of the second kind is also an R-proximal contraction of the second kind. Moreover, similar arguments as in Lemma 3.1 ensure us that each Geraghty proximal contraction of the second kind satisfies also the R-proximal contractive condition.*

The following result ensures the existence and uniqueness of a best proximity point for R-proximal contractive mappings of the second kind.

Theorem 3.2. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be an R-proximal contraction of the second kind with respect to $\varrho \in R_D$. Further, assume that T is continuous and $T(A_0) \subseteq B_0$. Then there exists a unique element $x \in A$ such that $d_v(x, Tx) = d_v(A, B)$, for each $v \in \mathfrak{A}$.*

Proof. Let $x_0 \in A_0$ be an arbitrary point and $\{x_n\}$ be a proximal Picard sequence of T based on x_0, that is, $d_v(x_{n+1}, Tx_n) = d_v(A, B)$, for all $n \in \mathbb{N}$ and $v \in \mathfrak{A}$. For $v \in \mathfrak{A}$, consider the sequence defined by $a_n^v = d_v(Tx_n, Tx_{n+1})$, for all $n \in \mathbb{N}$. We may assume from the beginning that $\{a_n^v\} \subseteq (0, \infty)$, otherwise the conclusion is trivial. Since T is an R-proximal contraction of the second kind with respect to $\varrho \in R_D$, we have

$$\varrho(a_{n+1}^v, a_n^v) = \varrho(d_v(Tx_{n+1}, Tx_{n+2}), d_v(Tx_n, Tx_{n+1})) > 0.$$

By applying condition (ϱ_1), we get

$$d_v(Tx_n, Tx_{n+1}) = a_n^v \to 0, \text{ as } n \to \infty \text{ for each } v \in \mathfrak{A}.$$

This shows that $\{Tx_n\}$ is an asymptotically regular sequence.

Next, we show that $\{Tx_n\}$ is a Cauchy sequence. Suppose the contrary, that $\{Tx_n\}$ is not a Cauchy sequence. Then there exist $L > 0$, some $v \in \mathfrak{A}$, and two subsequences $\{Tx_{m(k)}\}$, and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $k \leq n(k) \leq m(k)$ and $d_v(Tx_{n(k)-1}, Tx_{m(k)-1}) \leq L < d_v(Tx_{n(k)}, Tx_{m(k)})$, for all $k \in \mathbb{N}$. The asymptotical regularity leads to $\lim_{k\to\infty} d_v(Tx_{n(k)},$

$Tx_{m(k)}) = \lim_{k\to\infty} d_v(Tx_{n(k)-1}, Tx_{m(k)-1}) = L$. We have $a_k = d_v(Tx_{n(k)}, Tx_{m(k)}) \to L$ and $b_k = d_v(Tx_{n(k)-1}, Tx_{m(k)-1}) \to L$. It can be seen that

$$L < d_v(Tx_{n(k)}, Tx_{m(k)}) = a_k$$

and

$$\varrho(a_k, b_k) = \varrho(d_v(Tx_{n(k)}, Tx_{m(k)}), d_v(Tx_{n(k)-1}, Tx_{m(k)-1})) > 0$$

for all $k \in \mathbb{N}$. By using condition ϱ_2, we get $L = 0$, which is a contradiction. Therefore, $\{Tx_n\}$ is a Cauchy sequence in B. Since B is a closed subset of the complete space X, there exists $y^* \in B$ such that $Tx_n \to^{\mathfrak{F}} y^*$. By using the triangular inequality, we get

$$\begin{aligned} d_v(y^*, A) &\leq d_v(y^*, x_{n+1}) \\ &\leq d_v(y^*, Tx_n) + d_v(Tx_n, x_{n+1}) \\ &= d_v(y^*, Tx_n) + d_v(A, B) \\ &\leq d_v(y^*, Tx_n) + d_v(y^*, A), \text{ for all } v \in \mathfrak{A}. \end{aligned}$$

Therefore, $d_v(y^*, x_n) \to d_v(y^*, A)$, for each $v \in \mathfrak{A}$. Now, by using the fact that A is approximately compact with respect to B, the sequence $\{x_n\}$ has a subsequence $\{x_{n(k)}\}$ converging to an element x^* in A. By the hypotheses, T is continuous. Thus, we have $d_v(x^*, Tx^*) = \lim_{k\to\infty} d_v(x_{n(k)}, Tx_{n(k)-1}) = d_v(A, B)$, for each $v \in \mathfrak{A}$.

The uniqueness of the best proximity point of T can be proved as in Theorem 3.1. □

Based on Remark 3.2 and as direct consequence of the above result we state the following corollaries.

Corollary 3.3. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be a continuous $\mathcal{Z}$-proximal contraction of second kind such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

Corollary 3.4. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be a continuous Geraghty proximal contraction of the second kind satisfying $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

4. Conclusions

In this paper, we have defined the notions of R-proximal contraction of the first and second kind in the setting of gauge spaces. Some existence and uniqueness results of best proximity points have been stated and proved. Also, some results known in literature have been obtained as consequences of these results.

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