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θ-TYPE CALDERÓN-ZYGMUND OPERATORS ON MORREY AND MORREY-HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENTS

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In this paper, the authors establish the boundedness of the θ-type Caldern-Zygmund Operators T_θ and their commutators on the variable exponent Morrey spaces. Moreover, the various norm estimates for T_θ are also obtained on the variable exponents Morrey-Herz-type Hardy spaces.

Keywords: θ-type Calderón-Zygmund operator, variable exponent Morrey space, variable exponent Morrey-Herz-type Hardy space.

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1. Introduction and main theorems

In the last three decades, the interest for variable exponent spaces have been increasing year by year due to themselves and their applications in electrorheological fluid and differential equations[5, 12]. In [11, 15], the authors obtained the atomic and molecular decompositions of Hardy spaces with variable exponent and studied the boundedness of a class of singular integral operators. The results show that they are not only the generalized forms of the classical function spaces with invariable exponent, but also there are some new breakthroughs in the research techniques. These new real variable methods help people further understand those spaces. Rencently, Xu and Yang introduce the Morrey-Herz-type Hardy spaces with variable exponents and establish the boundedness of singular integral operators on these spaces in [17].

In 1985, Yabuta introduced certain θ-type Calderón-Zygmund operators to facilitate certain classes of pseudodifferential operators [18]. Following the terminology of Yabuta, we recall the so-called θ-type Caldern-Zygmund operators. Let θ be a non-negative and non-decreasing function on $(0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty. \tag{1.1}$$

A measurable function $K(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a θ-type Caldern-Zygmund kernel if it satisfies

$$|K(x,y)| \leq C|x-y|^{-n} \text{ and} \tag{1.2}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq \frac{C}{|x-y|^n}\theta(\frac{|x-z|}{|x-y|}), \text{ as } |x-y| \geq 2|x-z|. \tag{1.3}$$

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Definition 1.1. [18] Let T_θ be a linear operator from $\mathcal{S}$ into its dual $\mathcal{S}'$. One can say that T_θ is a θ-type Caldern-Zygmund operator (also called Dini-type singular integral) if

(i) T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;

(ii) there is a θ-type Calderón-Zygmund kernel $K(x,y)$ such that

$$T_\theta f(x) := \int_{\mathbb{R}^n} K(x,y)f(y)\mathrm{d}y, \text{ as } f \in C_c^\infty(\mathbb{R}^n) \text{ and } x \notin \operatorname{supp} f. \tag{1.4}$$

As $\theta(t) = t^\delta(\delta > 0)$, T_θ coincide with the classical Caldern-Zygmund operator with standard kernel.

Many authors concentrate on the properties of T_θ on various function spaces. In [9], J. Lan established the boundedness of T_θ on the weighted Lebesgue and Hardy spaces. Ri-Zhang obtained the bounedness of T_θ on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces in [13, 14]. Wang proved the boundedness of T_θ and $[b, T_\theta]$ on the generalized weighted Morrey spaces in [16]. After that, the authors established the boundedness of T_θ on variable exponent Herz spaces and weighted variable exponent Morrey spaces in [19, 20]. Inspired by the results mentioned previously, a natural and interesting problem is to consider whether T_θ and its commutator $[b, T_\theta]$ are bounded on Morrey and Morrey-Herz-type Hardy spaces with variable exponents or not. The purpose of this paper is to give an surely answer. We now formulate our other main results as follows.

Theorem 1.1. *Suppose that θ satisfies (1,1) and $K(\cdot,\cdot)$ satisfies (1.2)-(1.3). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that*

$$\|T_\theta(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.2. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and θ satisfies $\int_0^1 \frac{\theta(t)}{t}|\log t|\mathrm{d}t < \infty$, then there exists a constant $C > 0$ such that*

$$\|[b, T_\theta](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.3. *Suppose that θ satisfies (1,1). Let $u \in \mathbb{W}_{p(\cdot)}$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that*

$$\|T_\theta(f)\|_{\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)}.$$

Theorem 1.4. *Suppose that θ satisfies $\int_0^1 \frac{\theta(t)}{t}|\mathrm{log}t|\mathrm{d}t < \infty$ and u satisfies*

$$\sum_{j=0}^{\infty}(j+1)\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\frac{u(x,2^{j+1}r)}{u(x,r)} \le C. \tag{1.5}$$

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that

$$\|[b, T_\theta](f)\|_{\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)}.$$

Theorem 1.5. *Let $0 < q \le \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $0 \le \lambda < \infty$. Suppose that $\alpha(\cdot)$ is log-Hölder continuous both at the origin and infinity, $2\lambda \le \alpha(\cdot)$, $n\delta_2 \le \alpha(0)$, $\alpha_\infty < n\delta_1 + \delta$ with some $\delta > \max\{\alpha(0) - n\delta_1, \alpha_\infty - n\delta_1\}$, where δ_1 as in Lemma 2.1. If θ satisfies (1,1), then there exists a constant $C > 0$ independent of f such that*

$$\|T_\theta f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \le C\|f\|_{HM\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.$$

Throughout this paper, χ_E denotes the characteristic function of E and $f_E = 1/|E|\int_E f$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely, $1/p(x) + 1/p'(x) = 1$ holds. The symbol C stands for a positive constant, which may vary from line to line.

2. **Definitions and preliminaries**

In this section, some preliminary definitions and lemmas will be given.

Definition 2.1. [10] Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$. If there exist $C > 0$ such that for any $x, y \in \mathbb{R}^n$, $|x-y| < 1/2$,

$$|\alpha(x) - \alpha(y)| \leq \frac{C}{-\log(|x-y|)},$$

then $\alpha(\cdot)$ is said local log-Hölder continuous on $\mathbb{R}^n$. If

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{\log(e + 1/|x|)},$$

then $\alpha(\cdot)$ is said log-Hölder continuous at origin. If there exist $\alpha_\infty \in \mathbb{R}$ and a constant $C > 0$ such that all $x \in \mathbb{R}^n$,

$$|\alpha(x) - \alpha_\infty| \leq \frac{C}{\log(e + |x|)},$$

then $\alpha(\cdot)$ is said log-Hölder continuous at infinity.

Let $p : \mathbb{R}^n \longrightarrow [1, \infty)$ be a measurable function. We denote $p_- = \operatorname{ess}\inf_{x \in \mathbb{R}^n} p(x)$, $p_+ = \operatorname{ess}\sup_{x \in \mathbb{R}^n} p(x)$. Then $\mathcal{P}(\mathbb{R}^n)$ consists all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that the Hardy-littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, where $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by[8]

$$L^{p(\cdot)}(\mathbb{R}^n) = \{f \text{ is measurable} : \int_{\mathbb{R}^n} (\frac{|f(x)|}{\eta})^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0\}$$

with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\eta > 0 : \int_{\mathbb{R}^n} (\frac{|f(x)|}{\eta})^{p(x)} \mathrm{d}x \leq 1\}.$$

Definition 2.2. [4] Let $p(x) \in \mathcal{P}(\mathbb{R}^n)$ and $u(x, r) : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. If there exists a constant $C > 0$ such that

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x, 2^{j+1}r) \leq Cu(x, r), \tag{2.1}$$

then one says u is a Morrey weighted function. We denote the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$.

Definition 2.3. [4] Let $u(x, r) \in \mathbb{W}_{p(\cdot)}$. The Morrey spaces with variable exponent are defined by

$$\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n) := \{\text{f is measurable} : \|f\|_{\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)} < \infty\}, \tag{2.2}$$

where

$$\|f\|_{\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, R > 0} \frac{1}{u(x, R)} \|f\chi_{B(x,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{2.3}$$

Remark 2.1. (1) If $u(x, r) \equiv 1$, then $\mathcal{M}_{p(\cdot),\mu}(\mathbb{R}^n)$ is the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$. (2) Notice that if $p(x) \equiv p$, $1 < p < \infty$, is a constant function, then formula (2.1) can be written in the following form [4]

$$\int_r^\infty \frac{u(x, t)}{t^{n/p+1}} \mathrm{d}t \leq C \frac{u(x, r)}{r^{n/p}}, \ r > 0, \ x \in \mathbb{R}^n. \tag{2.4}$$

(3) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $u(x, r) = |B(x, r)|^{1/p(x) - 1/q(x)}$ with $p(x) \leq q(x)$. Denote $k_{p(\cdot)} = \sup\{q > 1 : p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)\}$ and $e_{p(.)}$ to be the conjugate exponent of $k_{p'(\cdot)}$. As $1 < s <$

$k_{p'(x)}$, $1/p(x) - 1/q(x) < 1 - 1/s$, it is easy to see $u(x, r)$ satisfying condition (2.1). That is because of the following fact [4]

$$\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C2^{jn(1/s-1)},\ r > 0,\ x \in \mathbb{R}^n,\ j \in \mathbb{N}. \tag{2.5}$$

Let $k \in \mathbb{Z},\ B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, C_k = B_k \backslash B_{k-1}$, and $\chi_k = \chi_{C_k}$.

Definition 2.4. [17] Let $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$ and $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous Morrey-Herz spaces spaces $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ is defined by $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\}$, where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\{\sum_{k=-\infty}^{L} \|2^{\alpha(\cdot)k} f\chi_k\|^q_{L^{p(\cdot)}}\}^{1/q}.$$

Next we recall the definition of Morrey-Herz-type Hardy spaces with variable exponents which was firstly introduced by Xu and Yang in [17]. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) = \sup_{\phi\in\mathcal{A}_N} |\phi^*_\nabla(f)(x)|,\ x \in \mathbb{R}^n,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\nu|,|\beta|\leq N, \forall x\in\mathbb{R}^n} |x^\nu D^\beta \phi(x)| \leq 1\}$ and $N > n + 1$ and ϕ^*_∇ is the nontangential maximal operator defined by $\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$, where $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$.

Definition 2.5. [17] Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n),\ 0 < q \leq \infty,\ p(\cdot) \in \mathcal{P}(\mathbb{R}^n),\ 0 \leq \lambda < \infty$, and $N > n + 1$. The homogeneous Morrey-Herz-type Hardy space with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ is defined by

$$HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} = \|G_N f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty\}.$$

Definition 2.6. [17] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot)$ be log-Hölder continuous both at the origin and infinity. Suppose that $\alpha_r = \alpha(0)$, as $r < 1$, and $\alpha_r = \alpha_\infty$, as $r \geq 1$, $n\delta_1 \leq \alpha_r < \infty$ and nonnegative integer $s \geq [\alpha_r - n\delta_1]$, where δ_1 is as in Lemma 2.1.

(1) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom, if it satisfies (a) supp $a \subset B(0, r)$; (b) $\|a\|_{L^{p(\cdot)}} \leq |B(0, r)|^{-\alpha_r/n}$; (c) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0,\ |\beta| \leq s$.

(2) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies (a) supp $a \subset B(0, r),\ r \geq 1$; (b) $\|a\|_{L^{p(\cdot)}} \leq |B(0, r)|^{-\alpha_r/n}$; (c) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0,\ |\beta| \leq s$

Lemma 2.1. [6] If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2 > 0$, such that for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|},\ \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1},\ \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Lemma 2.2. [17] Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, and $\alpha(\cdot) \in L^\infty$ be $2\lambda \leq \alpha(\cdot)$, $n\delta_1 \leq \alpha(0)$, $\alpha_\infty < \infty$ with δ_1 as in Lemma 2.1. Then $f \in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ if and only if $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_k and $\sup_{L\in\mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^{L} |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf \sup_{L\in\mathbb{Z}} 2^{-L\lambda}(\sum_{k=-\infty}^{L} |\lambda_k|^q)^{1/q}.$$

Lemma 2.3. [8] Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then

(1) For any $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq C_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},\ C_p = 1 + 1/p_- - 1/p_+,$$

where $p_- = \text{ess}\inf_{x\in\mathbb{R}^n} p(x),\ p_+ = \text{ess}\sup_{x\in\mathbb{R}^n} p(x)$.

(2) For any $f \in L^{p_1(\cdot)}$ and $g \in L^{p_2'(\cdot)}$, as $1/p(x) = 1/p_1(x) + 1/p_2(x)$, then

$$\|f(x)g(x)\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C_{p_1,p_2} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}},\ C_{p_1,p_2} = (1 + 1/p_- - 1/p_+)^{1/p_-}.$$

Lemma 2.4. [1] If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists constant $C > 0$, such that, for all balls $B \in \mathbb{R}^n$,

$$1/|B| \|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leq C.$$

Lemma 2.5. [7] Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and m be a positive integer. Then there exists constant $C > 0$, such that, for any $k, j \in \mathbb{Z}$ with $k > j$,

(1) $C^{-1}\|b\|_*^m \leq \sup_B (1/\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}) \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*^m$;

(2) $\|(b - b_{B_j})^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k - j)^m \|b\|_*^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 2.6. [17] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n),\ q \in (0, \infty]$, and $\lambda \in (0, \infty]$. If $\alpha(\cdot) \in L^\infty$ is log-Hölder continuous both at the origin and infinity, then

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \approx \max\{\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} (\textstyle\sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \|f\chi_k\|^q_{L^{p(\cdot)}})^{1/q},$$

$$\sup_{L>0, L\in\mathbb{Z}} [2^{-L\lambda} (\textstyle\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|f\chi_k\|^q_{L^{p(\cdot)}})^{1/q} + 2^{-L\lambda} (\textstyle\sum_{k=0}^{L} 2^{kq\alpha_\infty} \|f\chi_k\|^q_{L^{p(\cdot)}})^{1/q}]\}.$$

3. **Proofs of main theorems**

From the results of weighted norm inequalities and extension of Rubio de Francia extrapolation to the scale of variable Lebesgue spaces [2, 3], Theorems 1.1-1.2 are immediate consequences of known results. Thus we omit the details.

Proof of Theorem 1.3. Let $f \in \mathcal{M}_{p(\cdot),\mu}$. In order to prove Theorem 1.3, it is enough to show that the following inequality holds:

$$\frac{1}{u(z,r)} \|T_\theta(f)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{p(\cdot),\mu}}.$$

For any $z \in \mathbb{R}^n$ and $r > 0$, denote $f(x) = f_1(x) + f_2(x)$, where $f_1 = f\chi_{B(z,2r)}$, $f_2 = \sum_{j=1}^\infty f_j(x)$, $f_j(x) = f\chi_{B(z,2^{j+1}r)\backslash B(z,2^j r)}$, $j \in \mathbb{N}\backslash\{0\}$. Then

$$\begin{aligned}\tfrac{1}{u(z,r)} \|T_\theta(f)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \tfrac{1}{u(z,r)} \|T_\theta(f_1)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &+ \tfrac{1}{u(z,r)} \|T_\theta(f_2)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} =: D_1 + D_2.\end{aligned}$$

Noticing that T_θ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, it is easy to see by Lemma 2.1

$$\begin{aligned}D_1 &\leq C \frac{\|\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{1}{u(z,2r)} \|\chi_{B(z,2r)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C \frac{|B(z,2r)|}{|B(z,r)|} \frac{1}{u(z,2r)} \|\chi_{B(z,2r)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{p(\cdot),\mu}}\end{aligned}$$

We now turn to estimate D_2. Observing that if $x \in B(z,r), y \in B(z, 2^{j+1}r)\backslash B(z, 2^j r)$, then we have

$$|x - y| \geq |y - z| - |x - z| \geq C2^{-2}2^{(j+1)}r. \tag{3.1}$$

Thus, Lemma 2.3 yields

$$
\begin{aligned}
|T_\theta f_2(x)| &\le C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\int_{B(z,2^{j-1}r)\setminus B(z,2^j r)}|f(y)|\mathrm{d}y\\
&\le C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Hence, $\|T_\theta f_2(x)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$

$$
\le C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}.
$$

Applying Lemma 2.4 with $B=B(z,2^{j+1}r)$, we get

$$
\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C2^{(j+1)n}r^n/\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.2}
$$

Therefore, one has

$$
\begin{aligned}
\|T_\theta f_2(x)\chi_{B(z,r)}\|_{L^{p(\cdot)}} &\le C\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}}\cdot\frac{u(z,2^{(j+1)}r)}{u(z,2^{(j+1)}r)}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\\
\le C\sum_{j=1}^{\infty}&\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}u(z,2^{(j+1)}r)}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}}\cdot\sup_{z\in\mathbb{R}^n,r>0}\frac{1}{u(z,r)}\|\chi_{B(z,r)}f\|_{L^{p(\cdot)}}.
\end{aligned}
$$

Thus we have $D_2\le C\|f\|_{\mathcal{M}_{p(\cdot),\mu}}$. The proof of the **Theorem 1.3** is finished. □

Proof of Theorem 1.4. Let $b\in\mathrm{BMO}$, $f\in\mathcal{M}_{p(\cdot),\mu}$. We have

$$
\begin{aligned}
\frac{1}{u(z,r)}\|[b,T_\theta]f\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le \frac{1}{u(z,r)}\|[b,T_\theta]f_1\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad+\frac{1}{u(z,r)}\|[b,T_\theta]f_2\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}=:E_1+E_2,
\end{aligned}
$$

where $f(x)=f_1(x)+f_2(x)$, $f_1(x)$ and $f_2(x)$ are the same as in the proof of **Theorem 1.3**. With the similar argument for D_1 in the proof of Theorem 1.3, we have $E_1\le C\|b\|_*\|f\|_{\mathcal{M}_{p(\cdot),\mu}}$. Thus only need to estimate E_2.

Lemma 2.3 and (3.1) yield

$$
\begin{aligned}
|[b,T_\theta](f_2)(x)| &\le C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}|b(x)-b_{B(z,r)}|\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}\\
&\quad+C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|(b_{B(z,r)}-b)\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|[b,T_\theta](f_2)(x)\chi_{B(z,r)}\|_{L^{p(\cdot)}} &\le C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}\\
\cdot\|(b-b_{B(z,r)})\chi_{B(z,r)}\|_{L^{p(\cdot)}}&+C\sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\\
\cdot\|(b_{B(z,r)}-b)\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}&\cdot\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}
\end{aligned}
$$

Applying Lemma 2.5, we get

$$\|[b,T_\theta](f_2)(x)\chi_{B(z,r)}\|_{L^{p(\cdot)}} \le C\sum_{j=1}^{\infty}\frac{j+1}{(2^{j+1}r)^n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}}\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}.$$

Together the above inequality with (3.2), we have

$$\begin{aligned}
&\|[b,T_\theta](f_2)(x)\chi_{B(z,r)}\|_{L^{p(\cdot)}}\\
&\le C\sum_{j=1}^{\infty}(j+1)(2^{j+1}r)^{-n}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}\cdot\frac{2^{(j+1)n}r^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}}\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}\\
&\le C\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}}u(z,2^{(j+1)}r)}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}}}\cdot\sup_{y\in\mathbb{R}^n,r>0}\frac{1}{u(y,r)}\|\chi_{B(y,r)}f\|_{L^{p(\cdot)}}.
\end{aligned}$$

Then by (1.5), we have $E_2\le C\|b\|_*\|f\|_{\mathcal{M}_{p(\cdot),\mu}}$. Combining the estimates E_1,E_2, we complete the proof of **Theorem 1.4**. □

Proof of Theorem 1.5. Let $f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. By Lemma 2.2, $f=\sum_{j=-\infty}^{\infty}\lambda_jb_j$ (in $\mathcal{S}'(\mathbb{R}^n)$), where each b_j is a central $(\alpha(\cdot),q(\cdot))-$ atom with support contained in B_j. We denote $\Phi=\sup\limits_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q$. we have

$$\begin{aligned}
\|T_\theta f\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \max\{&\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|(T_\theta f)\chi_k\|^q_{L^{p(\cdot)}},\\
&\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|(T_\theta f)\chi_k\|^q_{L^{p(\cdot)}}+\sum_{k=0}^{L}2^{kq\alpha(\infty)}\|(T_\theta f)\chi_k\|^q_{L^{p(\cdot)}})\}\\
&=:\max\{\mathrm{I},\mathrm{II}+\mathrm{III}\}.
\end{aligned}$$

To complete our proof, it suffices to show that $\mathrm{I},\mathrm{II},\mathrm{III}\lesssim C\Phi$.

$$\begin{aligned}
\mathrm{I}&\lesssim\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}(\sum_{j=k}^{\infty}|\lambda_j|\|[T_\theta b_j]\chi_k\|_{L^{p(\cdot)}})^q\\
&+\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}(\sum_{j=-\infty}^{k-1}|\lambda_j|\|[T_\theta b_j]\chi_k\|_{L^{p(\cdot)}})^q=:\mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

By the result of Theorem 1.1, we have $\|[(T_\theta b_j)\chi_k]\|_{L^{p(\cdot)}}\le C\|b_j\|_{L^{p(\cdot)}}\le C2^{-\alpha_jj}$. Therefore, as $0<q\le1$, we get

$$\begin{aligned}
\mathrm{I}_1&\le Cm^{nq/2}\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\times\sum_{k=-\infty}^{L}2^{kq\alpha(0)}(\sum_{j=k}^{-1}|\lambda_j|^q2^{-\alpha(0)jq}+\sum_{j=0}^{\infty}|\lambda_j|^q2^{-\alpha_\infty jq})\\
&\le\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q+\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=L}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}\\
&+\Phi\sup_{L\le0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty)jq}\sum_{k=-\infty}^{L}2^{\alpha(0)kq-L\lambda q}\le C\Phi.
\end{aligned}$$

As $1<q<\infty$, we can obtain

$$\mathrm{I}_1 \le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q + \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q/2}$$
$$+ \Phi \sup_{L\le 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)jq} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq-L\lambda q} \le C\Phi.$$

Second, we estimate I_2. A simple computation shows that there exists a constant $\delta>0$ such that T_θ satisfies the following size condition

$$|T_\theta f| \le C(\mathrm{diam}(\mathrm{supp} f))^\delta |x|^{-(n+\delta)} \|f\|_1, \text{ when } \mathrm{dist}(x, \mathrm{supp} f) \ge \frac{|x|}{2}.$$

Thus with the help of Lemma 2.3, we get

$$\begin{aligned} |T_\theta b_j(x)| &\le C|x|^{-(n+\delta)} 2^{j\delta} \int_{B_j} |b_j(y)| \mathrm{d}y \\ &\le C 2^{-k(n+\delta)} 2^{j\delta} \|b_j\|_{L^{p(\cdot)}} \|\chi_{B_j}\|_{L^{p'}(\cdot)} \\ &\le C 2^{j(\delta-\alpha_j)-k(\delta+n)} \|\chi_{B_j}\|_{L^{p'}(\cdot)}. \end{aligned}$$

Thus, we have by Lemma 2.1 and 2.3

$$\begin{aligned} \|(T_\theta b_j)\chi_k\|_{L^{p(\cdot)}} &\le C 2^{j(\delta-\alpha_j)-k\delta} 2^{-kn} (|B_k| \|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}}) \|\chi_{B_j}\|_{L^{p'(\cdot)}} \\ &\le C 2^{(\delta+n\delta_2)(j-k)-j\alpha_j}. \end{aligned}$$

Therefore, when $0<q\le 1$, we get by noting the fact $n\delta_1 \le \alpha(0) < \delta + n\delta_1$

$$\begin{aligned} \mathrm{I}_2 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} (\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{[(\delta+n\delta_1)(j-k)-j\alpha(0)]q}) \\ &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(\delta+n\delta_1-\alpha(0))q} \le C\Phi. \end{aligned}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_1 \le \alpha(0) \le \delta + n\delta_1$, by Hölder's inequality, we have

$$\begin{aligned} \mathrm{I}_2 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} (\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{(\delta+n\delta_1)(j-k)-j\alpha(0)})^q \\ &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{\alpha(0)kq} (\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta+n\delta_1-\alpha(0))q/2}) \le C\Phi. \end{aligned}$$

Hence, we have $\mathrm{I} \le C\Phi$.

Third, we estimate II. Consider

$$\begin{aligned} \mathrm{II} \le C &\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} (\sum_{j=k}^{\infty} |\lambda_j| \|(T_\theta b_j)\chi_k\|_{L^{p(\cdot)}})^q \\ &+ \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} (\sum_{j=-\infty}^{k-1} |\lambda_j| \|(T_\theta b_j)\chi_k\|_{L^{p(\cdot)}})^q =: \mathrm{II}_1 + \mathrm{II}_2. \end{aligned}$$

When $0<q\le 1$, we get

$$
\begin{aligned}
\mathrm{II}_1 &\le C \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \Big(\sum_{j=k}^{-1} |\lambda_j|^q 2^{-\alpha(0)jq} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty jq}\Big) \\
&\le C \sum_{k=-\infty}^{-1} \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q} + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty - jq} \\
&\le C \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty jq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\le C \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{-\alpha_\infty jq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \le C\Phi.
\end{aligned}
$$

As $1 < q < \infty$, we have

$$
\begin{aligned}
\mathrm{II}_1 &\le C \sum_{k=-\infty}^{-1} \Big(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)}\Big)^q + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \Big(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_\infty j}\Big)^q \\
&\le C \sum_{k=-\infty}^{-1} \Big(\sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2}\Big)\Big(\sum_{j=k}^{-1} 2^{\alpha(0)(k-j)q'/2}\Big)^{q/q'} \\
&+ 2^{\alpha(0)kq} \Big(\sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty jq/2}\Big)\Big(\sum_{j=0}^{\infty} 2^{-\alpha_\infty jq'/2}\Big)^{q/q'} \\
&\le C \sum_{k=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q/2} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty jq/2} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\le C \sum_{k=-\infty}^{-1} |\lambda_j|^q + \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty/2}) jq} 2^{-j\lambda q} \sum_{i=-\infty}^{j} |\lambda_i|^q \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \le C\Phi.
\end{aligned}
$$

With a similar argument as in estimating I_2, we can obtain that $\mathrm{II}_2 \le C\Phi$.

Finally, we estimate III. Write

$$
\begin{aligned}
\mathrm{III} \le C &\sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \Big(\sum_{j=k}^{\infty} |\lambda_j| \|(T_\theta b_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q \\
&+ \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \Big(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(T_\theta b_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q =: \mathrm{III}_1 + \mathrm{III}_2.
\end{aligned}
$$

Also, with a similar argument as in estimating I_1 and I_2, we can get that $\mathrm{III}_1,\ \mathrm{III}_2 \le C\Phi$.

Combining the estimates of I, II, and III, we complete the proof of Theorem 1.5. □

4. **Conclusions**

With the help of the results obtained in [11] that the θ-type Calderón-Zygmund operator T_θ is bounded on the variable exponents Herz spaces, we show that T_θ is bounded on the variable Lebesgue spaces, Morrey spaces and Morrey-Herz-type Hardy spaces. Meanwhile, its commutator which generated by T_θ and a BMO function is also bounded respectively on the variable exponent Lebesgue spaces and Morrey spaces.

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