DIRECT SUMS OF S – DECOMPOSABLE AND S – SPECTRAL OPERATOR SYSTEMS

Cristina SERBĂNESCU[1], Mariana ZAMFIR[2]

In this paper is trying to extend and generalize several results of the spectral theory for a single S – decomposable (S – spectral) operator to S – decomposable (S – spectral) operator systems. The goal of the work is to establish the behaviour of S – decomposable (S – spectral) systems related to direct sums, by showing that the direct sum of two systems is an S – decomposable (S – spectral) system if and only if both systems are S – decomposable (S – spectral). These spectral decompositions are related to differential equations and to systems of differential equations ([9]) *and can have applications in quantum mechanics and fractal theory.*

Keywords: Taylor spectrum; S – spectral capacity; S – decomposable system; S – spectral measure; S – spectral system; direct sum.

1. Introduction

Along this paper, we consider $\mathbb{C}^n$ to be the space of all elements $z=(z_1,z_2,...,z_n)$, with $z_j \in \mathbb{C}$, $1 \le j \le n$, X and Y to be two complex Banach spaces, $\mathbf{B}(X)$ to be the algebra of all linear bounded operators on X, $\mathbf{P}(X)$ to be the set of all projectors on X and $\mathbf{S}(X)$ to be the family of all linear closed subspaces of X Moreover, if $S \subset \mathbb{C}^n$ is a compact fixed set, we denote by F_S^n (respectively, by B_S^n) the family of all closed subsets $F \subset \mathbb{C}^n$ (respectively, the family all Borelian subsets $B \subset \mathbb{C}^n$) which have the property that $F \cap S = \varnothing$ or $F \supset S$ (respectively, $B \cap S = \varnothing$ or $B \supset S$).

Let $a=(a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ be a system of commuting operators, i.e. $a_i a_j = a_j a_i$, $1 \le i, j \le n$. The system a is called *nonsingular* on X if the associated Koszul complex $E(X,a)$ is exact, where

[1] Lecturer, Faculty of Applied Sciences, University POLITEHNICA of Bucharest, Romania, email: mserbanescu@yahoo.com
[2] Lecturer, Department of Mathematics and Computer Science, Technical University of Civil Engineering of Bucharest, email: zamfirvmariana@yahoo.com

$$E(X,a):0\to X=\Lambda^n[\sigma,X]\overset{\delta_n}{\to}\Lambda^{n-1}[\sigma,X]\overset{\delta_{n-1}}{\to}\ldots$$
$$\ldots\overset{\delta_3}{\to}\Lambda^2[\sigma,X]\overset{\delta_2}{\to}\Lambda^1[\sigma,X]\overset{\delta_1}{\to}\Lambda^0[\sigma,X]=X\to 0$$

or, equivalent, the complex $F(X,a)$ is exact, where

$$F(X,a):0\to X=\Lambda^0[\sigma,X]\overset{\delta^0}{\to}\Lambda^1[\sigma,X]\overset{\delta^1}{\to}\Lambda^2[\sigma,X]\overset{\delta^2}{\to}\ldots$$
$$\ldots\overset{\delta^{n-2}}{\to}\Lambda^{n-1}[\sigma,X]\overset{\delta^{n-1}}{\to}\Lambda^n[\sigma,X]=X\to 0 \ ([6],[12]).$$

For an integer p, it can be defined the homology module of $E(X,a)$ as

$$H_p(X,a)=\mathrm{Ker}\left(\delta_{p+1}:\Lambda^{p+1}\to\Lambda^p\right)/\,\mathrm{Im}\left(\delta_p:\Lambda^p\to\Lambda^{p-1}\right)$$

and respectively, the cohomology module of chain complex $F(X,a)$ by

$$H^p(X,a)=\mathrm{Ker}\left(\delta^p:\Lambda^p\to\Lambda^{p+1}\right)/\,\mathrm{Im}\left(\delta^{p-1}:\Lambda^{p-1}\to\Lambda^p\right)\ ([6],[12]).$$

The Taylor spectrum of a *on* X is denoted by $\sigma(a,X)$ and it is the complementary in $\mathbb{C}^n$ of the set of $z=(z_1,z_2,\ldots,z_n)\in\mathbb{C}^n$ such that the system $z-a=(z_1-a_1,\ z_2-a_2,\ldots,z_n-a_n)$ is nonsingular on X ([6], [12]). *The analytic spectrum of* $x\in X$ *with respect to* a is denoted by $\sigma(a,x)$ and it is defined as the complementary in $\mathbb{C}^n$ of the set of $z=(z_1,z_2,\ldots,z_n)\in\mathbb{C}^n$ such that there are an open neighborhood V of z and n X-valued analytic functions $f_1,f_2,\ldots,f_n$ on V, satisfying the equation $x=(\zeta_1-a_1)f_1(\zeta)+\ldots+(\zeta_n-a_n)f_n(\zeta),\ \zeta\in V$ ([6]). *The spectrum* of $x\in X$ *with respect to* a is denoted by $sp(a,x)$ and it is the complementary in $\mathbb{C}^n$ of the reunion of all open sets $V\subset\mathbb{C}^n$ such that there is a form $\psi\in\Lambda^{n-1}\left[\sigma\cup d\,\bar{z},C^\infty(V,X)\right]$ satisfying the equality $s\,x=\left(\alpha\oplus\bar{\partial}\right)\psi$, i.e.

$$x(s_1\wedge s_2\wedge\ldots\wedge s_n)=\Big[(z_1-a_1)s_1+(z_2-a_2)s_2+\ldots+(z_n-a_n)s_n+\frac{\partial}{\partial\bar{z}_1}d\,\bar{z}_1+$$
$$+\frac{\partial}{\partial\bar{z}_2}d\,\bar{z}_2+\ldots+\frac{\partial}{\partial\bar{z}_n}d\,\bar{z}_n\Big]\wedge\psi(z)\ \ ([6],[7]).$$

In [5], J. Eschmeier proved that $\sigma(a,x)=sp(a,x)$.

The system $a=(a_1, a_2, ..., a_n)\subset \mathbf{B}(X)$ verifies *the cohomology condition* (L) (respectively, *the condition* (L_S)) if $H^{n-1}\left(C^{\infty}(G,X),\alpha\oplus\overline{\partial}\right)=0$, for any open set $G\subset\mathbb{C}^n$ ([6]) (respectively, for any open $G\subset\mathbb{C}^n$, with $G\bigcap S=\varnothing$, [11]), where $C^{\infty}(G,X)$ is the space of all continuous functions admitting partial derivatives of any order. We consider for $F\subset\mathbb{C}^n$ closed ([6]):

$X_a(F)=\{x;\, x\in X,\, \sigma(a,x)\subset F\}$ and $X_{[a]}(F)=\{x;\, x\in X,\, sp(a,x)\subset F\}$.

Let S_a be a compact minimal set having $H^{n-1}\left(C^{\infty}(G,X),\alpha\oplus\overline{\partial}\right)=0$, for any open set $G\subset\mathbb{C}^n$ with $G\bigcap S_a=\varnothing$ (minimal means that S_a is the intersection of all compact sets satisfying the specified property). The set S_a is called *the analytic spectral residuum* of the system a ([2], [14]); $S_a\subset\sigma(a,X)$. If $S_a=\varnothing$, then the system a verifies the cohomology property (L) ([14]).

2. Preliminaries

Definition 2.1. ([2]) An application $\mathcal{E}_S:\mathrm{F}_S^n\to\mathbf{S}(X)$ is said to be *S-spectral capacity* if it verifies the conditions:

(1) $\mathcal{E}_S(\varnothing)=\{0\}$, $\mathcal{E}_S\left(\mathbb{C}^n\right)=X$;

(2) $\mathcal{E}_S\left(\bigcap_{i=1}^{\infty}F_i\right)=\bigcap_{i=1}^{\infty}\mathcal{E}_S(F_i)$, for any family $\{F_i\}_{i\in\mathbb{N}}\subset\mathrm{F}_S^n$;

(3) for any open finite *S*-covering $G_S\cup\{G_j\}_{j=1}^{m}$ of $\mathbb{C}^n$ we have

$$X=\mathcal{E}_S\left(\overline{G}_S\right)+\sum_{j=1}^{m}\mathcal{E}_S\left(\overline{G}_j\right).$$

The system $a=(a_1, a_2, ..., a_n)\subset\mathbf{B}(X)$ is called $S-$*decomposable* if there is an *S*-spectral capacity $\mathcal{E}_S$ such that:

(4) $a_j\mathcal{E}_S(F)\subset\mathcal{E}_S(F)$, for any $F\in\mathrm{F}_S^n$ and for any $1\le j\le n$;

(5) $\sigma(a,\mathcal{E}_S(F))\subset F$, for any $F\in\mathrm{F}_S^n$.

Definition 2.2. ([1]) A mapping $E_S : \mathrm{B}_S^n \to \mathbf{P}(X)$ is called a $\left(\mathbb{C}^n, X\right)$ *type* $S-$*spectral measure* if it meets the properties:

(1) $E_S(\varnothing)=0$, $E_S\left(\mathbb{C}^n\right)=I_X$;

(2) $E_S(B_1 \cap B_2)=E_S(B_1)E_S(B_2)$, for B_1, $B_2 \in \mathrm{B}_S^n$;

(3) $E_S\left(\bigcup\limits_{m=1}^{\infty} B_m\right)x=\sum\limits_{m=1}^{\infty} E_S(B_m)x$, for $B_m \in \mathrm{B}_S^n$, $B_m \cap B_p=\varnothing$, if $m \neq p$, $x \in X$.

The system $a=(a_1,a_2,...,a_n) \subset \mathbf{B}(X)$ is called $S-$*spectral* if there is a $\left(\mathbb{C}^n, X\right)$ type $S-$spectral measure E_S such that:

(4) $a_j E_S(B)=E_S(B)a_j$, for any $B \in \mathrm{B}_S^n$ and for any $1 \leq j \leq n$;

(5) $\sigma(a, E_S(B)X) \subset \overline{B}$, for any $B \in \mathrm{B}_S^n$.

Lemma 2.1. ([2], [15]) *If* $a=(a_1,...,a_n) \subset \mathbf{B}(X)$ and $b=(b_1,...,b_n) \subset \subset \mathbf{B}(Y)$ *are two commuting operator systems, then the Taylor spectrum of the system* $a \oplus b=(a_1 \oplus b_1, a_2 \oplus b_2,...,a_n \oplus b_n) \subset B(X \oplus Y)$ *verifies the equality*

$$\sigma(a \oplus b, X \oplus Y)=\sigma(a,X) \cup \sigma(b,Y).$$

Proposition 2.1. ([2], [15]) *The operator systems* $a=(a_1,...,a_n) \subset \mathbf{B}(X)$ *and* $b=(b_1,...,b_n) \subset \mathbf{B}(Y)$ *verify the cohomology condition* (L) (*respectively*, (L_S)) *if and only if the system* $a \oplus b=(a_1 \oplus b_1,...,a_n \oplus b_n) \subset B(X \oplus Y)$ *verifies the same condition* (L) (*respectively*, (L_S)).

Proposition 2.2. ([2], [15]) *If* $a=(a_1,...,a_n) \subset \mathbf{B}(X)$ and $b=(b_1,...,b_n) \subset \subset \mathbf{B}(Y)$ *are two operator systems that verify condition* (L), *then the following equalities hold*:

1) $\sigma(a \oplus b, x \oplus y)=\sigma(a,x) \cup \sigma(b,y)$, $x \in X$, $y \in Y$;

2) $sp(a \oplus b, x \oplus y)=sp(a,x) \cup sp(b,y)$, $x \in X$, $y \in Y$;

3) $(X \oplus Y)_{a \oplus b}(F)=X_a(F) \oplus Y_b(F)$, $F \subset \mathbb{C}^n$ closed.

Proposition 2.3. ([11]) *An operator system* $a=(a_1,a_2,...,a_n) \subset \mathbf{B}(X)$ *is* $S-$*decomposable if and only if the following conditions are established*:

(I) *a verifies the cohomology condition $\left(L_S\right)$, the space $X_a(F)$ is closed and $\sigma\left(a, X_a(F)\right) \subset F$, for any $F \in \mathrm{F}_S^n$, $F \supset S$;*

(II) *for any open $S-$covering $G_S \cup \left\{G_j\right\}_{j=1}^{m}$* of $\mathbb{C}^n$ *and for any $x \in X$ we have*:

$$x = x_S + x_1 + x_2 + \ldots + x_m, \text{ with } \sigma\left(a, x_S\right) \subset G_S,\ \sigma\left(a, x_j\right) \subset G_j,\ 1 \le j \le m.$$

3. Direct sums of $S-$decomposable and $S-$spectral systems

Proposition 3.1. *Let $a = \left(a_1, a_2, \ldots, a_n\right) \subset \mathbf{B}(X)$* and *$b = \left(b_1, b_2, \ldots, b_n\right) \subset \subset \mathbf{B}(Y)$ be two commuting operator systems that verify condition (L), or condition $\left(L_S\right)$, or conditions $\left(L_{S_1}\right)$ for a and $\left(L_{S_2}\right)$ for b. Then the spaces $X_a(F)$ and $Y_b(F)$ are closed and $\sigma\left(a, X_a(F)\right) \subset F$, $\sigma\left(b, Y_b(F)\right) \subset F$, for $F \subset \mathbb{C}^n$ closed, or for $F \in \mathrm{F}_S^n$, with $F \supset S$, or for $F \in \mathrm{F}_S^n$, with $F \cap S = \varnothing$ (when $S_a = \varnothing$, $S_b = \varnothing$) if and only if the space $(X \oplus Y)_{a \oplus b}(F)$ is closed and $\sigma\left(a \oplus b, (X \oplus Y)_{a \oplus b}(F)\right) \subset F$.*

Proof. According to Proposition 2.1, when a and b verify condition (L) (respectively, $\left(L_S\right)$), then the system $a \oplus b$ verifies condition (L) (respectively $\left(L_S\right)$). Let us first suppose that the subspaces $X_a(F)$, $Y_b(F)$ are closed and

$$\sigma\left(a, X_a(F)\right) \subset F,\ \sigma\left(b, Y_b(F)\right) \subset F$$

for $F \subset \mathbb{C}^n$ closed (respectively, for $F \in \mathrm{F}_S^n$).

From the equality

$$X_a(F) \oplus Y_b(F) = (X \oplus Y)_{a \oplus b}(F) \qquad \mathbf{(1)}$$

(see Proposition 2.2 and [10]), we have that $(X \oplus Y)_{a \oplus b}(F)$ is closed. From Lemma 2.1, it results that

$$\sigma\left(a \oplus b, (X \oplus Y)_{a \oplus b}(F)\right) = \sigma\left(a, X_a(F)\right) \cup \sigma\left(b, Y_b(F)\right) \subset F.$$

Conversely, if the space $(X \oplus Y)_{a \oplus b}(F)$ is closed and

$$\sigma\left(a \oplus b, (X \oplus Y)_{a \oplus b}(F)\right) \subset F$$

by denoting with P_X and P_Y, respectively, the corresponding projections (i.e. $P_X(X \oplus Y) = X$, $P_Y(X \oplus Y) = Y$), then, according to equality (1), we have

$$P_X\left((X \oplus Y)_{a \oplus b}(F)\right) = X_a(F) \text{ and } P_Y\left((X \oplus Y)_{a \oplus b}(F)\right) = Y_b(F).$$

Let us now prove that the spaces $X_a(F)$ and $Y_b(F)$ are closed. One can easily verify that P_X and P_Y commute with every $a_j \oplus b_j$, $1 \le j \le n$, and since $(X \oplus Y)_{a \oplus b}(F)$ is ultrainvariant to $a \oplus b$, it follows that it is invariant to P_X and P_Y. Consequently, P_X, P_Y are also projections in $(X \oplus Y)_{a \oplus b}(F)$, hence the images X_1 and Y_1 through P_X and P_Y of the space $(X \oplus Y)_{a \oplus b}(F)$ are closed subspaces and

$$X_1 \oplus Y_1 = (X \oplus Y)_{a \oplus b}(F),\ \sigma(a, X_1) \subset F,\ \sigma(b, Y_1) \subset F.$$

It follows that $X_1 \subset X_a(F)$, $Y_1 \subset Y_b(F)$; furthermore, we have $X_a(F) \oplus Y_b(F) = X_1 \oplus Y_1$, whence $X_1 = X_a(F)$, $Y_1 = Y_b(F)$.

Theorem 3.1. *Let* $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ *and* $b = (b_1, b_2, ..., b_n) \subset \subset \mathbf{B}(Y)$ *be two systems. Then* $a \oplus b = (a_1 \oplus b_1, a_2 \oplus b_2, ..., a_n \oplus b_n) \subset \mathbf{B}(X \oplus Y)$ *is an* $S-$*decomposable system if and only if* a *and* b *are* $S-$*decomposable.*

Proof. From Proposition 2.1, it results that a and b verify (L_S) if and only if $a \oplus b$ verifies (L_S), and by Proposition 2.2 and Lemma 2.1 we have

$$(X \oplus Y)_{a \oplus b}(F) = X_a(F) \oplus Y_b(F)$$

$$\sigma\left(a \oplus b, (X \oplus Y)_{a \oplus b}(F)\right) = \sigma(a, X_a(F)) \cup \sigma(b, Y_b(F)) \subset F$$

for $F \in \mathrm{F}_S^n$, with $F \supset S$.

According to Proposition 3.1, we have that the systems a and b satisfy the condition (I) from the hypothesis of Proposition 2.3 (i.e. a and b verify (L_S), $X_a(F)$ and $Y_b(F)$ are closed, $\sigma(a, X_a(F)) \subset F$, $\sigma(b, Y_b(F)) \subset F$, $F \in \mathrm{F}_S^n$, with $F \supset S$) if and only if the system $a \oplus b$ verifies the same condition (I) (i.e. $a \oplus b$ verifies condition (L_S), the space $(X \oplus Y)_{a \oplus b}(F)$ is closed, $\sigma\left(a \oplus b, (X \oplus Y)_{a \oplus b}(F)\right) \subset F$, for $F \in \mathrm{F}_S^n$, with $F \supset S$).

In the same manner, if $G_S \cup \{G_j\}_{j=1}^{m}$ is an open $S-$covering of $\mathbb{C}^n$, then by using the equalities

$$(x_S \oplus y_S)+(x_1 \oplus y_1)+...+(x_m \oplus y_m)=(x_S+x_1+...+x_m)\oplus(y_S+y_1+...+y_m)$$

$$\sigma\left(a\oplus b, x_j \oplus y_j\right)=\sigma\left(a, x_j\right)\cup\sigma\left(b, y_j\right),\ 1\le j\le m$$

$$\sigma\left(a\oplus b, x_S \oplus y_S\right)=\sigma\left(a, x_S\right)\cup\sigma\left(b, y_S\right)$$

it follows that a and b verify the condition (II) from Proposition 2.3 if and only if $a\oplus b$ verifies the same condition. Consequently, according to Proposition 2.3, a and b are $S-$decomposable if and only if $a\oplus b$ is $S-$decomposable.

Remark 3.1. If $a\subset\mathbf{B}(X)$ and $b\subset\mathbf{B}(Y)$ are $S-$decomposable and $\mathcal{E}_{1S}:\mathrm{F}_S^n\to\mathbf{S}(X)$, $\mathcal{E}_{2S}:\mathrm{F}_S^n\to\mathbf{S}(Y)$ are their $S-$spectral capacities, then the application $\mathcal{E}_S:\mathrm{F}_S^n\to\mathbf{S}(X\oplus Y)$, $\mathcal{E}_S(F)=\mathcal{E}_{1S}(F)\oplus\mathcal{E}_{2S}(F)$, $F\in\mathrm{F}_S^n$, is the $S-$spectral capacity for $a\oplus b\subset\mathbf{B}(X\oplus Y)$, therefore $a\oplus b$ is $S-$decomposable (see Definition 2.1 and [10]).

Proposition 3.2. *Let E_{1S} be a $\left(\mathbb{C}^n, X\right)$ type $S-$spectral measure and let E_{2S} be a $\left(\mathbb{C}^n, Y\right)$ type $S-$spectral measure. Then $E_S=E_{1S}\oplus E_{2S}$ is a $\left(\mathbb{C}^n, X\oplus Y\right)$ type $S-$spectral measure. Conversely, if E_S is a $\left(\mathbb{C}^n, X\oplus Y\right)$ type $S-$spectral measure and $E_S=E_{1S}\oplus E_{2S}$, then E_{1S} is a $\left(\mathbb{C}^n, X\right)$ type $S-$spectral measure and E_{2S} is a $\left(\mathbb{C}^n, Y\right)$ type $S-$spectral measure, respectively.*

Proof. Since E_{1S} and E_{2S} are two $S-$spectral measures, we apply now Definition 2.2 and we deduce for $i=1,\ 2$:

$E_{iS}(\varnothing)=0$, $E_{1S}\left(\mathbb{C}^n\right)=I_X$, $E_{2S}\left(\mathbb{C}^n\right)=I_Y$;

$E_{iS}(B_1\cap B_2)=E_{iS}(B_1)E_{iS}(B_2)$, $B_1, B_2\in\mathrm{B}_S^n$;

$E_{iS}\left(\bigcup\limits_{m=1}^{\infty} B_m\right)x=\sum\limits_{m=1}^{\infty}E_{iS}(B_m)x$, $B_m\in\mathrm{B}_S^n$, $B_m\cap B_p=\varnothing$, if $m\ne p$,

$x\in X$ (for $i=1$) or $x\in Y$ (for $i=2$).

Let us prove that $E_{1S}\oplus E_{2S}$ is a $\left(\mathbb{C}^n, X\oplus Y\right)$ type $S-$spectral measure, where $E_{1S}\oplus E_{2S}$ is denoted by E_S and it is defined naturally by the equality

$$E_S(B)=\left(E_{1S}\oplus E_{2S}\right)(B)=E_{1S}(B)\oplus E_{2S}(B),\ B\in \mathrm{B}_S^n.$$

We obviously have, for $B_1,\ B_2\in \mathrm{B}_S^n$:

$$E_S(\varnothing)=\left(E_{1S}\oplus E_{2S}\right)(\varnothing)=E_{1S}(\varnothing)\oplus E_{2S}(\varnothing)=0\oplus 0=0;$$

$$E_S\left(\mathbb{C}^n\right)=\left(E_{1S}\oplus E_{2S}\right)\left(\mathbb{C}^n\right)=E_{1S}\left(\mathbb{C}^n\right)\oplus E_{2S}\left(\mathbb{C}^n\right)=I_X\oplus I_Y=I_{X\oplus Y};$$

$$E_S(B_1\cap B_2)=\left(E_{1S}\oplus E_{2S}\right)(B_1\cap B_2)=E_{1S}(B_1\cap B_2)\oplus E_{2S}(B_1\cap B_2)=$$

$$=\left(E_{1S}(B_1)E_{1S}(B_2)\right)\oplus\left(E_{2S}(B_1)E_{2S}(B_2)\right)=$$

$$=\left(E_{1S}(B_1)\oplus E_{2S}(B_1)\right)\left(E_{1S}(B_2)\oplus E_{2S}(B_2)\right)=$$

$$=\left(\left(E_{1S}\oplus E_{2S}\right)(B_1)\right)\left(\left(E_{1S}\oplus E_{2S}\right)(B_2)\right)=E_S(B_1)E_S(B_2).$$

In order to end the verifications, it remains to establish the countable additivity of E_S:

$$E_S\left(\bigcup_{m=1}^{\infty}B_m\right)(x\oplus y)=\left(E_{1S}\oplus E_{2S}\right)\left(\bigcup_{m=1}^{\infty}B_m\right)(x\oplus y)=$$

$$=\left(E_{1S}\left(\bigcup_{m=1}^{\infty}B_m\right)\oplus E_{2S}\left(\bigcup_{m=1}^{\infty}B_m\right)\right)(x\oplus y)=$$

$$=\left(E_{1S}\left(\bigcup_{m=1}^{\infty}B_m\right)x\right)\oplus\left(E_{2S}\left(\bigcup_{m=1}^{\infty}B_m\right)y\right)=$$

$$=\left(\sum_{m=1}^{\infty}E_{1S}(B_m)x\right)\oplus\left(\sum_{m=1}^{\infty}E_{2S}(B_m)y\right)=$$

$$= \sum_{m=1}^{\infty} \left(\left(E_{1S}(B_m)x \right) \oplus \left(E_{2S}(B_m)y \right) \right) = \sum_{m=1}^{\infty} \left(E_{1S}(B_m) \oplus E_{2S}(B_m) \right)(x \oplus y) =$$

$$= \sum_{m=1}^{\infty} \left(E_{1S} \oplus E_{2S} \right)(B_m)(x \oplus y) = \sum_{m=1}^{\infty} E_S(B_m)(x \oplus y),\ x \in X,\ y \in Y.$$

Conversely, we suppose that E_S is a $\left(\mathbb{C}^n, X \oplus Y\right)$ type S – spectral measure. Then E_S can be written as $E_S = E_{1S} \oplus E_{2S}$, where $E_{1S}(B) \in \mathbf{B}(X)$, $E_{2S}(B) \in \mathbf{B}(Y)$, for $B \in \mathrm{B}_S^n$.

The mappings $E_{1S} : \mathrm{B}_S^n \to \mathbf{P}(X)$ and $E_{2S} : \mathrm{B}_S^n \to \mathbf{P}(Y)$ are obviously linear and from the equalities

$$E_{1S} \oplus E_{2S} = \left(E_{1S} \oplus E_{2S} \right)^2 = E_{1S}^2 \oplus E_{2S}^2$$

it follows that $E_{1S} = E_{1S}^2$ and $E_{2S} = E_{2S}^2$, hence $E_{1S}(B)$ and $E_{2S}(B)$ are projectors in X and Y, respectively.

In the same manner, one can easily verify that E_{1S} is a $\left(\mathbb{C}^n, X\right)$ type S – spectral measure and E_{2S} is a $\left(\mathbb{C}^n, Y\right)$ type S – spectral measure; indeed:

$0 = E_S(\varnothing) = E_{1S}(\varnothing) \oplus E_{2S}(\varnothing)$, hence $E_{1S}(\varnothing) = 0$ and $E_{2S}(\varnothing) = 0$;

$I_{X \oplus Y} = E_S\left(\mathbb{C}^n\right) = E_{1S}\left(\mathbb{C}^n\right) \oplus E_{2S}\left(\mathbb{C}^n\right)$, hence $E_{1S}\left(\mathbb{C}^n\right) = I_X$ and $E_{2S}\left(\mathbb{C}^n\right) = I_Y$;

$$E_S(B_1 \cap B_2) = E_S(B_1) E_S(B_2) \Leftrightarrow$$

$$\left(E_{1S} \oplus E_{2S}\right)(B_1 \cap B_2) = \left(E_{1S} \oplus E_{2S}\right)(B_1)\left(E_{1S} \oplus E_{2S}\right)(B_2) \Leftrightarrow$$

$$E_{1S}(B_1 \cap B_2) \oplus E_{2S}(B_1 \cap B_2) =$$

$$= \left(E_{1S}(B_1) \oplus E_{2S}(B_1)\right)\left(E_{1S}(B_2) \oplus E_{2S}(B_2)\right) \Leftrightarrow$$

$$E_{1S}(B_1 \cap B_2) \oplus E_{2S}(B_1 \cap B_2) =$$

$$= \left(E_{1S}(B_1) E_{1S}(B_2)\right) \oplus \left(E_{2S}(B_1) E_{2S}(B_2)\right),$$

hence $E_{iS}(B_1 \cap B_2) = E_{iS}(B_1) E_{iS}(B_2)$, $B_1, B_2 \in \mathrm{B}_S^n$, $i = 1, 2$.

We can also write, for $B_m \in \mathrm{B}_S^n$, $B_m \cap B_p = \varnothing$, if $m \neq p$, $x \in X$, $y \in Y$:

$$E_S\left(\bigcup_{m=1}^{\infty} B_m\right)(x \oplus y) = \sum_{m=1}^{\infty} E_S(B_m)(x \oplus y) = \sum_{m=1}^{\infty} \left(E_{1S} \oplus E_{2S}\right)(B_m)(x \oplus y) =$$

$$= \sum_{m=1}^{\infty} \left(E_{1S}(B_m) \oplus E_{2S}(B_m)\right)(x \oplus y) = \sum_{m=1}^{\infty} \left(\left(E_{1S}(B_m)x\right) \oplus \left(E_{2S}(B_m)y\right)\right) =$$

$$= \left(\sum_{m=1}^{\infty} E_{1S}(B_m)x\right) \oplus \left(\sum_{m=1}^{\infty} E_{2S}(B_m)y\right)$$

and

$$E_S\left(\bigcup_{m=1}^{\infty} B_m\right)(x \oplus y) = \left(E_{1S} \oplus E_{2S}\right)\left(\bigcup_{m=1}^{\infty} B_m\right)(x \oplus y) =$$

$$= \left(E_{1S}\left(\bigcup_{m=1}^{\infty} B_m\right) \oplus E_{2S}\left(\bigcup_{m=1}^{\infty} B_m\right)\right)(x \oplus y) =$$

$$= \left(E_{1S}\left(\bigcup_{m=1}^{\infty} B_m\right)x\right) \oplus \left(E_{2S}\left(\bigcup_{m=1}^{\infty} B_m\right)y\right)$$

accordingly

$$E_{1S}\left(\bigcup_{m=1}^{\infty} B_m\right)x = \sum_{m=1}^{\infty} E_{1S}(B_m)x \text{ and } E_{2S}\left(\bigcup_{m=1}^{\infty} B_m\right)y = \sum_{m=1}^{\infty} E_{2S}(B_m)y.$$

Theorem 3.2. *Let* $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$, $b = (b_1, b_2, \dots, b_n) \subset \mathbf{B}(Y)$ *be two commuting operator systems. Then* a *and* b *are* $S-$*spectral if and only if* $a \oplus b = (a_1 \oplus b_1, a_2 \oplus b_2, \dots, a_n \oplus b_n) \subset \mathbf{B}(X \oplus Y)$ *is* $S-$*spectral.*

Proof. Let us suppose that the systems a and b are $S-$spectral and let E_{1S} and E_{2S} be their $S-$spectral measures. On account of the above proposition, the application E_S defined by the equality

$$E_S(B) = E_{1S}(B) \oplus E_{2S}(B), \text{ for } B \in \mathrm{B}_S^n$$

is an $S-$spectral measure of the system $a\oplus b$, which commutes with every operator $a_j\oplus b_j$, because a_j commutes with E_{1S} and b_j commutes with E_{2S}, for any $1\le j\le n$: For any Borelian set $B\in \mathrm{B}_S^n$, we have:

$$E_S(B)\left(a_j\oplus b_j\right)=\left(E_{1S}(B)\oplus E_{2S}(B)\right)\left(a_j\oplus b_j\right)=$$

$$=E_{1S}(B)a_j\oplus E_{2S}(B)b_j=a_j E_{1S}(B)\oplus b_j E_{2S}(B)=$$

$$=\left(a_j\oplus b_j\right)\left(E_{1S}(B)\oplus E_{2S}(B)\right)=\left(a_j\oplus b_j\right)E_S(B).$$

Furthermore, from the inclusions

$$\sigma\left(a,E_{1S}(B)X\right)\subset\overline{B},\ \sigma\left(b,E_{2S}(B)Y\right)\subset\overline{B},\ B\in \mathrm{B}_S^n$$

and by Lemma 2.1 we can conclude that

$$\sigma\left(a\oplus b,E_S(B)(X\oplus Y)\right)=\sigma\left(a\oplus b,E_{1S}(B)X\oplus E_{2S}(B)Y\right)=$$

$$=\sigma\left(a,E_{1S}(B)X\right)\cup\sigma\left(b,E_{2S}(B)Y\right)\subset\overline{B}$$

consequently the system $a\oplus b$ is $S-$spectral.

Conversely, if we assume that $a\oplus b$ is $S-$spectral and E_S is its $S-$spectral measure, then using once more the proof of Proposition 3.2, we have that E_S can be written as the form $E_S=E_{1S}\oplus E_{2S}$, where E_{1S}, E_{2S} are the $S-$spectral measures for a and b, respectively, and $E_{1S}(B)$, $E_{2S}(B)$ are projectors in X and Y, respectively.

From the inclusion (Lemma 2.1)

$$\sigma\left(a,E_{1S}(B)X\right)\cup\sigma\left(b,E_{2S}(B)Y\right)=\sigma\left(a\oplus b,E_S(B)(X\oplus Y)\right)\subset\overline{B}$$

it follows that

$$\sigma\left(a,E_{1S}(B)X\right)\subset\overline{B},\ \sigma\left(b,E_{2S}(B)Y\right)\subset\overline{B},\ B\in \mathrm{B}_S^n.$$

Moreover, it is easily seen that

$$\left(a_j\oplus b_j\right)E_S(B)=E_S(B)\left(a_j\oplus b_j\right)\Leftrightarrow$$

$$\left(a_j\oplus b_j\right)\left(E_{1S}(B)\oplus E_{2S}(B)\right)=\left(E_{1S}(B)\oplus E_{2S}(B)\right)\left(a_j\oplus b_j\right)\Leftrightarrow$$

$$a_j E_{1S}(B)\oplus b_j E_{2S}(B)=E_{1S}(B)a_j\oplus E_{2S}(B)b_j$$

for all $1\le j\le n$, $B\in \mathrm{B}_S^n$, therefore

$$a_j E_{1S}(B)=E_{1S}(B)a_j,\ b_j E_{2S}(B)=E_{2S}(B)b_j$$

and we deduce that a and b are $S-$spectral systems.

Corollary 3.1. *Let* $a=(a_1,a_2,...,a_n)\subset\mathbf{B}(X)$ *be an* $S-$*spectral system and let* $P\in\mathbf{B}(X)$ *be a projector which commutes with every operator* a_j, $1\leq j\leq n$. *Then the restrictions* $a\,|\,PX$ *and* $a\,|\,(I-P)X$ *are* S_i-*spectral systems*, $i=1,\,2$, *where* $S_1=S\cap\sigma(a,PX)X$ *and* $S_2=S\cap\sigma(a,(I-P)X)$.

Proof. The assertions of the hypothesis follow from Theorem 3.2, by the equality $a=(a\,|\,PX)\oplus(a\,|\,(I-P)X)$ and from the fact that the restriction of a spectral system to an invariant subspace is also a spectral system if and only if the subspace is also invariant to the spectral measure of the system (see [6], [1], [16]).

REFERENCES

[1]. *I. Bacalu*, Descompuneri spectrale reziduale (Residually spectral decompositions), Stud. Cerc. Mat. I (1980), II (1980), III (1981).

[2]. *I. Bacalu*, $S-$Spectral Decompositions, Ed. Politehnica Press, Bucharest, 2008.

[3]. *I. Colojoară and C. Foiaş*, Theory of generalized spectral operators, Gordon Breach, Science Publ., New York-London-Paris, 1968.

[4]. *N. Dunford and J.T. Schwartz*, Linear Operators, Interscience Publishers, New-York, Part I (1958), Part II (1963), Part III (1971).

[5]. *J. Eschmeier*, On two notions of the local spectrum for several commuting operators, Michigan Math. J., **30**, 1983, pp. 245-248.

[6]. *Şt. Frunză*, O teorie axiomatică a descompunerilor spectrale pentru sisteme de operatori I şi II (An axiomatic theory of spectral decompositions for systems of operators I and II), I, Stud. Cerc. Mat., **27**, 1975, pp. 655-711, and II, **29**, 1977, pp. 329-376.

[7]. *Şt. Frunză*, The Taylor spectrum and spectral decompositions, J. Func. Anal., **19**, 1977, pp. 390-421.

[8]. *K.B. Laursen and M.M. Neumann,* An Introduction to Local Spectral Theory, London Math. Soc. Monographs New Series, Oxford Univ. Press., New-York, 2000.

[9]. *B. Nagy,* Differential operators and spectral decomposition, Colloquia Mathematica Societatis Jonas Bolyai, **35**, Functions, Series, Operators, Budapest (Hungary), 1980.

[10]. *C. Şerbănescu and I. Bacalu*, On certain aspects of the multidimensional $S-$spectral theory, Mathematical Problems in Engineering, Hindawi Publishing Corporation, 2014.

[11]. *C. Şerbănescu and I. Bacalu*, Generalizing certain properties of decomposable systems, Recent Advances on Applied Mathematics an Computational Methods in Engineering, ISBN 978-1-61804-292-7, 2015, pp. 26-35.

[12]. *J.L. Taylor*, A joint spectrum for several commuting operators, J. Func. Anal., **6**, 1970, pp. 172-191.

[13]. *J.L. Taylor*, Analytic functional calculus for several commuting operators, Acta. Math., **125**, 1970, pp. 1-38.

[14]. *F.H. Vasilescu*, Analytic Functional Calculus and Spectral Decompositions, D. Reidel Publishing Company, Dordrecht; Editura Academiei, Bucharest, 1982.

[15]. *M. Zamfir and I. Bacalu*, Direct sums of decomposable systems, Scient. J. Math. Modelling in Civil Engineering, **7**, no 2 BIS, 2011, pp. 69-81.

[16]. *M. Zamfir and I. Bacalu*, On the local spectral properties of operator systems in Banach spaces, Applied Sciences, **14**, 2012, pp. 89-97.