LORENZ TYPE BEHAVIORS ASSOCIATED TO FRACTAL-NON-FRACTAL TRANSITION IN THE DYNAMICS OF THE COMPLEX SYSTEMS

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In the framework of Fractal Theory of Motion for the Scale Relativity Theory with arbitrary and constant fractal dimensions, dynamics in complex systems associated to the fractal-non-fractal transition are analyzed. Working with the assumption that these dynamics are described by means of fractal curves, Lorenz type behaviors become "operational" through a Galerkin method. Then Rayleigh and Prandtl effective numbers are specified both by means of classical kinetic coefficients and scale resolution while the dynamics variables act as the limit of a family of mathematical functions, non-differentiable for non-null scale resolution.

Keywords: Lorenz, Scale Relativity, fractal dimension

1. Introduction

Lorentz's classical model used in the description of the various dynamics [1,2] of the complex systems [3,4] are generally applied under the paradigm of differentiability of the core variables describing the physical system. The positive results of such a mathematical approach should be understood sequentially, on the regions where differentiability is still respected. However, when attempting to describe non-linear systems and chaotic behavior of the dynamics of the complex

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systems [3,4] the differentiable mathematical procedures should be altered. With such a change at task, the aim is to remain tributary to differentiable mathematical procedures in the description of the various dynamics of the complex systems. Thus, is becoming necessary to introduce the scale resolution explicitly in the expression of physical variables of the system and implicitly in the fundamental dynamics equations that govern the system. The final system will thus contain variables that will dependent on both spatial and time coordinates and scale resolution. Consequently, instead of operating for example with a single variable described by a strictly non-differentiable mathematical function, we will use only approximations of these mathematical functions obtained by averaging it at various scale resolutions. Moreover, any variable designed to describe various dynamics of the complex systems will work as the limit of a family of mathematical functions, this being non-differentiable for zero scale resolution and differentiable for non-zero scale resolution.

This approach obviously involves the development of new geometrical structures along with a new class of models for which the laws of motion, invariant to the spatial and temporal transformations, must be integrated in scale laws, which are invariant to the scale transformations. Such a geometrical framework can be based on the concept of a fractal the Fractal Theory of Motion either in the form of Scale Relativity Theory in Nottale's sense [5,6] or in the form of Scale Relativity Theory with arbitrary and constant fractal dimension and becomes functional in the description of the various dynamics of the complex systems. Fractal concepts have also been usefully incorporated into models of biological processes, including epithelial cell growth, blood vessel growth and configuration [7], bone and vascular pathology and neuropathology, modeling of biological processes using fractals and other miscellaneous applications [8,9], or integrative models for fractal description of the particular structure parameters [10].

So that, in the present paper, using the Scale Relativity Theory with an arbitrary but constant fractal dimension, Lorenz type behaviors associated to fractal-non-fractal transitions in the dynamics of the complex systems will be analyzed.

2.   Mathematical model

The mathematical model proposed here is based on the principle that the dynamics of thermal nature seen in complex systems, associated to the fractal-non-fractal transition are described by mean of fractal curves. Starting from this paradigm the time derivative d/dt is substituted by the scale covariant derivative [11-13]:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^i \partial_i + \frac{1}{4}(dt)^{[2/f(\alpha)]-1} D^{lk} \partial_l \partial_k \qquad (1)$$

where:

$$\hat{V}^l = V_D^l - iV_F^l$$
$$D^{lk} = d^{lk} - i\bar{d}^{lk}$$
$$d^{lk} = \lambda_+^l\lambda_+^k - \lambda_-^l\lambda_-^k \qquad (2)$$
$$\bar{d}^{lk} = \lambda_+^l\lambda_+^k + \lambda_-^l\lambda_-^k$$
$$\alpha = \alpha(D_F), i = \sqrt{-1}, \qquad l,k = 1,2,3$$

In the previous relations X^l are the fractal spatial coordinates, t is the non-fractal temporal coordinate and also affin parameter of the motion curves, $\hat{V}^l$ is the complex velocity with V_D^l the real part, differentiable and scale resolution independent and V_F^l the imaginary part, non-differentiable and scale resolution dependent, dt is the scale resolution, D^{lk} is the pseudo-tensor associated to the fractal-non-fractal transition, λ_+^l has constant coefficient through which the thermal type fractalization is imposed (λ_+^l the forward thermal processes and λ_-^l for the backwards ones), $f(\alpha)$ is the singularity spectrum of order α, α is the singularity index and D_F is the fractal dimension of the motion curves [11]. These are many modes and thus a various selection of definitions of fractal dimensions: fractal dimension in the Kolmogorov sense, fractal dimension in the Husdorff-Besikovici sense, etc. [11].

Selecting one of these definitions and operating with them in the thermal type fractal dynamics of the complex systems, the value of the fractal dimension must be constant and arbitrary: $D_F<2$ for correlative thermal type fractal processes, $D_F>2$ for non-correlative thermal type etc. In such a conjecture, we can identify not only the "areas" of the thermal type multifractal dynamics of the complex systems that are characterized by a certain fractal dimension, but also the number of "areas" whose fractal dimension are situated in an interval values. Moreover, through the singularity spectrum $f(\alpha)$ we can identify classes of universality in the thermal type multifractal dynamics of the complex systems, even when strange attractors have different aspects [2]. In such conjecture, if the fractalization of thermal type dynamics of the complex systems is accomplished by Markov stochastic processes:

$$\lambda_+^i\lambda_+^l = \lambda_-^i\lambda_-^l = 2\lambda\delta^{il} \qquad (3)$$

where λ is a coefficient specific to the multifractal – nonmultifractal transition and δ^{il} is the pseudo-tensor of Kronecker, then the scale covariant derivative (1) becomes [2-8]:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^i\partial_i - i\lambda(dt)^{[2/f(\alpha)]-1}\partial_l\partial_l \qquad (4)$$

If multifractalization of the thermal type dynamics of the complex systems is accomplished through non-Markov stochastic processes [2,8]:

$$d^{il} = \lambda_+^i\lambda_+^l - \lambda_-^i\lambda_-^l = 4\mu\delta^{il}$$
$$\bar{d}^{il} = \lambda_+^i\lambda_+^l + \lambda_-^i\lambda_-^l = 4\sigma\delta^{il} \qquad (5)$$

$$D^{il} = 4(\mu - i\sigma)\delta^{il}$$

where λ and σ are coefficients specific to the multifractal-non-multifractal transition, then the scale covariant derivative (1) takes the form:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^i \partial_i + (\mu - i\sigma)(dt)^{[2/f(\alpha)]-1}\partial_l \partial_l \qquad (6)$$

In such conjecture, the dynamics of momentum, mass and thermal transfer equations associated to fractal-non-fractal transition using the scale covariant derivative (4) are [11]:

$$\partial_t v + (v \cdot \nabla)v = -\frac{\nabla p}{\rho} + 2\left[v_F \cdot \nabla v_F + \lambda(dt)^{[2/f(\alpha)]-1}\Delta v_F\right] + \left[v - \lambda(dt)^{[2/f(\alpha)]-1}\right]\Delta v$$

$$\partial_t \rho + \nabla.(\rho v) = \rho\nabla.v - \lambda(dt)^{[2/f(\alpha)]-1}\Delta\rho$$

$$\partial_t T + (v \cdot \nabla)T = \left[\mu - \lambda(dt)^{[2/f(\alpha)]-1}\right]\Delta T$$

$$(7)$$

where

$$v = V_D - V_F \qquad (8)$$

is the velocity associated to the fractal-non-fractal transition,

$$f = 2\left[V_F \cdot \nabla V_F + \lambda(dt)^{[2/f(\alpha)]-1}\Delta V_F\right] \qquad (9)$$

is the multifractal specific force, p is the hydrostatic pressure, ρ is the density of states, $\left[v - \lambda(dt)^{[2/f(\alpha)]-1}\right]\Delta v$ is the effective viscous force associated with the fractal-non-fractal transition, T is the temperature, v is the kinematic viscosity coefficient, and μ is the thermal diffusibility coefficient.

Let us now consider the following assumptions:

i) the density of states is constant, $\rho = \rho_0 = const.$, excepting the momentum equation;

ii) we define the effective kinematic viscosity coefficient by means of relation:

$$\bar{v} = v - \lambda(dt)^{[2/f(\alpha)]-1} \qquad (10)$$

iii) between the velocity associated to fractal-non-fractal transition and the density of states there is the relationship:

$$\nabla.v = \lambda(dt)^{[2/f(\alpha)]-1}\frac{\Delta\rho}{\rho} \qquad (11)$$

iv) the thermal expansion associated to the fractal-non-fractal transition is linear,

$$\rho = \rho_0[1 + \bar{\alpha}(T - T_0)] \qquad (12)$$

where $\bar{\alpha}$ is the thermal expansion coefficient associated to the fractal-non-fractal transition, ρ_0 is the initial state density and T_0 is the initial temperature.

v)   we define the effective thermal diffusibility coefficient through the relationship:

$$\bar{\mu} = \mu - \lambda(dt)^{\left[2/f(\alpha)\right]-1} \tag{13}$$

vi)   The multifractal specific force is constant (its average is non-null)

With these assumptions the equations system (7) becomes:

$$\partial_t v + (v \cdot \nabla)v = -\frac{\nabla p}{\rho_0} + \left(1 + \frac{\delta\rho}{\rho_0}\right)f + \bar{v}\Delta v$$
$$\nabla.v = 0 \tag{14}$$
$$\partial_t T + (v \cdot \nabla)T = \bar{\mu}\Delta T$$

where $\delta\rho$ is density of perturbation states:

$$\rho = \rho_0 + \delta\rho$$
$$\delta\rho \ll \rho_0 \tag{15}$$

The thermal type fractal anomalous convection in complex systems occurs when the multifractal specific force resulting from the thermal expansion, $(\delta\rho/\rho_0)f$ exceeds the effective viscous force $\bar{v}\Delta v$. Then, we can define Rayleigh's effective number:

$$R = \frac{\left|\left(\delta\rho/\rho_0\right)f\right|}{\left|\bar{v}\Delta v\right|} \tag{16}$$

Its expression may be attributed to the relationship (12) given in the form:

$$\frac{\delta\rho}{\rho_0} = \bar{\alpha}\Delta T = \bar{\alpha}\beta d \tag{17}$$

and to the last relationship (14) given as form:

$$v \approx \frac{\bar{\mu}}{d} \tag{18}$$

where we considered that the thickness d of the complex systems is subject to gradient:

$$\beta = \frac{\Delta T}{d} \tag{19}$$

By substituting (17) and (18) into (16) the Rayleigh's effective number takes the form:

$$R = \frac{\bar{\alpha}\beta f}{\bar{v}\cdot\bar{\mu}}d^4 = \frac{\bar{\alpha}\beta f \cdot d^4}{\left[v-\lambda(dt)^{\left[2/f(\alpha)\right]-1}\right]\left[\mu-\lambda(dt)^{\left[2/f(\alpha)\right]-1}\right]} \tag{20}$$

The thermal type multifractal anomalous convection occurs for the condition:

$$R > R_{critic} \tag{21}$$

R being dictated mainly by the thermal gradient β and multifractal degree given both by $\lambda(dt)^{\left[2/f(\alpha)\right]-1}$ and multifractal specific force (9).

We choose as the reference state, the resting state, $v_s=0$, for which the first and last equation of the equations system (14) become:

$$\nabla p_s = -\rho_s f\hat{z} = -\rho_0[1 - \bar{\alpha}(T_s - T_0)]f\hat{z}$$
$$\Delta T_s = 0 \tag{22}$$

where $\hat{z}$ is the unit vector of direction Oz.

We also think that the pressure and temperature vary only along the Oz direction. For temperature, the boundary conditions are:

$$T(x, y, 0) = T_0, \quad T(x, y, d) = T_1 \tag{23}$$

Now integrating the second equation (22) with these boundary conditions, the dependence of the temperature in the direction Oz is linear and it has the expression:

$$T_s = T_0 - \beta z \tag{24}$$

Substituting (24) in the first equation (22) and integrating, we find:

$$p_s(z) = p_0 - \rho_0 f \left(1 + \frac{\bar{a}\beta z}{2}\right) z \tag{25}$$

The characteristics of the complex system in this state are independent of the "effective kinetic coefficients" $\bar{\nu}$ and $\bar{\mu}$ which occurring in the equations system (14).

We continue to analyze the stability of the reference state by the small perturbation method (Galerkin Method [1-3]). The perturbation state is explicitly explained by relationships:

$$\begin{aligned}
T &= T_s(z) + \theta(r, t, dt) \\
\rho &= \rho_s(z) + \delta\rho(r, t, dt) \\
p &= p_s(z) + \delta p(r, t, dt) \\
v &= \delta v(r, t, dt) = (u, v, w)
\end{aligned} \tag{26}$$

As it results from the above relationships, perturbations are functions that depend on position, time, and scale resolution. Substituting (26) in the equations system (14) and taking into account (24) and (25), the following equations system for perturbations, in linear approximation, is obtained:

$$\begin{aligned}
\partial_t \delta v &= -\frac{1}{\rho_0} \nabla \delta p + \bar{\nu} \Delta \delta v + f \bar{a} \theta \hat{z} \\
\nabla \cdot \delta v &= 0 \\
\partial_t \theta &= \beta w + \bar{\mu} \Delta \theta
\end{aligned} \tag{27}$$

We introduce the non-dimensional variables into the equations system (27) based on the relationships:

$$\vec{r}' = \frac{r}{d}, \ t' = \frac{t}{\frac{d^2}{\bar{\mu}}}, \theta' = \frac{\theta}{\left(\frac{\bar{\nu}\cdot\bar{\mu}}{f\bar{a}d^3}\right)}, \delta v' = \frac{\delta v}{\frac{\bar{\mu}}{d}}, \delta p' = \frac{\delta p}{\left(\frac{\rho_0 \bar{\nu}\bar{\mu}}{d^2}\right)} \tag{28}$$

Replacing these variables in the equations system (27), and renouncing for simplicity to the indexing with the symbol ",", it results:

$$P^{-1}(\partial_t v + (v \cdot \nabla)v) = -\nabla p + \theta \hat{z} + \Delta v$$

$$\nabla \cdot v = 0 \tag{29}$$

$$\partial_t \theta + (v \cdot \nabla)\theta = Rw + \Delta\theta$$

where

$$P = \frac{\bar{v}}{\bar{\mu}} = \frac{v - \lambda(dt)^{[2/f(\alpha)]-1}}{\mu - \lambda(dt)^{[2/f(\alpha)]-1}} \tag{30}$$

is Prandtl's effective number dependent on both classical kinetic coefficients v and μ the "multifractality degree" given by $\lambda(dt)^{[2/f(\alpha)]-1}$

For $R>R_c$, the reference state of the complex system becomes unstable. Choosing $v = 0$, the incompressibility condition of the complex system becomes:

$$u_x + w_z = 0 \tag{31}$$

The above equation is satisfied if and only if:

$$u = -\psi_z, \quad w = \psi_x \tag{32}$$

where $\psi(x, y, z)$ is Lagrange type current function. Velocity fields must satisfy the conditions imposed on the interior and exterior surfaces in the form:

$$w|_{z=\pm 1/2} = 0 \tag{33}$$

If the surfaces are assumed to be free, then the additional condition appears:

$$\left.\frac{\partial u}{\partial z}\right|_{z \to \pm 1/2} = 0 \tag{34}$$

Now using Lagrange type current function, boundary conditions (33) and (34) are written as:

$$\psi_x|_{z \to \pm 1/2} = 0, \quad \psi_{zz}|_{z \to \pm 1/2} = 0 \tag{35}$$

Consider ψ on the form:

$$\psi(x, z, t, dt) = \psi_1(t, dt) \cos(\pi z) \sin(qx) \tag{36}$$

According to (32), the components of the speed fields will be:

$$u = \pi\psi_1(t, dt) \sin(\pi z) \sin(qx) \tag{37}$$
$$w = q\psi_1(t, dt) \cos(\pi z) \cos(qx)$$

In these conditions, the first equation (29) for directions Ox and Oz will be:

$$P^{-1}(u_t + uu_x + wu_z) = -p_x + \Delta u \tag{38}$$
$$P^{-1}(w_t + uw_x + ww_z) = -p_z + \Delta w + \theta$$

Deriving the first equation (38) versus z and the second equation (38) versus x, it results:

$$P^{-1}\left[u_{tz} + \frac{\partial}{\partial z}(uu_x + wu_z)\right] = -p_{xz} + \frac{\partial}{\partial z}(\Delta u) \tag{39}$$
$$P^{-1}\left[w_{tx} + \frac{\partial}{\partial x}(uw_x + ww_z)\right] = -p_{zx} + \frac{\partial}{\partial x}(\Delta w) + \frac{\partial \theta}{\partial x}$$

Adding these equations, we will obtain:

$$P^{-1}\left[-(\Delta\psi)_t + \frac{\partial}{\partial z}(uu_x + wu_z) - \frac{\partial}{\partial x}(uw_x + ww_z)\right] = -\Delta^2\psi - \theta_x \tag{40}$$

The temperature being fixed on the two borders, we will have:

$$\theta|_{z \to \pm 1/2} = 0 \tag{41}$$

Let us consider θ of the form:

$$\theta(x, z, t, dt) = \theta_1(t, dt) \cos(\pi z) \cos(qx) + \theta_2(t, dt) \sin(2\pi z) \tag{42}$$

Substituting in (40) the expressions for u, w, θ and Ψ, it results:

$$P^{-1}\dot{\psi}_1 = \frac{q\theta_1}{\pi^2 + q^2} - (\pi^2 + q^2)\psi_1 \tag{43}$$

The equations of the multifractal thermal transfer will become:

$$\dot{\theta}_1 = -\pi q \psi_1 \theta_2 + qR\psi_1 - (\pi^2 + q^2) \cdot \theta_1 \tag{44}$$

$$\dot{\theta}_2 = \frac{1}{2}\pi q \psi_1 \theta_1 - 4\pi^2 \theta_2$$

3. Results and Discussion

In the equations (43) and (44), for the amplitudes we make the following variable changes:

$$t' = (\pi^2 + q^2)t, \quad X = \frac{\pi q}{\sqrt{2}(\pi^2 + q^2)}\psi_1 \tag{45}$$

$$Y = \frac{\pi q^2}{\sqrt{2}(\pi^2 + q^2)^3}\theta_1, \quad Z = \frac{\pi q^2}{(\pi^2 + q^2)^3}\theta_2$$

It results:

$$\dot{X} = P(Y - X)$$
$$\dot{Y} = -XZ + rX - Y \tag{46}$$
$$\dot{Z} = XY - bZ$$
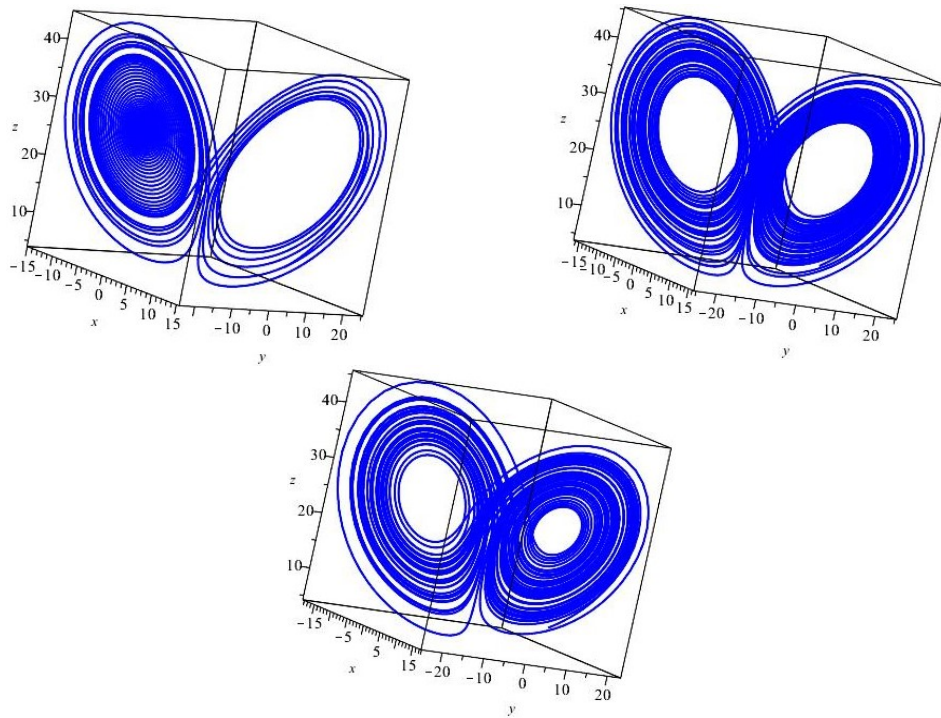
where:

$$r = \frac{q^2}{(\pi^2 + q^2)^3}R, \quad b = \frac{4\pi^2}{(\pi^2 + q^2)} \tag{47}$$

i.e Lorentz type multifractal system.

Lorenz type behaviors associated to fractal-non-fractal transition in the phase space are very complicated. In Figs .1 a-e only the influence of Prandtl effective coefficient is presented both for the same values of initial conditions ($X_0 = 1, Y_0 = 5, Z_0 = 10$) and the same values of parameters (r=28, b=8/3).
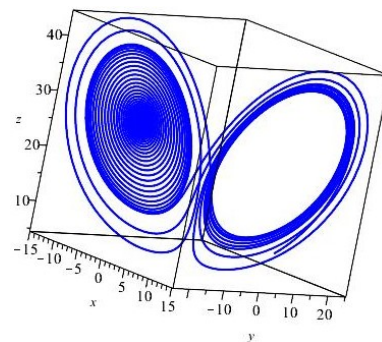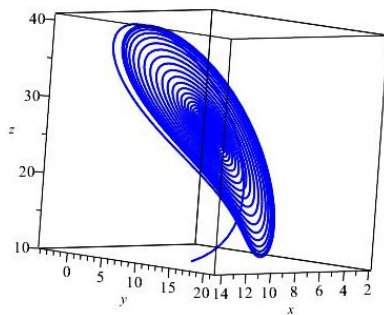
Figs. 1 a-e Lorenz type behaviors associated to fractal-non-fractal transition in the phase space for various values of Prandtl effective coefficient (P_1=4, P_2=10, P_3=24, P_4=30, P_5=80), for the same values of initial conditions ($X_0 = 1, Y_0 = 5, Z_0 = 10$) and the same values of parameters (r=28, b=8/3).

4. Conclusions

Assuming that the dynamics of the complex systems associated to fractal-non-fractal transition are described by continuous but non-differentiable curves (multifractal curves), Lorenz type behaviors became functional in the framework of the Scale Relativity Theory with arbitrary but constant fractal dimension. Then:

i) the dynamics are described by means of the scale covariant derivative;

ii) the equations systems of momentum, mass and thermal transfer associated to fractal-non-fractal transition are obtained;

iii) through some constraints and using the Galerking method, the equations systems of momentum, mass and thermal transfer Lorenz type multifractal involve;

iv) for Lorenz type multifractal system, Rayleigh and Prandtl effective numbers are specified both by means of classical kinetic coefficients (kinematic viscosity coefficient and thermal diffusion coefficient) and scale resolution;

vi) for Lorenz type multifractal system, the dynamics variables act as the limit of a family of mathematical functions, non-differentiable for null scale resolution;

vii) only the influence of Prandtl effective coefficient on the dynamics of Lorenz type multifractal system is analyzed given the complexity of such system in the phase space.

<div align="center">R E F E R E N C E S</div>

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