MULTI-GRANULATION VARIABLE PRECISION ROUGH SET BASED ON LIMITED TOLERANCE RELATION

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In this paper, the combination of the variable precision rough set and the limited tolerance relation under multi-granularity is explored. As an extension of rough set model, Multi-granularity variable precision limited tolerance rough set model is constructed. Properties of upper and lower approximation operator and detailed structure of object class through threshold are discussed. Theoretical proofs are given to show the rationality and superiority of the proposed model.

Keywords: Variable precision, mult-granulation, rough set, limited tolerance, incomplete information system.

MSC2020: 47H10, 55M20.

1. Introduction

The classical rough set model [11, 12] proposed by polish scholar Pawlak in the early 1980s is a powerful mathematical tool for data analysis. Rough set theory has been broadly applied in pattern recognition, machine learning, decision analysis, knowledge acquisition and data mining [7, 9, 17, 20, 27, 30, 31]. In the past few decades, due to the diversity of data and different requirements of analysis purposes, the extended rough set models, such as variable precision rough set [32], probability rough set [28, 29], Game-theoretic rough set [1, 4], fuzzy rough set [10, 23], local neighborhood rough set [22] and so on.

However, the classical rough set model and most of its extensions are basically based on the indiscernibility relation, actually, the indiscernibility relation is an equivalence relation which possesses reflexive, symmetric and transitive properties. While the equivalence relation is relatively strict condition in many practical application, and classes clustering on this relation cannot well reflect the natural characteristic of overlapping data set.

Many scholars have conducted research works for substitution of the equivalence relation [5, 19, 21, 26], some scholars also describe the concept of target through multiple indiscernibility relations, and propose a multi-granularity rough set model [14, 16, 24, 25]. In these research works, Skowron and Stepniuk [18] replaced the equivalence relation with the tolerance relation and proposed the tolerance approximation spaces, and Kryszkiewicz [8] defined a similarity relation in incomplete information systems. Kryszkiewicz's similarity relation is an extension of Skowron's tolerance relation, therefore, both of them are referred to as tolerance relation collectively by later researchers. The tolerance relation discards the transitivity requirement of indiscernibility relation in classical rough set and relaxes the

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symmetry requirement for incomplete information. Hence, the tolerance classes can well reflect the overlapping relation between groups of objects. Dai [2] defined the fuzzy tolerance relation in complete numerical data set and established the fuzzy tolerance rough set, Kang and Miao [6] proposed an extended version of the variable precision rough set model based on the granularity of tolerance relation. Xu et.al [24] extended the single-granulation tolerance rough set model to two types of tolerance multi-granulation rough set models from a granular computing view. Stefanowski and Tsoukias [19] introduced non symmetric similarity relation which can refine the results obtained using tolerance relation approach, they also proposed valued tolerance relation in order to provide more informative results, however, Wang [21] found that then on symmetric similarity relation may lose some important information, and valued tolerance relation requires accurate probability distribution of all attributes in advance, Wang then proposed the limited tolerance relation. Deris et.al [3] used conditional entropy to handle flexibility and precisely data classification in limited tolerance relation.

Combining the advantages of limited tolerance relation and probabilistic variable precision probabilistic rough set, this paper constructs a model of multi-granularity variable precision rough set based on limited tolerance relation in incomplete information system.Multi-granularity variable precision rough set is an effective unified and extended version of variable precision rough set, probabilistic rough set, tolerance rough set and multi-granularity rough set. In Section 2, some related concepts will be reviewed. In Section 3, multi-granulation variable precision rough set based limited tolerance relation is presented and properties of the proposed rough set model are analyzed. The relationships between the proposed rough set and others are discussed in Section 4. In Section 5, the method of measuring the uncertainty of the proposed roughed set model is given, and the superiority of the model is further verified. Finally, conclusions are made in section 6.

2. **Notations and Preliminaries**

In this section, some basic concepts such as information system, Pawlak's rough set, variable precision rough set, probabilistic rough set, tolerance rough set and multi-granularity rough set will be reviewed as preliminaries of the follows.

Definition 2.1 ([3, 8])**.** *An information system(IS) is a 4-tuple* $S = (U, TA, V, f)$*, where* $U = \{x_1, x_2, ..., x_{|U|}\}$ *is a non-empty finite set of objects,* $TA = \{a_1, a_2, ..., a_{|TA|}\}$ *is a non-empty finite set of attributes,* $V = \bigcup_{a \in TA} V_a$*,* V_a *is the value set of attribute,* $f : U \times TA \to V$ *is a total function such that* $f(x, a) \in V$*, for every* $(x, a) \in U \times TA$*, called information function. If* U *contains at least one object with an unknown or missing value (so-called null value), then* S *is called incomplete information system(IIS). The unknown value is denoted as "*" in incomplete information system. In this paper, we also use the quadruple* $S = (U, TA, V, f)$ *to denote an incomplete information system. If* $TA = C \cup D$*, where* C *is the set of condition attributes,* D *is the set of decision attributes, then* S *is called Decision Information System.*

Each subset of attributes $A \subseteq TA$ determines a binary indiscernibility relation $IND(A)$ as follows:

$$IND(A) = \{(x, y) \in U \times U | \forall a \in A, a(x) = a(y)\}.$$

The relation $IND(A)$ is an equivalence relation since it is reflexive, symmetric and transitive.

Example 2.1. Given descriptions of several cars in Table 2.1 and Table 2.2. They are an information system and an incomplete information system respectively. From tables, we have $U = \{1, 2, 3, 4, 5, 6\}$, $TA = \{\text{Car}, \text{Price}, \text{Mileage}, \text{Size}, \text{Max} - \text{Speed}\}$.

Table 2.1: An information system of Example 2.1.

Car	Price	Mileage	Size	Max-Speed
1	High	High	Full	Low
2	Low	Low	Full	Low
3	High	Low	Full	High
4	High	Low	Compact	High
5	Low	High	Full	High
6	Low	High	Full	High

Table 2.2: An incomplete information system of Example 2.1.

Car	Price	Mileage	Size	Max-Speed
1	High	High	Full	Low
2	Low	$*$	$*$	Low
3	$*$	$*$	Full	High
4	High	$*$w	Compact	High
5	$*$	$*$	Full	High
6	Low	High	$*$	$*$

Definition 2.2 ([11, 12]). *Let $S = (U, TA, V, f)$, be an IS, $A \subseteq TA$, the lower and upper approximations of an arbitrary subset X of U are defined as $\underline{A}(X) = \{x \in U : [x]_A \subseteq X\}$ and $\overline{A}(X) = \{x \in U : [x]_A \cap X \neq \emptyset\}$ respectively, where $[x]_A = \{y \in U(x, y) \in IND(A)\}$ is the -equivalence class containing. The pair $\left[\underline{A}(X), \overline{A}(X)\right]$is referred to as the Pawlak's rough set of with respect to the set of attributes A.*

Definition 2.3 ([32]). *Let $S = (U, TA, V, f)$ be an IS, $A \subseteq TA$, $0 < \beta \leq 0.5$, the lower and upper approximations of an arbitrary subset* X *of* U *are defined as $\underline{A}(X) = \{x \in U|P(X|[x]_A) \geq 1 - \beta\}$ and $\overline{A}^{\beta}(X) = \{x \in U|P(X|[x]_A) > \beta\}$ respectively, where $P(X|[x]_A)$ is the conditional probability of X given $[x]_A$. The pair $\left[\underline{A}^{\beta}(X), \overline{A}^{\beta}(X)\right]$ is referred to as the variable precision rough set of X with respect to the set of attributes A and the admissible error β.*

Definition 2.4 ([8]). *Let $S = (U, TA, V, f)$ be an IIS. For any subset $A \subseteq TA$, the tolerance relation is defined as $T(A) = \{(x, y) \in U \times U|\forall a \in A, a(x) = a(y) \vee a(x) = * \vee a(y) = *\}$.*

Obviously, T is reflexive and symmetric, but not transitive. The tolerance class $I_A^T(x)$ of an object with reference to an attribute subset is defined as $I_A^T(x)$=$\{y|y \in U \wedge T_A(x, y)\}$.

Definition 2.5 ([3]). *Let $S = (U, TA, V, f)$ be an IIS, $A \subseteq TA$. T is a tolerance relation, the lower and upper approximations of an arbitrary subset X of U with reference to attribute subset A respectively can defined as are defined as $\underline{A^T}(X) = \{x|x \in U \wedge I_A^T(x) \subseteq X\}$ and $\overline{A^T}(X) = \{x|x \in U \wedge I_A^T(x) \cap X \neq \phi\}$. The pair $\left[\underline{A^T}(X), \overline{A^T}(X)\right]$ is referred to as the tolerance rough set of X with respect to the set of attributes A.*

Definition 2.6 ([21]). *Let $S = (U, TA, V, f)$ be an IIS, $A \subseteq TA$, and $P_A(x) = \{a|a \in A \wedge a(x) \neq *\}$. A binary relation L (limited tolerance relation) defined on U is given as*

$$L(A) = \{(x, y) \in U \times U|\forall_{a \in A}(a(x) = a(y) = *) \vee ((P_A(x) \cap P_A(y) \neq \phi) \wedge \\ \forall_{a \in A}((a(x) \neq *) \wedge (a(y) \neq *) \to (a(x) = a(y))))\}.$$

L is reflexive and symmetric, but not transitive. The limited tolerance class $I_A^L(x)$ of an object x with reference to an attribute subset A is defined as

$$I_A^L(x)=\{y|y \in U \wedge L_A(x,y)\}.$$

Definition 2.7 ([3])**.** *Let $S=(U,TA,V,f)$ be an IIS, $A \subseteq TA$, L is a limited tolerance relation, the lower and upper approximations of an arbitrary subset X of U with reference to attribute subset A respectively can defined as are defined as $\underline{A^L}(X)=\{x|x \in U \wedge I_A^L(x) \subseteq X\}$ and øverline$A^L(X)=\{x|x \in U \wedge I_A^L(x) \cap X \neq \phi\}$. The pair $\left[\underline{A^L}(X), \overline{A^L}(X)\right]$ is referred to as the limited tolerance rough set of X with respect to the set of attributes A.*

Definition 2.8 ([14, 15])**.** *Let $S=(U,TA,V,f)$ be an IS,$A_1, A_2, \cdots, A_m \subseteq AT$, the optimistic multi-granulation lower and upper approximations of an arbitrary subset X of U are denoted by $\underline{\Sigma_{i=1}^m A_i^o}(X)$ and $\overline{\Sigma_{i=1}^m A_i^o}(X)$, respectively,*

$$\underline{\Sigma_{i=1}^m A_i^o}(X)=\{x \in U : [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \ldots \vee [x]_{A_m} \subseteq X\},$$

$$\overline{\Sigma_{i=1}^m A_i^o}(X)=\sim \underline{\Sigma_{i=1}^m A_i^o}(\sim X)$$

where $[x]_{A_i}(1 \leq i \leq m)$ is the equivalence class of x in terms of set of attributes A_i, and $\overline{\Sigma_{i=1}^m A_i^o}(X)=\sim \underline{\Sigma_{i=1}^m A_i^o}(\sim X)$ is the complement of X.

The pair $\left[\underline{\Sigma_{i=1}^m A_i^o}(X), \overline{\Sigma_{i=1}^m A_i^o}(X)\right]$ is referred to as the optimistic multi-granulation rough set.

Definition 2.9 ([13, 14])**.** *Let $S=(U,TA,V,f)$ be an IS, $A_1, A_2, \cdots, A_m \subseteq AT$, the pessimistic multi-granulation lower and upper approximations of an arbitrary subset X of U are denoted by $\underline{\Sigma_{i=1}^m A_i^P}(X)$ and $\overline{\Sigma_{i=1}^m A_i^P}(X)$, respectively,*

$$\underline{\Sigma_{i=1}^m A_i^p}(X)=\{x \in U : [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \ldots \wedge [x]_{A_m} \subseteq X\},$$

$$\overline{\Sigma_{i=1}^m A_i^p}(X)=\sim \underline{\Sigma_{i=1}^m A_i^p}(\sim X).$$

The pair $\left[\underline{\Sigma_{i=1}^m A_i^P}(X), \overline{\Sigma_{i=1}^m A_i^P}(X)\right]$ is referred to as the pessimistic multi-granulation rough set.

3. Multi-granulation variable precision rough set based limited tolerance relation

In this section, multi-granulation variable precision rough set based limited tolerance relation will be investigated in incomplete system.

Definition 3.1. *Let $S=(U,TA,V,f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$, $T$is a tolerance relation, $0.5 < \alpha \leq 1$, the optimistic multi-granulation variable precision tolerance lower and upper approximations of an arbitrary subset X of U are denoted by $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$ and $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$, respectively,*

$$\underline{\Sigma_{i=1}^m A_{iT_\alpha}^O}(X)=\{x \in U : P(X|T_{A_1}(X)) \geq \alpha \vee P(X|T_{A_2}(X)) \geq \alpha \vee \ldots \vee P(X|T_{A_m}(X)) \geq \alpha\}$$

$$\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)=\sim \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\sim X).$$

The pair $\left[\underline{\Sigma_{i=1}^m A_{iT_\alpha}^O}(X), \overline{\Sigma_{i=1}^m A_{iT_\alpha}^O}(X)\right]$ is referred to as the optimistic multi-granulation variable precision tolerance rough set.

Definition 3.2. *Let $S = (U, TA, V, f)$ be an IIS,$A_1, A_2, \cdots, A_m \subseteq AT$, T is a tolerance relation, $0.5 < \alpha \leq 1$, the pessimistic multi-granulation variable precision tolerance lower and upper approximations of an arbitrary subset of are denoted by $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X)$ and $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X)$, respectively,*

$\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X) = \{x \in U : P(X|T_{A_1}(X)) \geq \alpha \wedge P(X|T_{A_2}(X)) \geq \alpha \wedge \ldots \wedge P(X|T_{A_m}(X)) \geq \alpha\}$,

$$\overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X) = \tilde{}\,\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(\tilde{}X).$$

The pair $\left[\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X), \overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X)\right]$ is referred to as the pessimistic multi-granulation variable precision tolerance rough set.

Theorem 3.1. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$, T is a tolerance relation. Then $\forall X \subseteq U$, we have*

$$\begin{aligned}\overline{\Sigma_{i=1}^m A_{iT_\alpha}^O}(X) = \{x \in U : P(X|T_{A_1}(X)) &> 1-\alpha \wedge P(X|T_{A_2}(X)) \\ &> 1-\alpha \wedge \ldots \wedge P(X|T_{A_m}(X)) \\ &> 1-\alpha\}.\end{aligned}$$

Proof. By Definition 3.1, we have

$x \in \overline{\Sigma_{i=1}^m A_{iT_\alpha}^O}(X) \Leftrightarrow x \notin \underline{\Sigma_{i=1}^m A_{iT_\alpha}^O}(\tilde{}X)$

$\Leftrightarrow P(\sim X|T_{A_1}(X)) < \alpha \wedge P(\sim X|T_{A_2}(X)) < \alpha \wedge \ldots \wedge P(\sim X|T_{A_m}(X)) < \alpha$

$\Leftrightarrow \frac{|(\tilde{}X)\cap T_{A_1}(X)|}{|T_{A_1}(X)|} < \alpha \wedge \ldots \wedge \frac{|(\tilde{}X)\cap T_{A_m}(X)|}{|T_{A_m}(X)|} < \alpha$

$\Leftrightarrow 1 - \frac{|T_{A_1}(X)\cap(X)|}{|T_{A_1}(X)|} < \alpha \wedge \ldots \wedge 1 - \frac{|T_{A_m}(X)\cap(X)|}{|T_{A_m}(X)|} < \alpha$

$\Leftrightarrow P(X|T_{A_1}(X)) > 1-\alpha \wedge \ldots \wedge P(X|T_{A_m}(X)) > 1-\alpha\}$.

The proof is complete. □

Theorem 3.2. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$, T is a tolerance relation. Then $\forall X \subseteq U$, we have*

$$\begin{aligned}\overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X) = \{x \in U : P(X|T_{A_1}(X)) &> 1-\alpha \vee P(X|T_{A_2}(X)) \\ &> 1-\alpha \vee \ldots \vee P(X|T_{A_m}(X)) \\ &> 1-\alpha\}.\end{aligned}$$

Proof. The proof of the theorem is similar to Theorem 3.1. □

Proposition 3.1. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$, T is a tolerance relation, $0.5 < \alpha \leq 1$, $X \subseteq U$. Then the following properties hold:*

(1) $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\emptyset) = \overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\emptyset) = \emptyset$.

(2) $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\sim X) = \sim \overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$.

(3) $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(U) = \overline{\Sigma_{i=1}^m A_{i_\alpha}^o}(U) = U$.

(4) $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\sim X) = \sim \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$.

(5) *if* $X \subseteq Y$ *, then* $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y)$ *and* $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y)$.

(6) *if* $\alpha_1 \geq \alpha_2$ *, then*$\overline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iT_{\alpha_2}}^o}(Y)$ *and*$\underline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^o}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iT_{\alpha_2}}^o}(Y)$.

Proof. (1a) From Definition 3.1, we can easily know that $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\emptyset) \subseteq \emptyset$, and $\emptyset \subseteq \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\emptyset)$ because the empty set is included in every set. Therefore, $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\emptyset)$=$\emptyset$.

(1b) Suppose$\overline{\Sigma_{i=1}^m A_{i_\alpha}^o}(\emptyset) \neq \emptyset$. Then, there exists such that $x \in \overline{\Sigma_{i=1}^m A_{i\beta}^o}(\emptyset) \neq \emptyset$. For every $i \in \{1, 2, \cdots, m\}$, $I_{A_i}^T(x) \cap \emptyset \neq \emptyset$. But $I_{A_i}^T(x) \cap \emptyset \neq \emptyset$. It contradicts the assumption. So, $\overline{\Sigma_{i=1}^m A_{i_\alpha}^o}(\emptyset)$=$\emptyset$.

(2) From Definition 3.1, we know that, $\underline{\Sigma_{i=1}^m A_{i_\alpha}^o}(\sim X) =\sim \overline{\Sigma_{i=1}^m A_{i_\alpha}^o}(X)$. Let $X =\sim X$, then $\sim \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\sim X) = \overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$, namely, $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\sim X) =\sim \overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$.

(3) Let $X = \emptyset$, then ˜X=U. From (2), $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(U) = \tilde{}\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\emptyset) = \tilde{}\emptyset = U$.

(4) From (2), $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\tilde{}Y) = \tilde{}\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y)$holds, then $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y) = \tilde{}\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\tilde{}Y)$. Let X=˜Y, then we have $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(\sim X) =\sim \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$.

(5) $\forall x \in \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X)$, From Definition 3.1, there exists $A_i \in \{A_1, A_2, \cdots, A_n\}$ such that $P(X|T_{A_1}(X)) \geq \alpha$. Since $X \subseteq Y$, then $P(X|T_{A_i}(X)) \subseteq P(X|T_{A_i}(Y)) \geq \alpha$, hence $x \in \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y)$. Therefore, $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y)$.

Similarly, it is not difficult to prove that $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iT_\alpha}^o}(Y)$.

(6) $\forall x \in \Sigma_{i=1}^m A_{iT_{\alpha_2}}^o(X)$, From Definition 3.2, there exists $A_i \in \{A_1, A_2, \cdots, A_n\}$ such that $P(X|T_{A_i}(X)) \geq \alpha_2$. Since $\alpha_1 > \alpha_2$, then $P(X|T_{A_i}(X)) \geq \alpha_1$, i.e.$x \in \underline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^o}(X)$.

Therefore, $\underline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^o}(X) \supseteq \underline{\Sigma_{i=1}^m A_{iT_{\alpha_2}}^o}(Y)$.

Similarly, it is not difficult to prove that $\overline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iT_{\alpha_2}}^o}(Y)$. □

Proposition 3.2. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is a tolerance relation, $0.5 < \alpha \leq 1$, $X \subseteq U$. Then the following properties hold:*

(1) $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(U) = \overline{\Sigma_{i=1}^m A_{i_\alpha}^P}(U) = U$.

(2) $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(\emptyset) = \overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(\emptyset) = \emptyset$.

(3) $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(\tilde{}X) = \tilde{}\overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X)$.

(4) $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(\tilde{}X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X)$.

(5) *If $X \subseteq Y$, then $\overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(Y)$ and $\underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iT_\alpha}^P}(Y)$.*

(6) *If $\alpha_1 \geq \alpha_2$, then $\overline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^P}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iT_{\alpha_2}}^P}(Y)$ and $\underline{\Sigma_{i=1}^m A_{iT_{\alpha_1}}^P}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iT_{\alpha_2}}^P}(Y)$.*

Proof. The proofs of these terms are similar to Proposition 3.1. □

Definition 3.3. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is the limited tolerance relation, $0.5 < \alpha \leq 1$, the optimistic multi-granulation variable precision limited tolerance lower and upper approximations of an arbitrary subset X of U are denoted by $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$ and $\overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$, respectively,*

$$\underline{\Sigma_{i=1}^m A_{iL_\alpha}^O}(X) = \{x \in U : P(X|L_{A_1}(X)) \geq \alpha \vee P(X|L_{A_2}(X)) \geq \alpha \vee \ldots \vee P(X|L_{A_m}(X)) \geq \alpha\},$$

$$\overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(\tilde{}X).$$

The pair $\left[\underline{\Sigma_{i=1}^m A_{iL_\alpha}^O}(X), \overline{\Sigma_{i=1}^m A_{iL_\alpha}^O}(X)\right]$ is referred to as the optimistic multi-granulation variable precision limited tolerance rough set.

Definition 3.4. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is the tolerance relation, $0.5 < \alpha \leq 1$, the pessimistic multi-granulation variable precision limited tolerance lower and upper approximations of an arbitrary subset X of U are denoted by $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X)$ and $\overline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X)$, respectively,*

$$\underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X) = \{x \in U : P(X|L_{A_1}(X)) \geq \alpha \vee P(X|L_{A_2}(X)) \geq \alpha \vee \ldots \vee P(X|L_{A_m}(X)) \geq \alpha\},$$

$$\overline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(\tilde{}X).$$

The pair $\left[\underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X), \overline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X)\right]$ is referred to as the pessimistic multi-granulation variable precision limited tolerance rough set.

Theorem 3.3. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is a limited tolerance relation, $0.5 < \alpha \leq 1$. Then $\forall X \subseteq U$, we have*

$$\begin{aligned}\overline{\Sigma_{i=1}^m A_{iL_\alpha}^O}(X) = \{x \in U : P(X|L_{A_1}(X)) &> 1-\alpha \wedge P(X|L_{A_2}(X)) \\ &> 1-\alpha \wedge \ldots \wedge P(X|L_{A_m}(X)) \\ &> 1-\alpha\}.\end{aligned}$$

Proof. From Definition 3.3, we have

$$\begin{aligned}x \in \overline{\Sigma_{i=1}^m A_{iL_\alpha}^O}(X) &\Leftrightarrow x \notin \underline{\Sigma_{i=1}^m A_{iL_\alpha}^O}(\tilde{}X) \\ &\Leftrightarrow P(\tilde{}X|L_{A_1}(X)) < \alpha \wedge P(\tilde{}X|L_{A_2}(X)) < \alpha \wedge \ldots \wedge P(\tilde{}X|L_{A_m}(X)) < \alpha \\ &\Leftrightarrow \frac{|(\tilde{}X) \cap L_{A_1}(X)|}{|L_{A_1}(X)|} < \alpha \wedge \ldots \wedge \frac{|(\tilde{}X) \cap L_{A_m}(X)|}{|L_{A_m}(X)|} < \alpha \\ &\Leftrightarrow 1 - \frac{|L_{A_1}(X) \cap (X)|}{|L_{A_1}(X)|} < \alpha \wedge \ldots \wedge 1 - \frac{|L_{A_m}(X) \cap (X)|}{|L_{A_m}(X)|} < \alpha \\ &\Leftrightarrow P(X|L_{A_1}(X)) > 1-\alpha \wedge \ldots \wedge P(X|L_{A_m}(X)) > 1-\alpha\}.\end{aligned}$$

The proof is complete. □

Theorem 3.4. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is the limited tolerance relation, $0.5 < \alpha \leq 1$. Then $\forall X \subseteq U$, we have*

$$\begin{aligned}\overline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X) = \{x \in U : P(X|L_{A_1}(X)) &> 1-\alpha \vee P(X|L_{A_2}(X)) \\ &> 1-\alpha \vee \ldots \vee P(X|L_{A_m}(X)) \\ &> 1-\alpha\}.\end{aligned}$$

Proof. From Definition 3.4, we have

$$\begin{aligned}x \in \overline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X) &\Leftrightarrow x \notin \underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(\tilde{}X) \\ &\Leftrightarrow P(\tilde{}X|L_{A_1}(X)) < \alpha \vee P(\tilde{}X|L_{A_2}(X)) < \alpha \vee \ldots \vee P(\tilde{}X|L_{A_m}(X)) < \alpha \\ &\Leftrightarrow \frac{|(\tilde{}X) \cap L_{A_1}(X)|}{|L_{A_1}(X)|} < \alpha \vee \ldots \vee \frac{|(\tilde{}X) \cap L_{A_m}(X)|}{|L_{A_m}(X)|} < \alpha \\ &\Leftrightarrow 1 - \frac{|L_{A_1}(X) \cap (X)|}{|L_{A_1}(X)|} < \alpha \vee \ldots \vee 1 - \frac{|L_{A_m}(X) \cap (X)|}{|L_{A_m}(X)|} < \alpha \\ &\Leftrightarrow P(X|L_{A_1}(X)) > 1-\alpha \vee \ldots \vee P(X|L_{A_m}(X)) > 1-\alpha\}.\end{aligned}$$

The proof is complete. □

Proposition 3.3. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is the limited tolerance relation, $0.5 < \alpha \leq 1$, $\forall X, Y \subseteq U$. Then the following properties hold:*
(1) $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) \subseteq X \subseteq \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$.
(2) $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(\emptyset) = \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(\emptyset) = \emptyset$.
(3) $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(U) = \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(U) = U$.
(4) *If* $X \subseteq Y$ *, then* $\overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(Y)$ *and* $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(Y)$.
(5) *If* $\alpha_1 \geq \alpha_2$ *, then* $\overline{\Sigma_{i=1}^m A_{iL_{\alpha_1}}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL_{\alpha_2}}^o}(Y)$ *and* $\underline{\Sigma_{i=1}^m A_{iL_{\alpha_1}}^o}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iL_{\alpha_2}}^o}(Y)$.

Proof. (1) $\forall x \in \underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$, from Definition 3.3, $\exists A_i \in \{A_1, \cdots, A_m\}$, with $P(X|L_{A_i}(X)) \geq \alpha$, i.e., for each$\alpha$:$0.5 < \alpha \leq 1$, it holds $|X \cap I_{A_i}^L(x)| \geq \alpha |I_{A_i}^L(x)|$. Hence, $I_{A_i}^L(x) \subseteq X$. Thus $x \in X$. Therefore, $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) \subseteq X$; $\forall x \in X$, since limited tolerance relation is reflexive, then we have $x \in I_{A_i}^L(x)$. For each $A_i \in \{A_1, \cdots, A_m\}$, according to the above conclusion, it follows $|X \cap I^L_{A_i}(x)| \geq \alpha|I^L_{A_i}(x)|$, since $0.5 < \alpha \leq 1$, then it holds $|X \cap I^L_{A_i}(x)| \geq (1-\alpha)|I^L_{A_i}(x)|$, thus, $x \in \overline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X)$.

Therefore $X \subseteq \overline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X)$

(2) We can easily know that $\underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(\emptyset) \subseteq \emptyset$. Since the empty set is included in every set, then $\emptyset \subseteq \underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(\emptyset)$. Therefore, $\emptyset \subseteq \underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(\emptyset)$; $\emptyset \subseteq \overline{\Sigma^m_{i=1}A^o_{i_\alpha}}(\emptyset)$ holds obviously since the empty set is included in every set . Suppose$\overline{\Sigma^m_{i=1}A^o_{i_\alpha}}(\emptyset) \neq \emptyset$. Then, there exists x such that $x \in \overline{\Sigma^m_{i=1}A^o_{i\beta}}(\emptyset) \neq \emptyset$. Hence, according to Theorem 3.3, for each$i \in \{1, 2, \cdots, m\}$ and each α:$0.5 < \alpha \leq 1$, $|\emptyset \cap I^L_{A_i}(x)| \geq (1-\alpha)|I^L_{A_i}(x)|$, since $\emptyset \cap I^L_{A_i}(x) = \emptyset$, then we have $I^L_{A_i}(x) = \emptyset$. Since limited tolerance relation is reflexive, thus, $x \in I^L_{A_i}(x) = \emptyset$, it contradicts the assumption.

Therefore, $\overline{\Sigma^m_{i=1}A^o_{i_\alpha}}(\emptyset)$=$\emptyset$.

(3) Similar to (2), we can easily to prove the conclusion.

(4) $\forall x \in \underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X)$, from Definition 3.3, $\exists A_i \in \{A_1, \cdots, A_m\}$, with $P(X|L_{A_i}(X)) \geq \alpha$, i.e. for each α:$0.5 < \alpha \leq 1$, it holds $|X \cap I^L_{A_i}(x)| \geq \alpha|I^L_{A_i}(x)|$. Hence, $I^L_{A_i}(x) \subseteq X$. Since $X \subseteq Y$, then $I^L_{A_i}(x) \subseteq Y$, it holds $|Y \cap I^L_{A_i}(x)| \geq \alpha|I^L_{A_i}(x)|$, it means that $\forall x \in \underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(Y)$.

Therefore, $\underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X) \subseteq \underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(Y)$.

Similarly, it is not difficult to prove that $\overline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X) \subseteq \overline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(Y)$.

(5) $\forall x \in \underline{\Sigma^m_{i=1}A^o_{iL_{\alpha_2}}}(X)$, $\exists A_i \in \{A_1, \cdots, A_m\}$, such that $P(X|L_{A_i}(X)) \geq \alpha_2$, since$\alpha_1 \leq \alpha_2$, then we have$P(X|L_{A_i}(X)) \geq \alpha_1$. Hence, $\forall x \in \underline{\Sigma^m_{i=1}A^o_{iL_{\alpha_1}}}(X)$.

Therefore , $\underline{\Sigma^m_{i=1}A^o_{iL_{\alpha_1}}}(X) \subseteq \underline{\Sigma^m_{i=1}A^o_{iL_{\alpha_2}}}(X)$.

Similarly, it is not difficult to prove that $\overline{\Sigma^m_{i=1}A^o_{iL_{\alpha_1}}}(X) \subseteq \overline{\Sigma^m_{i=1}A^o_{iL_{\alpha_2}}}(Y)$. □

Proposition 3.4. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. Lis the limited tolerance relation, $0.5 < \alpha \leq 1$, $\forall X, Y \subseteq U$. Then the following properties hold:*

(1) $\underline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(X) \subseteq X \subseteq \overline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(X)$.

(2) $\underline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(\emptyset) = \overline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(\emptyset) = \emptyset$.

(3) $\underline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(U) = \overline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(U) = U$.

(4) *If $X \subseteq Y$, then $\overline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(X) \subseteq \overline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(Y)$ and $\underline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(X) \subseteq \underline{\Sigma^m_{i=1}A^P_{iL_\alpha}}(Y)$.*

(5) *If $\alpha_1 \geq \alpha_2$, then$\overline{\Sigma^m_{i=1}A^P_{iL_{\alpha_1}}}(X) \subseteq \overline{\Sigma^m_{i=1}A^P_{iL_{\alpha_2}}}(Y)$ and $\underline{\Sigma^m_{i=1}A^P_{iL_{\alpha_1}}}(X) \subseteq \underline{\Sigma^m_{i=1}A^P_{iL_{\alpha_2}}}(Y)$.*

Proof. The proofs of these terms are similar to Proposition 3.3. □

4. **The relationship between the proposed rough set and others**

From Definition 3.3 and Definition 3.4 we can see that, if the disjunctive condition is replaced with the conjunctive condition, the lower approximation of optimistic multi-granulation variable precision limited tolerance rough set will be converted to the lower approximation of pessimistic multi-granulation variable precision limited tolerance rough set. Conversely, if the conjunctive condition is replaced with the disjunctive condition, the upper approximation of optimistic multi-granulation variable precision limited tolerance rough set will be converted to the upper approximation of pessimistic multi-granulation variable precision limited tolerance rough set.

Theorem 4.1. *Let $S = (U, TA, V, f)$ be an IIS,$A_1, A_2, \cdots, A_m \subseteq AT$. Lis the limited tolerance relation, $0.5 < \alpha \leq 1$. Then $\forall X \subseteq U$, we have*

(1) $\underline{\Sigma^m_{i=1}A^p_{iL_\alpha}}(X) \subseteq \underline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X)$.

(2) $\overline{\Sigma^m_{i=1}A^o_{iL_\alpha}}(X) \subseteq \overline{\Sigma^m_{i=1}A^p_{iL_\alpha}}(X)$.

(3) $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$.
(4) $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X)$.

Proof. (1) $\forall x \in \underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X)$, from Definition 3.4, $\forall A_i \in \{A_1, \cdots, A_m\}$, with $P(X|L_{A_i}(X)) \geq \alpha$, according to Definition 3.3, we can say $x \in \underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$, hence, $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$.

(2) Similar to (1), we can easily to prove the conclusion.

(3) $\forall x \in \underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X)$, from Definition 3.4, $\forall A_i \in \{A_1, \cdots, A_m\}$, with $P(X|L_{A_i}(X)) \geq \alpha$, Since $0.5 < \alpha \leq 1$, then $P(X|L_{A_i}(X)) \geq 1 - \alpha$. Thus $x \in \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$. Therefore, $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X)$.

(4) Similar to (3), we can easily to prove the conclusion. □

From Theorem 4.1, we can easily produce the following corollary.

Corollary 4.1. $\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X)$.

Theorem 4.2. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is the limited tolerance relation, $0.5 < \alpha \leq 1$. Then $\forall X \subseteq U$, we have*
(1) $\underline{\Sigma_{i=1}^m A_i^o}(X) \subseteq \underline{\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}}(X)$.
(2) $\overline{\Sigma_{i=1}^m A_i^o}(X) \subseteq \overline{\overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}}(X)$.
(3) $\underline{\Sigma_{i=1}^m A_i^p}(X) \subseteq \underline{\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}}(X)$.
(4) $\overline{\Sigma_{i=1}^m A_i^p}(X) \subseteq \overline{\overline{\Sigma_{i=1}^m A_{iL_\alpha}^p}}(X)$.

Proof. The proofs of these terms are similar to Theorem 4.1. □

Let $S = (U, TA, V, f)$be an IIS,$A_1, A_2, \cdots, A_m \subseteq AT$. $L$is the limited tolerance relation, for any $X \subseteq U$, then when $\alpha = 1$, for the optimistic multi-granulation variable precision limited tolerance rough set, we have

$$\begin{aligned}
&\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) = \{x \in U : P(X|L_{A_1}(X)) \geq \alpha \vee \ldots \vee P(X|L_{A_m}(X)) \geq \alpha\} \\
&\Leftrightarrow \{x \in U : P(X|L_{A_1}(X)) \geq 1 \vee \ldots \vee P(X|L_{A_m}(X)) \geq 1\} \\
&\Leftrightarrow \{x \in U : L_{A_1}(X) \subseteq X \vee \ldots \vee L_{A_m}(X) \subseteq X\} = \underline{\Sigma_{i=1}^m A_{iL}^o}(X)
\end{aligned}$$

and

$$\overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(\tilde{}X) \Leftrightarrow \overline{\Sigma_{i=1}^m A_{iL}^o}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL}^o}(\tilde{}X).$$

For the pessimistic multi-granulation variable precision limited tolerance rough set, we also have

$$\begin{aligned}
&\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) = \{x \in U : P(X|L_{A_1}(X)) \geq \alpha \wedge \ldots \wedge P(X|L_{A_m}(X)) \geq \alpha\} \\
&\Leftrightarrow \{x \in U : P(X|L_{A_1}(X)) \geq 1 \wedge \ldots \wedge P(X|L_{A_m}(X)) \geq 1\} \\
&\Leftrightarrow \{x \in U : L_{A_1}(X) \subseteq X \wedge \ldots \wedge L_{A_m}(X) \subseteq X\} = \underline{\Sigma_{i=1}^m A_{iT}^p}(X)
\end{aligned}$$

and

$$\overline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(\tilde{}X) \Leftrightarrow \overline{\Sigma_{i=1}^m A_{iL}^p}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL}^p}(\tilde{}X).$$

It can be seen that, when $\alpha = 1$, the multi-granulation variable precision limited tolerance rough set will be converted into the multi-granularity limited tolerance rough set

$$\begin{aligned}
&\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) = \{x \in U : P(X|L_{A_1}(X)) \geq \alpha \vee \ldots \vee P(X|L_{A_m}(X)) \geq \alpha\} \\
&\Leftrightarrow \{x \in U : P(X|L_{A_1}(X)) \geq 1 - \beta \vee \ldots \vee P(X|L_{A_m}(X)) \geq 1 - \beta\} = \underline{\Sigma_{i=1}^m A_{iL_\beta}^o}(X)
\end{aligned}$$

and

$$\overline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{iL_\alpha}^o}(\tilde{}X) \Leftrightarrow \overline{\Sigma_{i=1}^m A_{i_{L_\beta}}^o}(X) = \tilde{}\underline{\Sigma_{i=1}^m A_{i_{L_\beta}}^o}(\tilde{}X).$$

$$\underline{\Sigma_{i=1}^m A_{iL_\alpha}^P}(X) = \{x \in U : P(X|L_{A_1}(X)) \geq \alpha \wedge \ldots \wedge P(X|L_{A_m}(X)) \geq \alpha\}$$
$$\Leftrightarrow \{x \in U : P(X|L_{A_1}(X)) \geq 1-\beta \wedge \ldots \wedge P(X|L_{A_m}(X)) \geq 1-\beta\} = \underline{\Sigma_{i=1}^m A_{i_\beta}^P}(X)$$

and

$$\overline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(X) = \tilde{}\,\underline{\Sigma_{i=1}^m A_{iL_\alpha}^p}(\tilde{}X) \Leftrightarrow \overline{\Sigma_{i=1}^m A_{i_{L\beta}}^p}(X) = \tilde{}\,\underline{\Sigma_{i=1}^m A_{i_{L\beta}}^p}(\tilde{}X).$$

It can be seen that, when $\beta = 1-\alpha$, the multi-granulation variable precision limited tolerance rough set will be converted into the multi-granularity variable precision probability limited tolerance rough set.

5. Measurements for the proposed rough set

In this section, we will investigate several elementary measures in multi-granulation limited tolerance rough set and their properties. Uncertainty of a set (category) is due to the existence of a borderline region. The greater the borderline region of set, the lower is the accuracy of the set. In order to express this idea more precisely, similar to reference [15, 24], we introduce the accuracy measures to the multi-granulation limited tolerance rough set.

Definition 5.1. *Let $S^* = (U, TA, V_*, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. Lis the tolerance relation, $0.5 < \alpha \leq 1$, $\forall X \subseteq U$ $X \neq \emptyset$. The accuracy measures of X by $\Sigma_{i=1}^m A_{iL}(X)$, $\Sigma_{i=1}^m A_{iL_\alpha}(X)$ are respectively defined as*

$$\alpha^o(\Sigma_{i=1}^m A_i, X) = \frac{\left|\underline{\Sigma_{i=1}^m A_{iL}^o}(X)\right|}{\left|\overline{\Sigma_{i=1}^m A_{iL}^o}(X)\right|}.$$

$$\alpha^p(\Sigma_{i=1}^m A_i, X) = \frac{\left|\underline{\Sigma_{i=1}^m A_{iL}^p}(X)\right|}{\left|\overline{\Sigma_{i=1}^m A_{iL}^p}(X)\right|}.$$

$$\alpha_\alpha^o(\Sigma_{i=1}^m A_i, X) = \frac{\left|\underline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)\right|}{\left|\overline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)\right|}.$$

$$\alpha_\alpha^p(\Sigma_{i=1}^m A_i, X) = \frac{\left|\underline{\Sigma_{i=1}^m A_{iL\alpha}^p}(X)\right|}{\left|\overline{\Sigma_{i=1}^m A_{iL\alpha}^p}(X)\right|}.$$

Theorem 5.1. *Let $S = (U, TA, V, f)$ be an IIS, $A_1, A_2, \cdots, A_m \subseteq AT$. L is the limited tolerance relation,$0.5 < \alpha \leq 1$. Then $\forall X \subseteq U$, we have*

$$\alpha^o(\Sigma_{i=1}^m A_i, X) \leq \alpha_\alpha^o(\Sigma_{i=1}^m A_i, X)$$

and

$$\alpha^p(\Sigma_{i=1}^m A_i, X) \leq \alpha_\alpha^p(\Sigma_{i=1}^m A_i, X).$$

Proof. From previous definitions and $0.5 < \alpha \leq 1$, we know that $\underline{\Sigma_{i=1}^m A_{iL}^o}(X) \subseteq \underline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)$ and $\overline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X) \subseteq \overline{\Sigma_{i=1}^m A_{iL}^o}(X)$, hence

$$\left|\underline{\Sigma_{i=1}^m A_{iL}^o}(X)\right| \leq \left|\underline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)\right|$$

and

$$\left|\overline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)\right| \leq \left|\overline{\Sigma_{i=1}^m A_{iL}^o}(X)\right|$$

hold.

Thus,

$$\frac{\left|\underline{\Sigma_{i=1}^m A_{iL}^o}(X)\right|}{\left|\overline{\Sigma_{i=1}^m A_{iL}^o}(X)\right|} \leq \frac{\left|\underline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)\right|}{\left|\overline{\Sigma_{i=1}^m A_{iL\alpha}^o}(X)\right|}.$$

Therefore,

$$\alpha^{o}(\Sigma_{i=1}^{m}A_i, X) \leq \alpha_{\alpha}^{o}(\Sigma_{i=1}^{m}A_i, X).$$

Similarly, we can easily to prove the conclusion $\alpha^{p}(\Sigma_{i=1}^{m}A_i, X) \leq \alpha_{\alpha}^{p}(\Sigma_{i=1}^{m}A_i, X)$. □

6. Conclusions

In this paper, we propose a combined rough set model of the limited tolerance relation and the variable precision rough set under multi-granulation. The proposed rough set not only inherits the merits of the limited tolerance relation and the variable precision rough set, but also provides a more detailed multi-granulation structure of object class through threshold. Theoretical proofs show the rationality and superiority of the proposed model.

Funding

This work was supported in part by National Natural Science Foundation of China (61662001, 61762002), Young and Middle-aged Talents Training Program of National Ethnic Affair Commission (2016GQR06), Ningxia First-class Construction Discipline Program (NXYLXK2017B09), Open Foundation of Ningxia Key Laboratory of Intelligent Information and Big Data Processing(2019KLBD006).

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