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A SERIES METHOD APPLIED TO ENGINEERING CALCULATIONS IN STRUCTURAL DYNAMICS

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This paper shows an application of the Φ-functions series method to calculate the response of structures in face of an earthquake, modelled by a 2DOF.

The Φ-functions series method is an adaptation of the ideas of Scheifele to integrate forced and damped oscillators. This algorithm presents the advantage of integrating precisely the perturbed problem with only two Φ-functions. Method coefficients are calculated by simple algebraic recurrences in which the perturbation function is involved.

Results show the good precision compared to those obtained by other well-known integrators implemented in MAPLE. Results are also contrasted with classic methods of Structural Engineering.

Keywords: Numerical solutions, series methods, structural dynamics, earthquake response.

1. Introduction

Modern computational techniques, in particular those including sophisticated matrix structural analysis methods and numerical analysis can be applied, in a practical and efficient manner, to calculations relating to the theory of structural dynamics, such as building and civil engineering. Particularly, it is very interesting to know the response of a structure under the effect of an earthquake. This motivates the research in the design of computer algorithms.

A resistant structure that is resting and is subjected to an external force undergoes oscillations. These oscillations can be modeled by matrices through a system of equations with *n* degrees of freedom, MDOF:

$$M\ddot{\boldsymbol{x}}(t)+C\dot{\boldsymbol{x}}(t)+K\boldsymbol{x}(t)=-M\boldsymbol{a}(t),\quad \boldsymbol{x}(0)=\boldsymbol{x}_0=\underline{\boldsymbol{0}},\quad \dot{\boldsymbol{x}}(0)=\dot{\boldsymbol{x}}_0=\underline{\boldsymbol{0}}, \tag{1}$$

where *M, C* and *K* are the mass, damping and stiffness matrices, respectively. The column vector $\boldsymbol{a}(t)$ contains the acceleration values.

In most cases, only horizontal translation of the earthquake ground motion is considered by structural engineers.

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There are different numerical algorithms specially adapted for the integration of this type of PVI's, highlighting among them the Scheifele's method [1,2] , which presents the good property of integrating, without truncation error and accurately, the non-perturbed problem.

Refinements of Scheifele`s method for the integration of this type of oscillators have been designed following different ideas and motivations [3-8].

Based on the Scheifele`s method, the authors have obtained an algorithm for integration damped and forced systems [9-12], which maintains the accuracy and the good properties of previous method.

The basic idea of this code is remove the function of perturbation, integrating exactly the non-perturbed problem. To achieve this the differential operator *D*+*B* is applied to the system (1), where *B* is an element of real matrices of order *m*, obtaining a system of second-order unperturbed with the same solution.

The method consists of defining a sequence of $\{\Phi_j\}_{j\in\mathbb{N}}$ matrices which serves as a basis for constructing the solution as a linear combination thereof. Said solution shall be used for obtaining a numerical integration method, the Φ-functions series method [12].

This has an advantage over the Scheifele`s method in that it exactly integrates the perturbed problem with only the two first terms of the series of Φ-functions. The coefficients of this series are obtained through recurrence relations involving the perturbation function.

In this paper is shown an application of the Φ-functions series method to calculate the response of structures modelled as 2DOF system to an earthquake.

The precision and efficiency of the Φ-function series method is contrasted with the results obtained by other well-known integrators.

2. Notation and preliminary ideas

In this section it focus the attention to numerical integration of equations of the form:

$$\boldsymbol{x}' + A\boldsymbol{x} = \varepsilon \boldsymbol{f}(\boldsymbol{x}(t),t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ t \in [a,b] = I, \qquad (2)$$

where, $\boldsymbol{x}:\mathbb{R}\to\mathbb{R}^m$, $A\in\mathcal{M}(m,\mathbb{R})$, ε being a small parameter of perturbation and $\boldsymbol{f}:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^m$. The components of the vector perturbation field $\boldsymbol{f}(\boldsymbol{x}(t),t)$ are $f_i(\boldsymbol{x}(t),t)$ with $i=1\ldots m$ and the field is continuous, with continuous derivatives until a certain order that satisfies the conditions for existence and uniqueness of solution. This type of system is called a perturbed linear system.

Assuming that $\boldsymbol{g}(t) = \boldsymbol{f}(\boldsymbol{x}(t),t)$ is analytical in *I* with regard to *t*, where it is sufficient that $\boldsymbol{f}$ is analytical in its arguments. In terms of the linear operator derivation D, with respect to the variable t, (2) can be written as follows:

$$(D+A)\boldsymbol{x} = \varepsilon \boldsymbol{g}(t), \boldsymbol{x}(0) = \boldsymbol{x}_0, t \in [0,T] = I, \tag{3}$$

for which it is supposed that $\boldsymbol{x}(t)$ will be the only solution, in *I*, which can be developed in a power series.

Applying the operator $(D+B)$ to (3), where $B \in \mathcal{M}(m,\mathbb{R})$, and noting $L_2 = D^2 + (A+B)D + BA$, the new IVP is obtained:

$$L_2(\boldsymbol{x}) = (D+B)\varepsilon \boldsymbol{g}(t), \boldsymbol{x}(0) = \boldsymbol{x}_0, \boldsymbol{x}'(0) = -A\boldsymbol{x}_0 + \varepsilon \boldsymbol{g}(0) = \boldsymbol{x}_0', \tag{4}$$

whose exact solution $\boldsymbol{x}(t)$ is the same as that of (2) and (3).

The idea that leads us to consider this "enlarged" IVP, is that of cancelling the perturbation with the operator $(D+B)$.

Given that $\boldsymbol{g}(t)$ is analytical in its arguments, we can write:

$$\boldsymbol{g}(t) = \boldsymbol{f}(\boldsymbol{x}(t),t) = \sum_{n=0}^{\infty} \frac{\boldsymbol{g}^{n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \boldsymbol{c}_n, \tag{5}$$

with:

$$L_2(\boldsymbol{x}) = \varepsilon \sum_{n=1}^{\infty} \frac{t^n}{n!} \left(\boldsymbol{c}_{n+1} + B\boldsymbol{c}_n\right), \boldsymbol{x}(0) = \boldsymbol{x}_0, \boldsymbol{x}'(0) = -A\boldsymbol{x}_0 + \varepsilon \boldsymbol{g}(0) = \boldsymbol{x}_0'. \tag{6}$$

The solution of the IVP (4) is obtained by adding a specific unperturbed IVP solution with null initial conditions to the general solution of the perturbed IVP with given initial conditions. The former can be obtained by resolving the following specific IVPs:

$$X_j'' + (A+B)X_j' + BAX_j = \frac{t^j}{j!} I_m, \ X_j(0) = \underline{0}, \ X_j'(0) = \underline{0}, j \geq 0, \tag{7}$$

where X_j is a real function with values in the ring $\mathcal{M}(m,\mathbb{R})$ of the squared matrices of order *m*, with I_m and $\underline{0}$ being, respectively, the unit and neutral elements of said ring.

The solutions of (7) are the so-called Φ-functions [12].

2.1 The Φ-functions

Definition 1

$$\Phi_{j+2}(t) = X_j(t) \text{ with } j \geq 0, j \in \mathbb{N}. \tag{8}$$

Proposition 1 (Law of derivation) The Φ-functions verify:

$$\Phi'_j(t) = \Phi_{j-1}(t) \text{ with } j \geq 3, j \in \mathbb{N} \tag{9}$$

Proposition 2 (Law of recurrence) The Φ-functions verify the following recurrence law:

$$\Phi_{j-2}(t) + (A+B)\Phi_{j-1}(t) + BA\Phi_j(t) = \frac{t^{j-2}}{(j-2)!} I_m \text{ with } j \geq 4, j \in \mathbb{N}. \tag{10}$$

In order to complete the construction of the Φ-functions, given in (8), are defined $\Phi_0(t)$ and $\Phi_1(t)$.

Definition 2

$\Phi_0(t)$ and $\Phi_1(t)$, are respectively, the solutions of the following IVP:

$$X''(t) + (A+B)X'(t) + BAX(t) = \underline{0}, X(0) = I_m, X'(0) = \underline{0}, \tag{11}$$

$$X''(t) + (A+B)X'(t) + BAX(t) = \underline{0}, X(0) = \underline{0}, X'(0) = I_m. \tag{12}$$

The law of derivation presented in Proposition 1, is completed by the proposition below.

Proposition 3

$$\Phi'_2(t) = \Phi_1(t). \tag{13}$$

Theorem 1

The solution of the IVP $L_2(\mathbf{x}) = \underline{0}, \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}'(0) = -A\mathbf{x}_0 + \varepsilon \mathbf{g}(0) = \mathbf{x}'_0$ is

$$\Phi_0(t)\mathbf{x}_0 + \Phi_1(t)\mathbf{x}'_0. \tag{14}$$

Theorem 2

The solution of the IVP (4), in terms of Φ -functions, is given by:

$$\mathbf{x}(t) = \Phi_0(t)\mathbf{x}_0 + \Phi_1(t)\mathbf{x}'_0 + \varepsilon \sum_{n=2}^{\infty} \Phi_n(t)\left(\mathbf{c}_{n-1} + B\mathbf{c}_{n-2}\right). \tag{15}$$

2.2 Φ-functions Series Method

It is assumed that the IVP (2) the perturbation function $\mathbf{g}(t) = \mathbf{f}(\mathbf{x}(t), t)$ admits an absolutely convergent power series expansion in $[0, T]$: $\mathbf{g}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{c}_k$. The solution can be expressed as $\mathbf{x}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{a}_k$.

By substituting these developments in (2), recurrent formulae are established between the coefficients $\mathbf{a}_k$ and the coefficient $\mathbf{c}_k$, based on the initial condition. We assume that we have calculated an approximation to the solution and to its derivative at a point $t = nh$, which are noted as $\mathbf{x}_n$ and $\mathbf{x}'_n$, the approximation to the solution [12] at a point $t = (n+1)h$ is:

$$\boldsymbol{x}_{n+1} = \Phi_0(h)\boldsymbol{b}_0 + \Phi_1(h)\boldsymbol{b}_1 + \sum_{n=0}^{m-2} \Phi_{n+2}(h)\boldsymbol{b}_{n+2} \ , \tag{16}$$

where:

$$\begin{aligned} &\boldsymbol{a}_0 = \boldsymbol{x}_n, \boldsymbol{a}_{k+1} + A\boldsymbol{a}_k = \varepsilon \boldsymbol{c}_k, \quad \text{with } k \ge 0, \\ &\boldsymbol{b}_0 = \boldsymbol{a}_0, \boldsymbol{b}_1 = \boldsymbol{a}_1, \boldsymbol{b}_k = \boldsymbol{a}_k + (A+B)\boldsymbol{a}_{k-1} + BA\boldsymbol{a}_{k-2}, \quad \text{with } k \ge 2, \end{aligned} \tag{17}$$

which is the numerical integration method, based on Φ-functions series, for linear perturbed systems.

Proposition 4 (Truncation error)

Carrying out a truncation of $m+1$ Φ-functions, with $m \ge 2$ $\boldsymbol{x}_m(t) = \Phi_0(t)\boldsymbol{x}_0 + \Phi_1(t)\boldsymbol{x}'_0 + \varepsilon \sum_{n=0}^{m-2} \Phi_{n+2}(t)\left(\boldsymbol{c}_{n+1} + B\boldsymbol{c}_n\right)$, the truncation error corresponding to $\boldsymbol{x}_m(t)$, shall be given by:

$$E_m(t) = \varepsilon \sum_{n=m-1}^{\infty} \frac{t^n}{n!}\left(\boldsymbol{c}_{n+1} + B\boldsymbol{c}_n\right). \tag{18}$$

As a result the truncation error is small with ε. If $\varepsilon = 0$, that is, if the perturbation disappear in an arbitrary instant of the independent variable t, the Φ-functions integrates without discretisation error (4) [12].

3. Resolution of 2DOF by Φ-functions series method

The 2DOF system is represented in Fig. 1 and it is used to study the dynamic forces acting on this system. Four types of forces act on each floor mass, the stiffness force, the damping force, the external force and inertial force [13].

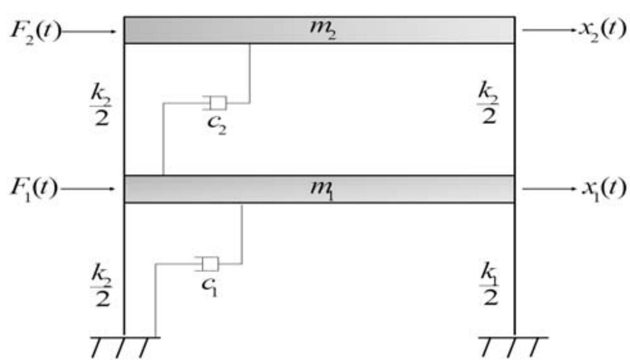

Fig. 1 Two Degrees of Freedom System (2DOF)

The dynamic equilibrium equations of motion are:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}+\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix}, \quad (19)$$

defining: $M=\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$, $C=\begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}$, $K=\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$,

$\boldsymbol{x}(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $F(t)=\begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix}$, when the symmetrical and positive definite matrices *M*, *C* and *K*, are the mass, damping and stiffness matrix, respectively; the system (19) can be expressed by: $M\ddot{\boldsymbol{x}}(t)+C\dot{\boldsymbol{x}}(t)+K\boldsymbol{x}(t)=F(t)$.

Considering that the structure is subjected to an earthquake ground motion, where only horizontal translation of the earthquake ground motion is considered. Applying the Newton´s second law and given that the external force is zero, are:

$$\begin{aligned} &m_1\ddot{y}_1+c_2\left(\dot{y}_1-\dot{y}_2\right)+c_1\left(\dot{y}_1-\dot{u}_g\right)+k_2\left(y_1-y_2\right)+k_1\left(y_1-u_g\right)=0,\\ &m_2\ddot{y}_2+c_2\left(\dot{y}_2-\dot{y}_1\right)+k_2\left(y_2-y_1\right)=0, \end{aligned} \quad (20)$$

where u_g and $\dot{u}_g$ are the absolute ground displacement and the absolute ground velocity, respectively and, y_1, y_2 are the absolute displacements of the masses respectively.

We define $x_1(t)=y_1(t)-u_g(t)$ and $x_2(t)=y_2(t)-u_g(t)$, as relative displacement between the mass and the ground. In this manner the equations (20) are:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}+\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=-\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\ddot{u}_g . \quad (21)$$

If $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\ddot{u}_g$ is a harmonic matrix forcing function the equation (21) is:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}+\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=-\begin{pmatrix} F_0\sin(\omega_0 t) \\ F_0\sin(\omega_0 t) \end{pmatrix}, \quad (22)$$

at the moment that the earthquake occurs, it is very reasonable to assume that the structure is at rest.

Therefore or normalized form, the IVP is:

$$\begin{aligned} &\ddot{\boldsymbol{x}}(t)+M^{-1}C\dot{\boldsymbol{x}}(t)+M^{-1}K\boldsymbol{x}(t)=-\begin{pmatrix} \dfrac{F_0\sin(\omega_0 t)}{m_1} & \dfrac{F_0\sin(\omega_0 t)}{m_2} \end{pmatrix}^t=-\begin{pmatrix} F_1(t) & F_2(t) \end{pmatrix}^t,\\ &\boldsymbol{x}(0)=\boldsymbol{x}_0=\begin{pmatrix} 0 & 0 \end{pmatrix}^t, \dot{\boldsymbol{x}}(0)=\dot{\boldsymbol{x}}_0=\begin{pmatrix} 0 & 0 \end{pmatrix}^t. \end{aligned} \quad (23)$$

In order to apply the Φ-function series method, is effected the change of variable: $x_1 = u_1$, $\dot{x}_1 = u_3$, $\ddot{x}_1 = \dot{u}_3$ and $x_2 = u_2$, $\dot{x}_2 = u_4$, $\ddot{x}_2 = \dot{u}_4$.

The IVP (23) can be expressed as:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{pmatrix} + \begin{pmatrix} O_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & M^{-1}C \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \begin{pmatrix} O_{2\times 2} & I_{2\times 2} \\ M^{-1}K & O_{2\times 2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ F_0 \sin(\omega_0 t)/m_1 \\ F_0 \sin(\omega_0 t)/m_2 \end{pmatrix} \quad (24)$$

with $\begin{pmatrix} u_1(0) & u_2(0) & u_3(0) & u_4(0) \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^t$.

Consequently

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{pmatrix} + \begin{pmatrix} O_{2\times 2} & I_{2\times 2} \\ M^{-1}K & M^{-1}C \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ F_0 \sin(\omega_0 t)/m_1 \\ F_0 \sin(\omega_0 t)/m_2 \end{pmatrix}, \text{with } \boldsymbol{u}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (25)$$

The variable is introduced in order to make easier the elimination the disturbance's function of the IVP (25), following the Steffensen´s techniques [14,15].

$u_5 = -\frac{F_0}{m_1}\sin(\omega_0 t)$, obtaining a new IVP.

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \\ \dot{u}_5(t) \end{pmatrix} + \begin{pmatrix} O_{2\times 2} & I_{2\times 2} & O_{2\times 1} \\ M^{-1}K & M^{-1}C & O_{2\times 1} \\ O_{1\times 2} & O_{1\times 2} & O_{1\times 1} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ u_5(t) \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ F_0 \sin(\omega_0 t)/m_1 \\ F_0 \sin(\omega_0 t)/m_2 \\ F_0 \omega_0 \cos(\omega_0 t)/m_1 \end{pmatrix}, \text{with } \boldsymbol{u}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (26)$$

To invalidate the function of disturbance, the differential operator $(D+B)$ is applied to (26), where B is the following matrix

$$B = \begin{pmatrix} O_{2\times 2} & O_{2\times 2} & O_{2\times 1} \\ O_{2\times 2} & O_{2\times 2} & \bar{\Omega}_{2\times 1} \\ O_{1\times 2} & \Omega_{1\times 2} & O_{1\times 1} \end{pmatrix} \text{ with } \bar{\Omega}_{2\times 1} = \begin{pmatrix} -1 \\ -\frac{m_1}{m_2} \end{pmatrix} \text{ and } \Omega_{1\times 2} = \begin{pmatrix} \omega_0^2 & 0 \end{pmatrix}, \quad (27)$$

obtaining the extended IVP

$$\begin{pmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \\ \ddot{u}_5(t) \end{pmatrix} + \begin{pmatrix} O_{2\times2} & I_{2\times2} & O_{2\times1} \\ M^{-1}K & M^{-1}C & \bar{\Omega}_{2\times1} \\ O_{1\times2} & \Omega_{1\times2} & O_{1\times1} \end{pmatrix} \begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \\ \dot{u}_5(t) \end{pmatrix} + \begin{pmatrix} O_{2\times2} & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & O_{2\times2} & O_{2\times1} \\ \Omega_{1\times2}M^{-1}K & \Omega_{1\times2}M^{-1}C & O_{1\times1} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ u_5(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (28)$$

$$\boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t, \dot{\boldsymbol{u}}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{F_0}{m_1}\omega_0 \end{pmatrix}^t,$$

which is integrated exactly using the Φ-functions' series algorithm described in (16).

3.1 Numerical results

In this section the Φ-functions series method is applied to calculate the solution of 2DOF.

The good behavior of the method is shown compared with other known codes, implemented in the DSOLVE NUMERIC package in the MAPLE program, such as ROSENBROCK, GEAR, and TAYLORSERIES. Moreover, it has been programmed other codes like NEWMARK β-METHOD and WILSON θ-METHOD to compare them with the Φ-functions series method.

Consider the two-story frame subjected to an earthquake ground motion [13], Fig. 2.

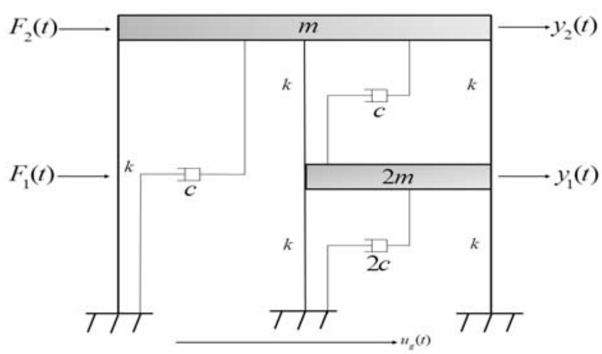

Fig. 2 Two-Story Frame

The dynamic equilibrium equation of motion is:

$$\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 3c & -c \\ -c & 2c \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} 4k & -2k \\ -2k & 3k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ddot{u}_g(t), \quad (29)$$

If $\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\ddot{u}_g(t)$ is a harmonic matrix forcing function, i.e. $\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\ddot{u}_g(t) = \begin{pmatrix} F_0\sin(\omega_0 t) \\ F_0\sin(\omega_0 t) \end{pmatrix}$ then the equation (29) is:

$$\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 3c & -c \\ -c & 2c \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} 4k & -2k \\ -2k & 3k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} F_0\sin(\omega_0 t) \\ F_0\sin(\omega_0 t) \end{pmatrix}. \quad (30)$$

In notation more compact and normalizing the equation (30), is obtained:

$$\ddot{\boldsymbol{x}}(t) + M^{-1}C\dot{\boldsymbol{x}}(t) + M^{-1}K\boldsymbol{x}(t) = -\begin{pmatrix} \dfrac{F_0\sin(\omega_0 t)}{2m} & \dfrac{F_0\sin(\omega_0 t)}{m} \end{pmatrix}^t \quad (31)$$

at the moment that the earthquake occurs, it is very reasonable to assume that the structure is at rest.

To solve the IVP:

$$\ddot{\boldsymbol{x}}(t) + M^{-1}C\dot{\boldsymbol{x}}(t) + M^{-1}K\boldsymbol{x}(t) = -\begin{pmatrix} \dfrac{F_0\sin(\omega_0 t)}{2m} & \dfrac{F_0\sin(\omega_0 t)}{m} \end{pmatrix}^t, \quad (32)$$

with $\boldsymbol{x}(0) = 0$, $\dot{\boldsymbol{x}}(0) = 0$, $t \in [0, T]$, using the methodology of the Φ-functions, the new expression for the IVP

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \end{pmatrix} + \begin{pmatrix} O_{2\times 2} & I_{2\times 2} \\ M^{-1}K & M^{-1}C \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix} = -\begin{pmatrix} 0 & 0 & \dfrac{F_0}{2m}\sin(\omega_0 t) & \dfrac{F_0}{m}\sin(\omega_0 t) \end{pmatrix}^t, \quad (33)$$

with $\boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^t$, $M^{-1}K = \dfrac{k}{m}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ and $M^{-1}C = \dfrac{c}{2m}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$.

The variable is introduced in order to make easier the elimination the disturbance's function of the IVP (33), following the Steffensen´s techniques [14,15].

$u_5 = -\dfrac{F_0}{2m}\sin(\omega_0 t)$, obtaining a new IVP.

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \\ \dot{u}_5(t) \end{pmatrix} + \begin{pmatrix} O_{2\times2} & I_{2\times2} & O_{2\times1} \\ M^{-1}K & M^{-1}C & O_{2\times1} \\ O_{1\times2} & O_{1\times2} & O_{1\times1} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ u_5(t) \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ \dfrac{F_0}{2m}\sin(\omega_0 t) \\ \dfrac{F_0}{m}\sin(\omega_0 t) \\ \dfrac{F_0}{2m}\omega_0\cos(\omega_0 t) \end{pmatrix}, \tag{34}$$

with $\boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$.

To invalidate the function of disturbance, the differential operator $(D+B)$ is applied to (34), where B is the following matrix:

$$B = \begin{pmatrix} O_{2\times2} & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & O_{2\times2} & \bar{\Omega}_{2\times1} \\ O_{1\times2} & \Omega_{1\times2} & O_{1\times1} \end{pmatrix} \text{ with } \bar{\Omega}_{2\times1} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \text{ and } \Omega_{1\times2} = \begin{pmatrix} \omega_0^2 & 0 \end{pmatrix}. \tag{35}$$

Choosing the following values for the structural variables [13] $m = 1.5\ \dfrac{k\cdot s^2}{in.}$, $\zeta = 5\%$, $F_0 = 15\ kip$, $\omega_0 = \omega_n = \dfrac{3\pi}{2}\ \dfrac{rad}{s}$.

The IVP is:

$$\begin{pmatrix} 3 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} \dfrac{3\pi}{10} & -\dfrac{\pi}{5} \\ -\dfrac{\pi}{5} & \dfrac{2\pi}{5} \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} \dfrac{27\pi^2}{2} & -\dfrac{27\pi^2}{4} \\ -\dfrac{27\pi^2}{4} & \dfrac{81\pi^2}{8} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} 15\sin\left(\dfrac{3\pi}{2}t\right) \\ 15\sin\left(\dfrac{3\pi}{2}t\right) \end{pmatrix}, \tag{36}$$

with $\begin{pmatrix} x_1(0) & x_2(0) \end{pmatrix}^t = \begin{pmatrix} 0 & 0 \end{pmatrix}^t$, making the change of variable $x_1 = u_1$, $\dot{x}_1 = u_3$, $\ddot{x}_1 = \dot{u}_3$ and $x_2 = u_2$, $\dot{x}_2 = u_4$, $\ddot{x}_2 = \dot{u}_4$, the new IVP is:

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \\ \dot{u}_5(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \frac{9\pi^2}{2} & -\frac{9\pi^2}{4} & \frac{9\pi}{40} & \frac{-3\pi}{40} & 0 \\ -\frac{9\pi^2}{2} & \frac{27\pi^2}{4} & \frac{-3\pi}{20} & \frac{3\pi}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ u_5(t) \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ 5\sin\left(\frac{3\pi}{2}t\right) \\ 10\sin\left(\frac{3\pi}{2}t\right) \\ \frac{15\pi}{2}\cos\left(\frac{3\pi}{2}t\right) \end{pmatrix} \tag{37}$$

with $\boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$.

Applying the operator $(D+B)$ to the system (37) we obtain the extended IVP:

$$\begin{pmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \\ \ddot{u}_5(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{9\pi^2}{2} & -\frac{9\pi^2}{4} & \frac{9\pi}{40} & \frac{-3\pi}{40} & -1 \\ -\frac{9\pi^2}{2} & \frac{27\pi^2}{4} & \frac{-3\pi}{20} & \frac{3\pi}{10} & -2 \\ 0 & 0 & \frac{9\pi^2}{4} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \\ \dot{u}_5(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{81\pi^4}{8} & -\frac{81\pi^4}{16} & \frac{81\pi^3}{160} & -\frac{27\pi^3}{160} & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ u_5(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{38}$$

$$\text{with } \boldsymbol{u}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t, \dot{\boldsymbol{u}}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{15\pi}{2} \end{pmatrix}^t,$$

this is integrated exactly by the following algorithm, particularized for this problem.

$$\boldsymbol{a}_0 = \boldsymbol{u}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$$

$$\boldsymbol{a}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{9\pi^2}{2} & \frac{9\pi^2}{4} & -\frac{9\pi}{40} & \frac{3\pi}{40} & 0 \\ \frac{9\pi^2}{2} & -\frac{27\pi^2}{4} & \frac{3\pi}{20} & -\frac{3\pi}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{a}_0 - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{15\pi}{2} \end{pmatrix} \tag{39}$$

from $k = 1$ up to n calculates

$$\boldsymbol{u}_k = \Phi_0(h)\boldsymbol{a}_0 + \Phi_1(h)\boldsymbol{a}_1$$

$$\boldsymbol{a}_0 = \boldsymbol{u}_k$$

$$\boldsymbol{a}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{9\pi^2}{2} & \frac{9\pi^2}{4} & -\frac{9\pi}{40} & \frac{3\pi}{40} & 0 \\ \frac{9\pi^2}{2} & -\frac{27\pi^2}{4} & \frac{3\pi}{20} & -\frac{3\pi}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{u}_k - \begin{pmatrix} 0 \\ 0 \\ 5\sin\left(\frac{3\pi}{2}t\right) \\ 10\sin\left(\frac{3\pi}{2}t\right) \\ \frac{15\pi}{2}\cos\left(\frac{3\pi}{2}t\right) \end{pmatrix}$$

next k.

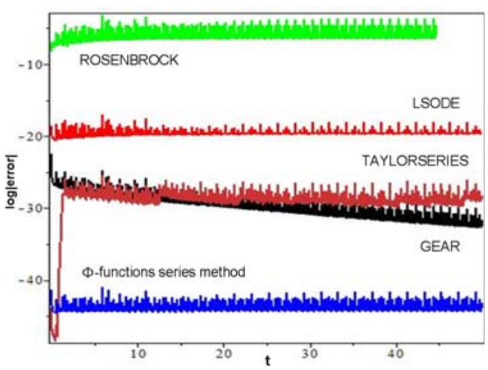

Fig. 3 The decimal logarithm of module of the relative error of the solution $\boldsymbol{u}(t)$.

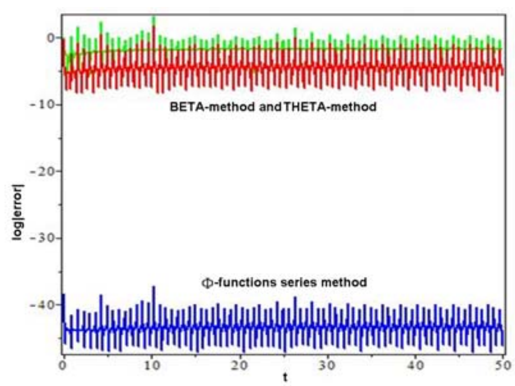

Fig. 4 The decimal logarithm of module of the relative error of the position $\boldsymbol{x}(t)$.

The Fig. 3 shows the graph of the decimal logarithm of module of the relative error of the solution $\boldsymbol{u}(t)$, vs. t, calculate using Φ-functions series method with two Φ-functions, step size $h = 0.01$ and 50 digits, with the numerical integration codes LSODE with $tol = 10^{-25}$, ROSENBROCK with *abserr* = 10^{-30}, GEAR with $errorper = 10^{-25}$ and TAYLORSERIES with *abserr* = 10^{-25}.

The Fig. 4 shows the logarithm graph for the absolute value of the relative error of solution $x(t)$, vs. t, obtained with 50 digits, calculated by means of (39), with two Φ-functions and step size $h = 0.001$, compared with the numerical integration codes NEWMARK β-METHOD with $\delta = 1/2$, $\alpha = 1/4$, $h = 0.001$ and WILSON θ-METHOD with $\delta = 1/2$, $\alpha = 1/6$, $\theta = 1.4$, $h = 0.001$.

Analogous results are obtained for velocity $\dot{x}(t)$.

4. Conclusions

In this paper we have applied a numerical integration algorithm based on series of Φ-functions, which generalises Scheifele´s original method.

The Φ-functions series method has an advantage over the Scheifele´s method in that it exactly integrates the perturbed problem, transforming it into second-order homogeneous problem which is able to integrate exactly with two first Φ-functions.

An application of the method has been developed for the analysis of an earthquake modeled by 2DOF. The accuracy in the resolution of a 2DOF through the Φ-functions series method could successfully compete with well-known integrators.

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