APPLICATION OF MAX MIN APPROACH AND AMPLITUDE FREQUENCY FORMULATION TO NONLINEAR OSCILLATION SYSTEMS

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This paper applied the Max-Min Approach (MMA) and Amplitude Frequency Formulation (AFF) to derive the approximate analytical solution for motion of nonlinear free vibration of conservative, single degree of freedom systems. In both nonlinear problems MMA and AFF yields the same results. In comparison to forth-order runge-kutta method which is powerful numerical solution, the results show that these methods are very convenient for solving nonlinear equations and also can be used for the wide range of time and boundary conditions for nonlinear oscillators.

Keywords: nonlinear oscillation, max min approach, amplitude frequency formulation, analytical solution.

1. Introduction

This study has clarified the motion equation of two oscillators by Max min approach and Amplitude frequency formulation to obtain the relationship between Amplitude and angular frequency. As it can be illustrated in Fig.1, m_1 is mass of the block on the horizontal surface, m_2 is the mass of block which is just slipped in the vertical and is linked to m_1, L is length of link, g is gravitational acceleration, and k is spring constant.

By assuming $u = \frac{x}{l}, |u| << 1$, the equation of motion can be yield as following terms:

$$\ddot{u} + \left(\frac{m_2}{m_1}\right)u^2\ddot{u} + \left(\frac{m_2}{m_1}\right)u\dot{u}^2 + \left(\frac{k}{m_1} + \frac{m_2 g}{l m_1}\right)u + \frac{m_2 g}{2 l m_1}u^3 = 0 \quad (1)$$

In which u and t are generalized dimensionless displacement and time variables.

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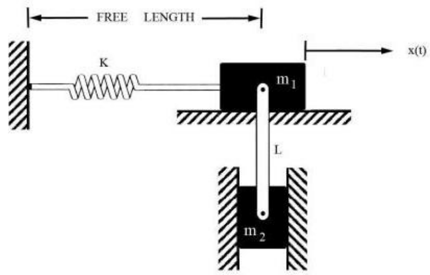

Fig. 1. Geometry of first problem

The second problem is derived from the motion of simple pendulum attached to a rotating rigid frame that is shown in fig.2 which has following nonlinear differential equation:

$$\ddot{\theta}+\left(1-\Lambda\cos\left(\theta\right)\right)\sin\left(\theta\right)=0 \quad (2)$$

In which θ and t are generalized dimensionless displacements and time variables, and $\Lambda=\frac{\Omega^2 r}{g}$.

As it can be considered both of the problem are strongly nonlinear. In recent years, many powerful methods such as Max Min Approach [1, 2], Amplitude Frequency Formulation [3-6], Harmonic Balance Method (HBM) [7], Homotopy Perturbation Method (HPM) [8-12], Variational Iteration Method (VIM) [13, 14], Parameter Expansion Method (PEM) [15] and Energy Balance Method (EBM) [16-19], are used to find approximate solution to the nonlinear differential equations

This study has investigated two nonlinear oscillators by Max min approach and Amplitude frequency formulations which are two methods come from Chinese mathematics. The comparison between approximate solutions and the forth-order runge kutta method assures us about accuracy and validity of solution.

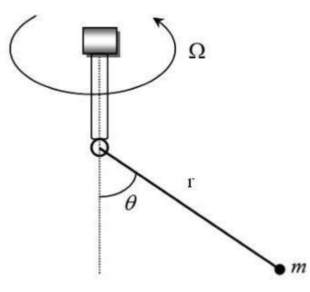

Fig. 2. Geometry of second problem

2. Application of MMA to the first problem:

In order to solve first problem with the max min approach, it can be rewritten as:

$$\ddot{u}+\left(\frac{m_2}{m_1}\right)u^2\ddot{u}+\left(\frac{m_2}{m_1}\right)u\dot{u}^2+\omega_0^2 u+\frac{m_2 g}{2lm_1}u^3=0, u(0)=A, \dot{u}(0)=0$$
$$\omega_0^2=\left(\frac{k}{m_1}+\frac{m_2 g}{lm_1}\right) \tag{3}$$

We can rewrite Eq. (3) in the following form:

$$\ddot{u}+f\left(u,\dot{u},\ddot{u},t\right)u=0 \tag{4}$$

Where $f\left(u,\dot{u},\ddot{u},t\right)=\left(\left(\frac{m_2}{m_1}\right)u\ddot{u}+\left(\frac{m_2}{m_1}\right)\dot{u}^2+\omega_0^2+\frac{m_2 g}{2lm_1}u^2\right)u$.

Inserting $u=A\cos(\omega t)$ an initial assumption and inserting this trial function into Eq. (4), the maximum and minimum value of $f\left(u,\dot{u},\ddot{u},t\right)$ can be yield:

$$0<\omega^2=f\left(u,\dot{u},\ddot{u},t\right)<\omega_0^2-\left(\frac{m_2}{m_1}\right)A^2\omega^2+\frac{m_2 g}{2lm_1}A^2 \tag{5}$$

Using He Chengtian's average [1], gives:

$$\omega^2=\frac{n\left(\omega_0^2-\left(\frac{m_2}{m_1}\right)A^2\omega^2+\frac{m_2 g}{2lm_1}A^2\right)}{m+n}=k\left(\omega_0^2-\left(\frac{m_2}{m_1}\right)A^2\omega^2+\frac{m_2 g}{2lm_1}A^2\right) \tag{6}$$

Where *m, n* are weighting factors and *k= n/(m+n).* Substituting approximate angular frequency into Eq. (4) yields:

$$\ddot{u}+\omega^2 u=\ddot{u}+uf\left(u,\dot{u},\ddot{u},t\right)+\rho\left(u,\dot{u},\ddot{u},t\right) \tag{7}$$

Where:

$$\rho\left(u,\dot{u},\ddot{u},t\right)=-\left(\frac{m_2}{m_1}\right)u^2\ddot{u}-\left(\frac{m_2}{m_1}\right)u\dot{u}^2-u\omega_0^2-\frac{m_2 g}{2lm_1}u^3$$
$$+ku\left(\omega_0^2-\left(\frac{m_2}{m_1}\right)A^2\omega^2+\frac{m_2 g}{2lm_1}A^2\right) \tag{8}$$

Substituting $u(t)=A\cos(\omega t)$ into ρ , gives:

$$\rho=\left(\frac{m_2}{m_1}\right)A^3\cos(\omega t)^3\omega^2-\left(\frac{m}{m}\right)A^3\cos(\omega t)\sin(\omega t)^2\omega^2-A\omega_0^2\cos(\omega t)$$
$$-\frac{m_2g}{2lm_1}A^3\cos(\omega t)^3+kA\cos(\omega t)\left(\omega_0^2-\left(\frac{m_2}{m_1}\right)A^2\omega^2+\frac{m_2g}{2lm_1}A\right) \quad (9)$$

Using Fourier expansion series, secular term can be obtained:

$$\rho(\omega t)=\sum_{n=0}^{\infty}b_{2n+1}\cos[(2n+1)\omega t]\approx b_1\cos(\omega t)$$

$$b_1=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\left(\begin{array}{l}\left(\frac{m_2}{m_1}\right)A^3\cos(\varphi)^3\omega^2-\left(\frac{m_2}{m_1}\right)A^3\cos(\varphi)\sin(\varphi)^2\omega^2\\ -A\omega_0^2\cos(\omega t)-\frac{m_2g}{2lm_1}A^3\cos(\varphi)^3\\ +kA\cos(\varphi)\left(\omega_0^2-\left(\frac{m_2}{m_1}\right)A^2\omega^2+\frac{m_2g}{2lm_1}A\right)\end{array}\right)\cos(\varphi)d\varphi \quad (10)$$

Avoiding secular term requires $\delta_1=0$, therefore angular frequency can be yield:

$$\omega_{MMA}=\frac{1}{2}\sqrt{\frac{8\omega_0^2lm_1+3m_2gA^2}{l\left(m_2A^2+2m_1\right)}} \quad (11)$$

3. Application of AFF to the first problem:

According to He's frequency formulation [3], we can assume for the amplitude frequency formulation:

$$\omega^2=\frac{\omega_1^2R_2(t_2)-\omega_2^2R_1(t_1)}{R_2(t_2)-R_1(t_1)} \quad (12)$$

In order to solve Eq. (1), we first use two trial functions as follows:

$$u_1=A\cos(t) \quad (13)$$

$$u_2=A\cos(\omega t) \quad (14)$$

Inserting Eqs. (12) and (13) into Eq. (1) , gives the following Residuals:

$$R_1(t_1) = -A\cos(t) - A^3\left(\frac{m_2}{m_1}\right)\cos(t)^3 + A^3\left(\frac{m_2}{m_1}\right)\cos(t)\sin(t)^2$$
$$+A\omega_0^2\cos(t) + A^3\left(\frac{m_2 g}{2lm_1}\right)\cos(t)^3 \tag{15}$$

$$R_2(t) = -A\cos(\omega t)\omega^2 - A^3\left(\frac{m_2}{m_1}\right)\cos(\omega t)^3\omega^2 + A\omega_0^2\cos(\omega t)$$
$$+A^3\left(\frac{m_2}{m_1}\right)\cos(\omega t)\sin(\omega t)^2\omega^2 + A^3\left(\frac{m_2 g}{2lm_1}\right)\cos(\omega t)^3 \tag{16}$$

Weighted residuals can be introduced as follows:

$$\tilde{R}_1(t_1) = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1(t)\cos(t)\,dt, T_1 = \frac{2\pi}{\omega_1} \tag{17}$$

$$\tilde{R}_2(t_2) = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2(t_2)\cos(\omega t), T_2 = \frac{2\pi}{\omega_2} \tag{18}$$

Equating $\omega_1 = 1, \omega_2 = \omega$, we can obtain weighted residuals :

$$\tilde{R}_1(t_1) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(-A\cos(t) - A^3\left(\frac{m_2}{m_1}\right)\cos(t)^3 + A^3\left(\frac{m_2}{m_1}\right)\cos(t)\sin(t)^2\right.$$
$$\left.+A\omega_0^2\cos(t) + A^3\left(\frac{m_2 g}{2lm_1}\right)\cos(t)^3\right)\cos(t)\,dt \tag{19}$$
$$= \frac{1}{2}A\omega_0^2 + \frac{3}{8}A^3\left(\frac{m_2 g}{2lm_1}\right) - \frac{1}{4}\left(\frac{m_2}{m_1}\right)A^3 - \frac{1}{2}A$$

$$\tilde{R}_2(t_2) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(-A\cos(\omega t)\omega^2 - A^3\left(\frac{m_2}{m_1}\right)\cos(\omega t)^3\omega^2 + A\omega_0^2\cos(\omega t)\right.$$
$$\left.+A^3\left(\frac{m_2}{m_1}\right)\cos(\omega t)\sin(\omega t)^2\omega^2 + A^3\left(\frac{m_2 g}{2lm_1}\right)\cos(\omega t)^3\right)\cos(\varphi)\,d\varphi \tag{20}$$
$$= -\frac{1}{2}A\omega^2 + \frac{3}{8}\left(\frac{m_2 g}{2lm_1}\right)A^3 - \frac{1}{4}A^3\left(\frac{m_2}{m_1}\right)\omega^2 + \frac{1}{2}\omega_0^2 A$$

The substitution of Eqs. (19) and (20) into Eq. (12), angular frequency can be yield:

$$\omega^2 = \left(\frac{A}{2} - \frac{A^3\omega^2\left(m_1/m_2\right)}{4} - \frac{A\omega^2}{2} + \frac{A^3\left(m_2/m_1\right)}{4}\right)^{-1} \times$$
$$\left(\frac{\omega_0^2 A}{2} + \frac{3m_2 g}{16lm_1} - \frac{A^3 m_2\omega^2}{4m_1} - \frac{A\omega^2}{2} - \omega^2\left(\frac{\omega_0^2 A}{2} + \frac{3m_2 g}{16lm_1} - \frac{A^3 m_2}{4m_1} - \frac{A}{2}\right)\right) \quad (21)$$

Solving Equation (21), Amplitude-frequency relationship can be obtained:

$$\omega_{AFF} = \frac{1}{2}\sqrt{\frac{8\omega_0^2 lm_1 + 3m_2 gA^2}{l\left(m_2A^2 + 2m_1\right)}} \quad (22)$$

4. Application of MMA to the second problem:

Eq. (2) can be rewritten as follows:

$$\ddot{\theta} + \sin(\theta) - \frac{1}{2}\Lambda\sin(2\theta) = 0, \dot{\theta}(0) = 0, \theta(0) = A \quad (23)$$

Substitution of the relatively accurate approximations: $\sin(\theta) \approx \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$ and $\sin(2\theta) \approx 2\theta - \frac{4\theta^3}{3} + \frac{4\theta^5}{15}$ into Eq. (23), yields:

$$\ddot{\theta} + (1-\Lambda)\theta + \left(-\frac{1}{6} + \frac{2\Lambda}{3}\right)\theta^3 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^5 = 0 \quad (24)$$

To attack Eq. (24) by the max min approach, we rewrite it in the following form:

$$\ddot{\theta} = -f\left(\theta, \dot{\theta}, \ddot{\theta}, t\right)\theta \quad (25)$$

Where $f\left(\theta, \dot{\theta}, \ddot{\theta}, t\right) = (1-\Lambda) + \left(\frac{-1}{6} + \frac{2\Lambda}{3}\right)\theta^2 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^4$. Like Section.2, It can be written:

$$1 - \Lambda < \omega^2 = f\left(\theta, \dot{\theta}, \ddot{\theta}, t\right) < 1 - \Lambda + \frac{1}{6}A^2 + \frac{2}{3}\Lambda A^2 + \frac{1}{120}A^4 - \frac{2}{15}\Lambda A^4 \quad (26)$$

Similar of pervious example, angular frequency can be yields:

$$\omega^2 = \frac{n\left(1 - \Lambda + \frac{1}{6}A^2 + \frac{2}{3}\Lambda A^2 + \frac{1}{120}A^4 - \frac{2}{15}\Lambda A^4\right) + m(1-\Lambda)}{m+n} = (1-k)(1-\Lambda) \quad (27)$$
$$+k\left(1 - \Lambda + \frac{1}{6}A^2 + \frac{2}{3}\Lambda A^2 + \frac{1}{120}A^4 - \frac{2}{15}\Lambda A^4\right)$$

Where *m*, *n* are weighting factors and *k*= *n/(m+n)*. Substituting approximate angular frequency into Eq. (25) obtains:

$$\ddot{\theta}+\omega^2\theta=\ddot{\theta}+\theta f\left(\theta,\dot{\theta},\ddot{\theta},t\right)+\rho \tag{28}$$

Inserting $u\left(t\right)=A\cos\left(\omega t\right)$ as a trial function into ρ can be yield:

$$\rho=A\cos(\omega t)\left(1-b-\frac{kA^2}{6}+\frac{2k\Lambda A^2}{3}+\frac{kA^4}{120}-\frac{2k\Lambda A^4}{15}\right)+\left(\begin{array}{l}1-\frac{A^2\cos(\omega t)^2}{6}+\frac{A^4\cos(\omega t)^4}{120}-\Lambda\\+\frac{4\Lambda A^2\cos(\omega t)^2}{6}-\frac{4\Lambda A^4\cos(\omega t)^4}{30}\end{array}\right)A\cos(\omega t) \tag{29}$$

Using Fourier series and avoiding secular term, angular frequency can be obtained:

$$\omega_{MMA}=\sqrt{1-\frac{1}{8}A^2-\frac{1}{12}\Lambda A^4+\frac{1}{2}\Lambda A^2-\Lambda+\frac{1}{192}A^4} \tag{30}$$

5. Application of AFF to the second problem:

Like pervious example, the trial functions $\theta_1\left(\tau\right)=A\cos\left(\tau\right)$ and $\theta_2\left(\tau\right)=A\cos\left(\omega\tau\right)$ are inserted into Eq. (2) to yield the residuals:

$$R_1\left(t\right)=-\frac{1}{6}A^3\cos\left(t\right)^3+\frac{1}{120}A^5\cos\left(t\right)^5-\frac{\Lambda}{2}\left(2A\cos\left(t\right)-\frac{4}{3}A^3\cos\left(t\right)^3+\frac{4}{15}A^5\cos\left(t\right)^5\right) \tag{31}$$

$$R_2\left(t\right)=-A\cos\left(\omega t\right)\omega^2+A\cos\left(\omega t\right)-\frac{1}{6}A^3\cos\left(\omega t\right)^3+\frac{1}{120}A^5\cos\left(\omega t\right)^5-\frac{\Lambda}{2}\left(2A\cos\left(\omega t\right)-\frac{4}{3}A^3\cos\left(\omega t\right)^3+\frac{4}{15}A^5\cos\left(\omega t\right)^5\right) \tag{32}$$

Weighted residuals can be yield as:

$$\tilde{R}_1\left(t_1\right)=-\frac{1}{16}A^3-\frac{1}{24}\Lambda A^5+\frac{1}{4}\Lambda A^3-\frac{1}{2}A\Lambda+\frac{1}{384}A^5 \tag{33}$$

$$\tilde{R}_2\left(t_2\right)=\frac{1}{2}A-\frac{1}{16}A^3+\frac{1}{384}A^5-\frac{1}{2}A\Lambda-\frac{1}{2}A\omega^2-\frac{1}{24}\Lambda A^5+\frac{1}{4}\Lambda A^3 \tag{34}$$

Inserting Eq. (33) and Eq. (34) into Eq. (12), angular frequency can be yield:

$$\omega_{AFF}=\sqrt{1-\frac{1}{8}A^2-\frac{1}{12}\Lambda A^4+\frac{1}{2}\Lambda A^2-\Lambda+\frac{1}{192}A^4} \tag{35}$$

6. Results and Disscussions

In this section, the obtained results will be studied in some numerical cases. In Fig.3 the comparison between Analytical solutions and runge-kutta result is shown. As we see, amplitude frequency formulation and Max min approach obtain same results and have a high validity in comparison with runge-kutta method.

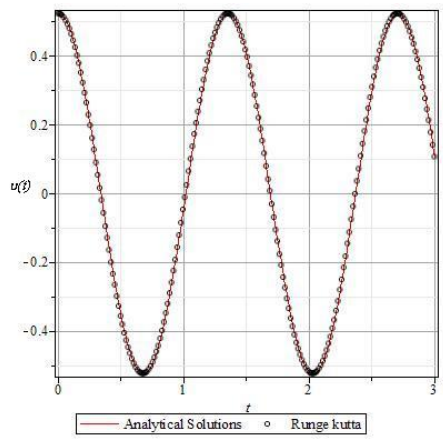

Fig.3. Comparison between AFF & MMA & Runge-kutta forth order in first problem for $A=\frac{\pi}{6}, g=9.81\, m/s^2, k=100\, N/m^2, m_1=5kg, m_2=1kg, l=1m$

Fig.4 shows the comparison between forth-order runge kutta and analitycal solutions in second problem for $A=\frac{\pi}{3}, \Lambda=0.25$. Good agreement can be illustrated in Fig.4.

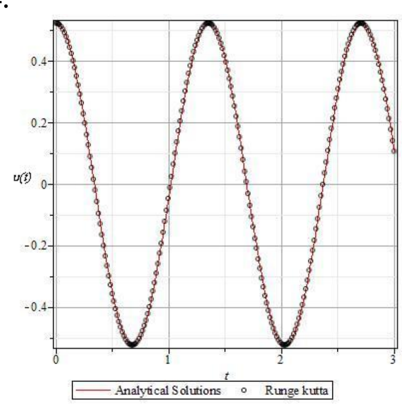

Fig.4. Comparison between AFF & MMA & Runge-kutta forth order in second problem for $A=\frac{\pi}{3}, \Lambda=0.25$

6. Conclusions

In this paper, Max min approach and Amplitude frequency formulation which are two powerful methods and derived from Chinese mathematics are applied to the motion equations of two nonlinear oscillators. At first the equation of motion was derived for both problems. Max min approach and Amplitude frequency formulation was applied to the nonlinear ordinary differential equations and finally the results compared with forth order runge kutta method. Simple procedure and high accuracy and validity are the advantages of these methods.

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