ON SOME PROBLEMS IN THE SPACE $C^{(n)}[0,1]$

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We consider the so called $\alpha -$Duhamel product, denoted $\underset{\alpha}{\otimes}$, on the space $C^{(n)}[0,1]$ and prove that, with this product, this space has the structure of a unital Banach algebra, and then show that its maximal ideal space consists of the homomorphism φ_α defined by $\varphi_\alpha(f) = f(\alpha)$. Moreover, we consider the usual convolution product $\underset{\alpha}{*}$ and study the $\underset{\alpha}{*}$-generators of the Banach algebra $\left(C^{(n)}[0,1], \underset{\alpha}{*}\right)$. Some other related questions are also discussed. Our results improve the work of [2,3,4,5] where the case $\alpha = 0$ was considered.

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1. Introduction

Let $C^{(n)}[0,1]$ be the space of continuous functions with n derivatives and the n^{th} derivative continuous. Set $\|f\| = \sup\{|f(x)|, x \in [0,1]\}$ for any continuous function, and consider the norm

$$\|f\|_n = \max\left\{\|f\|, \|f'\|, ..., \|f^{(n)}\|\right\}, \text{ for } f \in C^{(n)}[0,1].$$

For $\alpha \in [0,1]$, we define the $\alpha -$Duhamel product on $C^{(n)}[0,1]$ as follows:

$$(f \underset{\alpha}{\otimes} g)(x) = \frac{d}{dx}\int_\alpha^x f(x+\alpha-t)g(t)dt = \int_\alpha^x f'(x+\alpha-t)g(t)dt + f(\alpha)g(x). \quad (1.1)$$

Moreover, define the following convolution product on $C^{(n)}[0,1]$ by:

$$K_{\alpha,f}\, g(x) := (f \underset{\alpha}{*} g)(x) = \int_\alpha^x f(x+\alpha-t)g(t)dt. \quad (1.2)$$

Observe that for $\alpha = 0$, we recover the classical convolution products $\otimes$ and $*$ given respectively by

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$$(f \otimes g)(x) = \int_0^x f'(x-t)g(t)dt + f(0)g(x) \text{ and } (f * g)(x) = \int_0^x f(x-t)g(t)dt.$$

Various algebras of functions with respect to the products $*$ and $\otimes$ have been investigated in the literature; we refer to the papers [1,2,3,4,5,7] for more details and related references. For instance, the space $C^{(n)}[0,1]$ endowed with the convolution products $*$ and $\otimes$ yields two closely related algebras that have been considered and well elaborated in [4]. In particular, its $*$-generators have been characterized. Moreover, extended eigenvalues, extended eigenoperators and spectral multiplicity of some concrete operators, such as the Volterra integration operator, have been determined using the resulting tools. The point is that the methods used there hinge heavily on the Banach algebras structures of these algebras. Note that algebras of analytic functions on the unit disk have also been analogously studied, and many similar interesting results have been established; for more details, we refer to [2,3,4,7,6] and the references therein.

In [1], Ginsberg and Newman have shown that a necessary condition for $f \in C_0$, (here $C_0 := \{f \in C[0,1], f(0) = 0\}$), to generate the radical algebra $(C_0, *)$ is that f does not vanish throughout any interval $[0,\lambda], \lambda > 0$; while it is not yet known whether this condition is sufficient. Nevertheless, they proved that the necessary condition $f(0) \neq 0$ is also sufficient in the Banach algebra T of functions $f(x) = \sum_{k=0}^{\infty} a_k x^k$, with the norm $\|f\|_T = \sum_{k=0}^{\infty} |a_k|$ and multiplication $*$.

In this paper we prove that $\left(C^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$ is in fact a commutative Banach algebra, and carry out an invertibility criterion with respect to this product, and then describe its maximal ideal space. Besides, we characterize the $\underset{\alpha}{*}$-generators of the algebra $C^{(n)}[0,1]$ with respect to this convolution product. The above techniques have certain interesting applications, namely that we are able to exploit the underlying structure in order to establish an estimate for the solutions of some integral equation of Volterra type in terms of the kernel function $K \in C^{(n)}[0,1]$.

2. A Banach algebra structure for $C^{(n)}[0,1]$ and its maximal ideal space

In this section we investigate the Banach algebra structure of the space $C^{(n)}[0,1]$ with respect to the α-Duhamel product and describe its maximal ideal space. We begin with the following lemma.

Lemma 2.1 $\left(C^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$ *is a commutative Banach algebra with the unity* $f = \mathbf{1}$.

Proof. Only the norm product inequality needs to be shown. For, we have

$$(f \underset{\alpha}{\otimes} g)'(x) = \int_{\alpha}^{x} f''(x+\alpha-t)g(t)dt + f(\alpha)g'(x) + f'(\alpha)g(x),$$

and by induction we get

$$(f \underset{\alpha}{\otimes} g)^{(k)}(x) = \int_{\alpha}^{x} f'^{(k+1)}(x+\alpha-t)g(t)dt + \sum_{m=0}^{k} f^{(m)}(\alpha)g^{(k-m)}(x).$$

An integration by parts leads to

$$(f \underset{\alpha}{\otimes} g)^{(k)}(x) = \int_{\alpha}^{x} f'^{(k)}(x+\alpha-t)g'(t)dt + \sum_{m=0}^{k-1} f^{(m)}(\alpha)g^{(k-m)}(x) + g(\alpha)f^{(k)}(x).$$

So we obtain

$$\left|(f \underset{\alpha}{\otimes} g)^{(k)}(x)\right| \le \left\|f^{(k)}\right\|\left\|g'\right\| + \sum_{m=0}^{k-1}\left\|f^{(m)}\right\|\left\|g^{(k-m)}\right\| + \|g\|\left\|f^{(k)}\right\|$$

and thus

$$\left\|(f \underset{\alpha}{\otimes} g)^{(k)}\right\| \le (k+2)\|f\|_n\|g\|_n,$$

which yields

$$\left\|(f \underset{\alpha}{\otimes} g)\right\|_n \le (n+2)\|f\|_n\|g\|_n.$$

This completes the proof.

Now, we establish an invertibility criterion with respect to the Duhamel product:

Lemma 2.2 $f \in C^{(n)}[0,1]$ *is* $\underset{\alpha}{\otimes}$*-invertible if and only if* $f(\alpha) \ne 0$.

Proof. Indeed we have

$$(f \underset{\alpha}{\otimes} g)(x) = \int_{\alpha}^{x} f'(x+\alpha-t)g(t)dt + f(\alpha)g(x).$$

If g is the inverse of f we get $(f \underset{\alpha}{\otimes} g)(\alpha) = f(\alpha)g(\alpha) = 1,$ whence $f(\alpha) \ne 0.$

Conversely, if $f(\alpha) \ne 0$, set $D_{\alpha,f}(g) = f \underset{\alpha}{\otimes} g$. We will prove that $D_{\alpha,f}$ is an invertible operator. To this aim, write f as $f = F + f(\alpha)$, where $F = f - f(\alpha)$. So $D_{\alpha,f} = f(\alpha)I + D_{\alpha,F}$. Since $f(\alpha) \ne 0$, it suffices to show that $D_{\alpha,F}$ is quasinilpotent, i.e that $\sigma(D_{\alpha,F}) = \{0\}$. For this, we will show that

$\lim_{k\to\infty}\left\|D^k_{\alpha,F}\right\|^{1/k}=0.$ In fact, in terms of the operator $K_{\alpha,f}$ defined above, by considering $F(\alpha)=0,$ we have

$$(D_{\alpha,F}g)(x)=\frac{d}{dx}\int_\alpha^x F(x+\alpha-t)g(t)dt$$
$$=\int_\alpha^x F'(x+\alpha-t)g(t)dt=K_{\alpha,F'}g(x).$$

Thus we get

$$(K^2_{\alpha,F'}g)(x)=K_{\alpha,F'}(K_{\alpha,F'}g)(x)$$
$$=\int_\alpha^x F'(x+\alpha-t)(K_{\alpha,F'}g)(t)dt$$
$$=\int_\alpha^x F'(x+\alpha-t)\left(\int_\alpha^t F'(t+\alpha-s)g(s)ds\right)dt.$$

Hence, we obtain

$$|(K^2_{\alpha,F'}g)(x)|\le\|F\|_n^2\|g\|_n\frac{(x-\alpha)^2}{2}.$$

By induction we easily get

$$|(K^k_{\alpha,F'}g)(x)|\le\|F\|_n^k\|g\|_n\frac{(x-\alpha)^k}{k!}.$$

On the other hand, we have:

$$(K^2_{\alpha,F'}g)'(x)=\int_\alpha^x F''(x+\alpha-t)\left(\int_\alpha^t F'(t+\alpha-s)g(s)ds\right)dt$$
$$+F'(\alpha)\int_\alpha^x F'(x+\alpha-s)g(s)ds.$$

Thus, we get

$$|(K^2_{\alpha,F'}g)'(x)|\le\|F\|_n^2\|g\|_n\left(\frac{(x-\alpha)^2}{2}+x-\alpha\right)$$
$$\le\|F\|_n^2\|g\|_n\frac{(x-\alpha+1)^2}{2}.$$

So, assume by induction that

$$|(K^k_{\alpha,F'}g)'(x)|\le\|F\|_n^k\|g\|_n\frac{(x-\alpha+1)^k}{k!}.$$

By differentiation we get

$$(K^{k+1}_{\alpha,F'}g)'(x)=\int_\alpha^x F''(x+\alpha-t)(K^k_{\alpha,F'}g)(t)dt+F''(\alpha)(K^k_{\alpha,F'}g)(x).$$

We deduce that

$$|(K_{\alpha,F'}^{k+1}g)'(x)| \le \|F\|_n^{k+1}\|g\|_n\left(\frac{(x-\alpha)^{k+1}}{(k+1)!}+\frac{(x-\alpha)^k}{k!}\right) \le \|F\|_n^{k+1}\|g\|_n\frac{(x-\alpha+1)^{k+1}}{(k+1)!}.$$

Now, from the equality

$$(K_{\alpha,F'}^2 g)'(x) = \int_\alpha^x F''(x+\alpha-t)(K_{\alpha,F'}g)(t)dt + F'(\alpha)(K_{\alpha,F'}g)(x).$$

we infer that

$$(K_{\alpha,F'}^2 g)''(x) = \int_\alpha^x F'''(x+\alpha-t)(K_{\alpha,F'}g)(t)dt + F''(\alpha)(K_{\alpha,F'}g)(x) + F'(\alpha)(K_{\alpha,F'}g)'(x),$$

which leads to

$$|(K_{\alpha,F'}^2 g)''(x)| \le \|F\|_n^2\|g\|_n\left(\frac{(x-\alpha)^2}{2}+x-\alpha+\frac{(x-\alpha+1)^2}{2}\right)$$

$$\le \|F\|_n^2\|g\|_n\frac{(x-\alpha+2)^2}{2}.$$

By induction we thus obtain

$$|(K_{\alpha,F'}^k g)^{(j)}(x)| \le \|F\|_n^k\|g\|_n\frac{(x-\alpha+j)^k}{k!}, \quad \text{for all } j\in\{2,\ldots,n\}.$$

It follows that

$$\left\|K_{\alpha,F'}^k g\right\|_n \le \|F\|_n^k\|g\|_n\frac{(1+n)^k}{k!},$$

and thus

$$\left\|K_{\alpha,F'}^k\right\|^{1/k} \le \|F\|_n\frac{1+n}{(k!)^{1/k}} \underset{k\to\infty}{\longrightarrow} 0.$$

So $K_{\alpha,F'}$ is quasinilpotent, which implies that $D_{\alpha,f}$ is invertible.

Theorem 2.3 $\left(C^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$ *is a unital commutative Banach algebra with maximal ideal space* $M=\{\varphi_\alpha\}$ *where* $\varphi_\alpha : C^{(n)}[0,1]\to\mathbb{C}$ *and* $\varphi_\alpha(f)=f(\alpha)$.

Proof. We set here by $\sigma(f)$ the spectrum of the element f in the Banach algebra $\left(C^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$ with respect to the multiplication $\underset{\alpha}{\otimes}$. It follows from Lemma 2.2 that $\sigma(f)=\{f(\alpha)\}$ and by Gelfand's theory we see that $M=\{\varphi_\alpha\}$. Indeed, the functions which vanish at the point α form a maximal ideal. Any other proper ideal cannot have an element which does not vanish at α; hence there is only one maximal ideal. Therefore the maximal ideal space M of $\left(C^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$ consists of one homomorphism, namely, evaluation at α, and the Gelfand transform is trivial. This proves the theorem.

3. The $\underset{\alpha}{*}$-generators of the radical algebra $C^{(n)}[0,1]$

Recall that for a Banach algebra **B** the radical $\Re$ of **B** is equal to the intersection of the kernel of all (strictly) irreducible representations of **B**. If $\Re = \{0\}$, then **B** is said to be semi-simple and if $\Re = \mathbf{B}$, then **B** is called a radicial algebra. Equivalently, **B** is a radicial Banach algebra, if for every element $b \in \mathbf{B}$ the associated multiplication operator M_b, $M_b a := ba$ $(a \in \mathbf{B})$, is quasinilpotent on **B** (i.e., $\sigma(M_b) = \{0\}$).

It is classical that $\lim_{k\to\infty} \left\| f^{\underset{\alpha}{*}k} \right\|_n^{1/k} = 0$ and so $(C^{(n)}[0,1], \underset{\alpha}{*})$ is a radical Banach algebra with respect to the convolution $\underset{\alpha}{*}$ defined by means of Formula (1.2). Here $f^{\underset{\alpha}{*}k} := f \underset{\alpha}{*} f \underset{\alpha}{*} \ldots \underset{\alpha}{*} f$ is the k^{th} iterated convolution of the function f in $C^{(n)}[0,1]$. For any $f \in C^{(n)}[0,1]$, we have that $(f \underset{\alpha}{*} f)(\alpha) = 0$. Also $f \underset{\alpha}{*} f \underset{\alpha}{*} f = \left(\int_\alpha^x f(x+\alpha-t) \int_\alpha^t f(t+\alpha-\tau) f(\tau) d\tau\right)|_{x=\alpha} = 0$. Thus, it is easy to see that $f^{\underset{\alpha}{*}k}(\alpha) = 0, k = 1,2 \ldots,$ and therefore we see that a necessary condition for $f \in C^{(n)}[0,1]$ to generate $(C^{(n)}[0,1], \underset{\alpha}{*})$, (that is to yield $\mathrm{Span}\left[f, f \underset{\alpha}{*} f, f \underset{\alpha}{*} f \underset{\alpha}{*} f, \ldots\right] = C^{(n)}[0,1]$), is that $f(\alpha) \neq 0$. However, it is not yet known whether this condition is sufficient, even for $\alpha = 0$, (see eg. Ginsberg and Newman [1], as well as [2]). Note that for $\alpha = 0$, (i.e. for the classical convolution product $*$), this problem was first considered by Ginsberg and Newman [ginsburg-newman] for the space $C[0,1]$ of all continuous functions on the segment $[0,1]$. They considered the subalgebra C_0, defined by $C_0 := \{f \in C[0,1], f(0) = 0\}$, and showed that a necessary condition for $f \in C_0$ to generate C_0 is that f does not vanish throughout any interval $[0,\lambda], \lambda > 0$. However, it is proved there that this condition is not sufficient. Furthermore, they proved that the necessary condition $f(0) \neq 0$ is also sufficient in the Banach algebra T of functions $f(x) = \sum_{k=0}^{\infty} a_k x^k$, with the norm $\|f\|_T = \sum_{k=0}^{\infty} |a_k|$ and multiplication $*$ defined above.

In the present section, we discuss the above stated question in the Banach algebra $(C^{(n)}[0,1], \underset{\alpha}{*})$. Namely we prove the following theorem, which reduces this question to the case of the subalgebra

$$C_\alpha^{(n)}[0,1] := \left\{ f \in C^{(n)}[0,1]: \ f(\alpha) = 0 \right\}.$$

Theorem 3.1 *Let* $f \in C^{(n)}[0,1]$ *be a function such that* $f(\alpha) \neq 0$. *Let* $F(x) = \int_\alpha^x f(t)dt$. *Then* f *is a* $\underset{\alpha}{*}$*-generator of the algebra* $\left(C^{(n)}[0,1], \underset{\alpha}{*}\right)$ *if and only if* F *is a* $\underset{\alpha}{\otimes}$*-generator of the subalgebra* $\left(C_\alpha^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$.

Proof. Indeed, since $F(x) = \int_\alpha^x f(t)dt$, for all $g \in C^{(n)}[0,1]$, we obtain

$$D_{\alpha,F}\, g(x) = \frac{d}{dx}\int_\alpha^x F(x+\alpha-t)g(t)dt = \int_\alpha^x f(x+\alpha-t)g(t)dt.$$

This means that $D_{\alpha,F} = K_{\alpha,f}$, where $K_{\alpha,f}$ is the convolution operator defined above. Hence $F \underset{\alpha}{\otimes} f = f \underset{\alpha}{*} f$. Moreover, we have

$$(F \underset{\alpha}{\otimes} F) \underset{\alpha}{\otimes} f = D_{\alpha,F}^2 f = D_{\alpha,F}(D_{\alpha,F} f) = D_{\alpha,F}(K_{\alpha,f} f) = K_{\alpha,f}^2 f.$$

By induction we get $K_{\alpha,f}^k f = D_{\alpha,F}^k f$, for any $k \geq 0$. These equalities show that

$$\begin{aligned}
\mathrm{Span}\left[f, f \underset{\alpha}{*} f, f \underset{\alpha}{*} f \underset{\alpha}{*} f, \ldots\right] &= \mathrm{Span}\left[f, F \underset{\alpha}{\otimes} f, F \underset{\alpha}{\otimes} F \underset{\alpha}{\otimes} f, \ldots\right] \\
&= \mathrm{Span}\left[D_{\alpha,f}\left(F^{\overset{\otimes k}{\alpha}}\right) : \ k = 0,1,2\ldots\right] \\
&= \mathrm{Clos}\left\{D_{\alpha,f}\left(\mathrm{Span}\left[F^{\overset{\otimes k}{\alpha}} \ : \ k = 0,1,2\ldots\right]\right)\right\} \\
&= \mathrm{Clos}\left\{D_{\alpha,f}\left(\mathrm{Span}\left[1, F, F \underset{\alpha}{\otimes} F, F^{\overset{\otimes 3}{\alpha}}, \ldots\right]\right)\right\}.
\end{aligned}$$

Now, using the fact that

$$\mathrm{Span}\left[1, F, F \underset{\alpha}{\otimes} F, F^{\overset{\otimes 3}{\alpha}}, \ldots\right] = \mathrm{Span}[\lambda 1, \lambda \in \mathbb{C}] \oplus \mathrm{Span}\left[F, F \underset{\alpha}{\otimes} F, F^{\overset{\otimes 3}{\alpha}}, \ldots\right],$$

where $\oplus$ stands for the direct sum of subspaces, we see that

$$\mathrm{Span}\left[f, f^{\overset{*2}{\alpha}}, \ldots\right] = \mathrm{Clos}\left\{D_{\alpha,f}\left(\mathrm{Span}[\lambda 1, \lambda \in \mathbb{C}] \oplus \mathrm{Span}\left[F, F^{\overset{?2}{\alpha}}, \ldots\right]\right)\right\}. \tag{3.3}$$

Since $f(\alpha) \neq 0$, by Lemma 2.2 the $\alpha-$Duhamel operator $D_{\alpha,f}$ is invertible on the space $C^{(n)}[0,1]$. On the other hand, by considering that

$$C^{(n)}[0,1] = \mathrm{Span}[\lambda 1, \lambda \in \mathbb{C}] \oplus C_\alpha^{(n)}[0,1], \tag{3.4}$$

where $C_{\alpha}^{(n)}[0,1]$ is the subspace defined above, the assertions of the theorem now follow from the invertibility of the operator $D_{\alpha,f}$ and the representations (3.3) and (3.4). The proof is complete.

4. An estimate for the solutions of the Volterra integral equation

In this section we give an estimate for the solutions of the Volterra integral equation

$$\left(\Re_{\alpha,K} f\right)(x) = \int_{\alpha}^{x} K(x+\alpha-t) f(t)dt = g(x), \tag{4.5}$$

in terms of the kernel function $K \in C^{(n)}[0,1]$. It is classical that Equation (4.5) has a solution in the space $C^{(n)}[0,1]$ for any given function $g \in C^{(n)}[0,1]$. Let us set

$$G_g := \left\{u \in C^{(n)}[0,1] : u \text{ is a solution of equation (4.5)}\right\}.$$

It can be proved that $\sigma(\Re_{\alpha,K}) = \{0\}$ (i.e. $\Re_{\alpha,K}$ is a quasinilpotent operator on $C^{(n)}[0,1]$). Let $\sigma_p(\Re_{\alpha,K})$ denote the point spectrum of the operator $\Re_{\alpha,K}$ (i.e., the set of eigenvalues of $\Re_{\alpha,K}$). Since $\Re_{\alpha,K}$ is also compact, it is easy to see then that $\sigma_p(\Re_{\alpha,K}) = \phi.$ This implies that $g \notin G_g$ for any nonzero $g \in C^{(n)}[0,1]$.

Let $G_g^1 = \left\{u \in G_g : \|u\|_n = 1\right\}$ be the unit sphere of the set G_g. Here we are interested in the following natural question:

Calculate the distance between g and G_g^1, denoted $\mathtt{dist}\left(g, G_g^1\right)$.

Our assertion estimates $\mathtt{dist}\left(g, G_g^1\right)$ in terms of the kernel function K.

Proposition 4.1 *The following inequality holds*

$$\inf\left\{dist\left(g, G_g^1\right), g \in C^{(n)}[0,1] \setminus \{0\}\right\} \geq \frac{1}{n+2} \left\| \left(-1 + \int_{\alpha}^{x} K(t)dt\right)^{-1?}_{\alpha} \right\|_n^{-1},$$

where the symbol $-1\underset{\alpha}{\otimes}$ denotes the $\underset{\alpha}{\otimes}$-inverse in the algebra $\left(C^{(n)}[0,1], \underset{\alpha}{\otimes}\right)$.

Proof. We set $F(x) = -1 + \int_{\alpha}^{x} K(t)dt.$ Then, the equation $K \underset{\alpha}{*} u = g$ can be rewritten as

$$\frac{d}{dx} \int_{\alpha}^{x} F(x+\alpha-t)u(t)dt + u(x) = g(x),$$

or in brief as $F \underset{\alpha}{\otimes} u = g - u.$ Then, by considering that $F(\alpha) = -1 (\neq 0)$, by Lemma 2.2, there exists a function $f \in C^{(n)}[0,1]$ such that $f \underset{\alpha}{\otimes} F = 1,$ which implies that

$f \underset{\alpha}{\otimes} F \underset{\alpha}{\otimes} u = f \underset{\alpha}{\otimes}(g-u)$ that is $u = f \underset{\alpha}{\otimes}(g-u)$. Hence by making use of Lemma 2.1, we obtain for any $u \in G_g^1$

$$1 = \|u\|_n = \left\| f \underset{\alpha}{\otimes}(g-u)\right\|_n \leq (n+2)\|f\|_n \|g-u\|_n,$$

which implies that

$$\|g-u\|_n \geq \frac{1}{n+2}\frac{1}{\|f\|_n}, \ \forall u \in G_g^1.$$

Since $f = F^{-1\underset{\alpha}{\otimes}}$, we infer that

$$\|g-u\|_n \geq \frac{1}{n+2}\frac{1}{\left\|F^{-1\underset{\alpha}{\otimes}}\right\|_n} = (n+2)^{-1}\left\|\left(-1+\int_\alpha^x K(t)dt\right)^{-1\underset{\alpha}{\otimes}}\right\|_n^{-1}, \ \forall u \in G_g^1.$$

Hence

$$dist\left(g, G_g^1\right) \geq (n+2)^{-1}\left\|\left(-1+\int_\alpha^x K(t)dt\right)^{-1\underset{\alpha}{\otimes}}\right\|_n^{-1}. \qquad (4.6)$$

Since $g \in C^{(n)}[0,1] \setminus \{0\}$ is an arbitrary function, Inequality (4.6) implies that

$$\inf\left\{dist\left(g, G_g^1\right), g \in C^{(n)}[0,1] \setminus \{0\}\right\} \geq (n+2)^{-1}\left\|\left(-1+\int_\alpha^x K(t)dt\right)^{-1\underset{\alpha}{\otimes}}\right\|_n^{-1},$$

as desired.

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