SOME NOVEL RESULTS OF T-PERIODIC SOLUTIONS FOR RAYLEIGH TYPE EQUATION WITH DOUBLE DEVIATING ARGUMENTS

by Yong Wang[1], Zhengwu Tao[2], Donghong Tian[3], Xin Ma[4], Mingjun Li[5] and Zonghong Feng[6]

In this work, we study the following Rayleigh equation with double deviating arguments:

$$x''(t) + f(t, x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t).$$

Some criteria to guarantee the existence and uniqueness of periodic solutions of this equation is given by using Mawhin's continuation theorem and some new techniques. Our results are new and complement some known results.

Keywords: Periodic solution; Existence and uniqueness; Deviating argument; Rayleigh equation.

MSC2010: 34B15; 34K13.

1. Introduction

In this present paper, we investigate the existence and uniqueness of the periodic solutions of the following Rayleigh equation with double deviating arguments

$$x''(t) + f(t, x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t), \tag{1.1}$$

where $\tau_1, \tau_2 \in C(\mathbb{R}, \mathbb{R})$; $f, g_1, g_2 \in C(\mathbb{R}^2, \mathbb{R})$; $\tau_1(t)$, $\tau_2(t)$, $f(t, x)$, $g_1(t, x)$, $g_2(t, x)$ are T-periodic functions with respect to t, $T > 0$; $f(t, 0) = 0$ for all $t \in \mathbb{R}$; $e \in C(\mathbb{R}, \mathbb{R})$, and $e(t)$ is a T-periodic function.

As it is well known, the Rayleigh equation can be derived from many fields, such as physics, mechanics and engineering technique fields, and an important question is whether this equation can support periodic solutions. During the past several years, many authors have contributed to the theory of this equation with respect to existence of periodic solutions. For example, in 1977, Gaines and Mawhin [2] introduced some continuation theorems and applied them to discussing the existence of solutions of differential equations. In particular, a specific example is provided in [2, p. 99] on how T-periodic solutions can be obtained by means of these theorems for the Rayleigh equation

$$x''(t) + f(x'(t)) + g(t, x(t)) = 0. \tag{1.2}$$

In this direction, the researchers in [1, 3, 5–9, 11–16, 18, 19] continued to discuss the Rayleigh equation and got some new results on the existence of periodic solutions of (1.1), and

[1]Associate Professor, School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China, e-mail: ywangsc@foxmail.com

[2]Assistant Researcher, Research Institute of Exploration and Development, Tarim Oilfield Company, PetroChina, Korla, Xinjiang 84100, China

[3]Lecturer, School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China

[4]Associate Professor, School of Science, Southwest University of Science and Technology, Mianyang 621010, China

[5]Assistant Researcher, CNOOC–Ltd. Zhanjiang, Zhanjiang, Guangdong 524057, China

[6]Lecturer, School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou, Gansu 730070, China

generalized the results in [2]. Wang et al. [20–36] studied the existence and uniqueness of the solution of the seepage equation. Ma et al. [37–60] studied the existence of the solution of the seepage equation. However, as far as we know, much fewer authors have considered the uniqueness of periodic solutions for Rayleigh equation (**1**.1), the main difficulty lies in the middle term $f(t, x'(t))$ of (**1**.1), the existence of which obstructs the usual method of finding some sufficient conditions to guarantee the uniqueness of periodic solutions for Liénard equation or Duffing equation from working. For instance, if $x(t)$ is a T-periodic solution to Duffing type p-Laplacian equation

$$(\varphi_p(x'(t)))' + Cx'(t) + g(t, x(t)) = e(t), \qquad (\mathbf{1}.3)$$

then $\int_0^T Cx'(t)dt = 0$, which is very useful to get some criteria for securing the uniqueness of T-periodic solutions to (**1**.3) in [17]. But for (**1**.1), the corresponding middle term formula $\int_0^T f(t, x'(t))dt = 0$ no longer holds, generally. Hence, it is essential to continue to study the periodic solutions of (**1**.1) in this case.

The main purpose in this work is to give some criteria to guarantee the existence and uniqueness of periodic solutions to (**1**.1). Some sufficient conditions for securing the existence and uniqueness of T-periodic solutions of (**1**.1) are obtained by using Mawhin's continuation theorem and some new techniques. Our results are new and complement the previously known results. An illustrative example will be provided to demonstrate the applications of our results in Section 4.

2. Lemmas

Let us start with some notations. Define

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad |x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k}.$$

Let

$$C_T^1 := \{x \in C^1(\mathbb{R}, \mathbb{R}) : x \text{ is } T\text{-periodic}\}$$

and

$$C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x \text{ is } T\text{-periodic}\},$$

which are two Banach spaces with the norms

$$||x||_{C_T^1} = max\{|x|_\infty, |x'|_\infty\}, \quad ||x||_{C_T} = |x|_\infty.$$

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.1. *If $x \in C^2(\mathbb{R}, \mathbb{R})$ with $x(t + T) = x(t)$, then*

$$|x'|_2^2 \le \left(\frac{T}{2\pi}\right)^2 |x''|_2^2.$$

Proof. Lemma 2.1 is a direct consequence of the Wirtinger inequality, and see [4, 10] for its proof.

Consider the homotopic equation of (**1**.1)

$$x''(t) + \lambda f(t, x'(t)) + \lambda g_1(t, x(t - \tau_1(t))) + \lambda g_2(t, x(t - \tau_2(t))) = \lambda e(t), \qquad (\mathbf{2}.1)$$

where $\lambda \in (0, 1)$.

We have the following results.

Lemma 2.2. *Assume that the following conditions are satisfied:*

(H_1): *one of the following conditions holds*

(1): $(g_i(t, u) - g_i(t, v))(u - v) > 0$ *for all* $t, u, v \in \mathbb{R}, u \ne v, i = 1, 2,$

(2): $(g_i(t, u) - g_i(t, v))(u - v) < 0$ *for all* $t, u, v \in \mathbb{R}, u \ne v, i = 1, 2;$

(H_2): *there exists a constant $D \ge 0$ such that one of the following conditions holds*

(1): $x(g_1(t,x)+g_2(t,x)-e(t))>0$, *for all* $t\in\mathbb{R}, |x|\geq D$,

(2): $x(g_1(t,x)+g_2(t,x)-e(t))<0$, *for all* $t\in\mathbb{R}, |x|\geq D$.

*If $x(t)$ is a T-periodic solution of (**2**.1), then***:**

$$|x|_\infty \leq D+\frac{\sqrt{T}}{2}|x'|_2. \tag{2.2}$$

Proof. Let $x(t)$ be an arbitrary T-periodic solution of (**2**.1). Set

$$x(\bar{t}_{max})=\max_{t\in\mathbb{R}} x(t), x(\bar{t}_{min})=\min_{t\in\mathbb{R}} x(t), \quad \text{where } \bar{t}_{max},\bar{t}_{min}\in\mathbb{R}.$$

Then we have

$$x'(\bar{t}_{max})=0, x''(\bar{t}_{max})\leq 0; x'(\bar{t}_{min})=0, x''(\bar{t}_{min})\geq 0. \tag{2.3}$$

By (**2**.1) and $f(t,0)=0$ for all $t\in\mathbb{R}$, (**2**.3) leads to

$$\begin{aligned}&g_1(\bar{t}_{max},x(\bar{t}_{max}-\tau_1(\bar{t}_{max})))+g_2(\bar{t}_{max},x(\bar{t}_{max}-\tau_2(\bar{t}_{max})))-e(\bar{t}_{max})\\&=-\frac{x''(\bar{t}_{max})}{\lambda}\geq 0,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&g_1(\bar{t}_{min},x(\bar{t}_{min}-\tau_1(\bar{t}_{min})))+g_2(\bar{t}_{min},x(\bar{t}_{min}-\tau_2(\bar{t}_{min})))-e(\bar{t}_{min})\\&=-\frac{x''(\bar{t}_{min})}{\lambda}\leq 0.\end{aligned} \tag{2.5}$$

Since $g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)$ is a continuous function in $\mathbb{R}$, it follows from (**2**.4) and (**2**.5) that there exists a constant $t_1\in\mathbb{R}$ such that

$$g_1(t_1,x(t_1-\tau_1(t_1)))+g_2(t_1,x(t_1-\tau_2(t_1)))-e(t_1)=0. \tag{2.6}$$

Now we show that the following claim is true.

Claim. *If $x(t)$ is a T-periodic solution of (**2**.1), then there exists a constant $t_2\in\mathbb{R}$ such that*

$$|x(t_2)|\leq D. \tag{2.7}$$

Assume, by way of contradiction, that (**2**.7) does not hold. Then

$$|x(t)|>D \quad \text{for all } t\in\mathbb{R}, \tag{2.8}$$

which, together with H_2 and (**2**.6), implies that one of the following relations holds:

$$x(t_1-\tau_1(t_1))>x(t_1-\tau_2(t_1))>D, \tag{2.9}$$

$$x(t_1-\tau_2(t_1))>x(t_1-\tau_1(t_1))>D, \tag{2.10}$$

$$x(t_1-\tau_1(t_1))<x(t_1-\tau_2(t_1))<-D, \tag{2.11}$$

$$x(t_1-\tau_2(t_1))<x(t_1-\tau_1(t_1))<-D. \tag{2.12}$$

Suppose that (2.9) holds, in view of $(H_1)(1)$, $(H_1)(2)$, $(H_2)(1)$ and $(H_2)(2)$, we will consider fours cases as follows:

Case (i): If $(H_1)(1)$ and $(H_2)(1)$ hold, according to (2.9), we have

$$\begin{aligned}0 &< g_1(t_1,x(t_1-\tau_2(t_1)))+g_2(t_1,x(t_1-\tau_2(t_1)))-e(t_1)\\&< g_1(t_1,x(t_1-\tau_1(t_1)))+g_2(t_1,x(t_1-\tau_2(t_1)))-e(t_1),\end{aligned}$$

which contradicts that (**2**.6). This contradiction implies that (**2**.7) is true.

Case (ii): If $(H_1)(2)$ and $(H_2)(1)$ hold, according to (2.9), we have

$$\begin{aligned}0 &< g_1(t_1,x(t_1-\tau_1(t_1)))+g_2(t_1,x(t_1-\tau_1(t_1)))-e(t_1)\\&< g_1(t_1,x(t_1-\tau_1(t_1)))+g_2(t_1,x(t_1-\tau_2(t_1)))-e(t_1),\end{aligned}$$

which contradicts that (**2**.6). This contradiction implies that (**2**.7) is true.

Case (iii): If $(H_1)(1)$ and $(H_2)(2)$ hold, according to (2.9), we have

$$\begin{aligned} 0 &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) - e(t_1) \\ &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - e(t_1), \end{aligned}$$

which contradicts that (**2**.6). This contradiction implies that (**2**.7) is true.

Case (iv): If $(H_1)(2)$ and $(H_2)(2)$ hold, according to (2.9), we have

$$\begin{aligned} 0 &> g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - e(t_1) \\ &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - e(t_1), \end{aligned}$$

which contradicts that (**2**.6). This contradiction implies that (**2**.7) is true.

Suppose that (2.10)(or (2.11), or (2.12)) holds; using methods similar to those used in Case(i)–(iv), we can show that (**2**.7) is true. This completes the proof of the above claim.

Let $t_2 = kT + \widetilde{t}_2$, where $\widetilde{t}_2 \in [0,T]$ and k is an integer. Then noticing $x(t) = x(t+T)$ and (**2**.7), for any $t \in [\widetilde{t}_2, \widetilde{t}_2 + T]$, we obtain

$$|x(t)| = \left| x(\widetilde{t}_2) + \int_{\widetilde{t}_2}^{t} x'(s)ds \right| \le D + \int_{\widetilde{t}_2}^{t} |x'(s)|ds$$

and

$$\begin{aligned} |x(t)| &= \left| x(\widetilde{t}_2 + T) + \int_{\widetilde{t}_2+T}^{t} x'(s)ds \right| \\ &\le D + \left| -\int_{t}^{\widetilde{t}_2+T} x'(s)ds \right| \\ &\le D + \int_{t}^{\widetilde{t}_2+T} |x'(s)|ds. \end{aligned}$$

Combining above two inequalities, we get

$$|x(t)| \le D + \frac{1}{2}\int_0^T |x'(s)|ds.$$

Using Schwarz inequality yields

$$\begin{aligned} |x|_\infty &= \max_{t\in[\widetilde{t}_2, \widetilde{t}_2+T]} |x(t)| \\ &\le D + \frac{1}{2}\int_0^T |x'(s)|ds \\ &\le D + \frac{1}{2}|1|_2|x'|_2 = D + \frac{1}{2}\sqrt{T}|x'|_2. \end{aligned} \qquad (\mathbf{2}.13)$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let* (H_1) *hold. Suppose there exist some nonnegative constants* C_0*,* C_1 *and* C_2 *such that*

(H_3): $|f(t,u) - f(t,v)| \le C_0|u-v|$, *for all* $t, u, v \in \mathbb{R}$,

(H_4): $|g_i(t,u) - g_i(t,v)| \le C_i|u-v|$, *for all* $t, u, v \in \mathbb{R}, i = 1, 2$,

(H_5): $C_0\frac{T}{2\pi} + (C_1+C_2)\frac{T^2}{4\pi} < 1$.

*Then (**1**.1) has at most one T-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (**1**.1). Then, we have

$$\begin{aligned} &[x_1(t) - x_2(t)]'' + [f(t, x_1'(t)) - f(t, x_2'(t))] \\ &+[g_1(t, x_1(t-\tau_1(t))) - g_1(t, x_2(t-\tau_1(t)))] \\ &+[g_2(t, x_1(t-\tau_2(t))) - g_2(t, x_2(t-\tau_2(t)))] = 0. \end{aligned} \qquad (\mathbf{2}.14)$$

Set $Z(t) = x_1(t) - x_2(t)$, then, from (**2**.14), we obtain

$$\begin{aligned} &Z''(t) + [f(t, x_1'(t)) - f(t, x_2'(t))] \\ &+[g_1(t, x_1(t - \tau_1(t))) - g_1(t, x_2(t - \tau_1(t)))] \\ &+[g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t)))] = 0. \end{aligned} \qquad (\mathbf{2}.15)$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous T-periodic function in $\mathbb{R}$, there exist two constants $t_{max}, t_{min} \in \mathbb{R}$ such that

$$Z(t_{max}) = \max_{t\in[0,T]} Z(t) = \max_{t\in\mathbb{R}} Z(t), \quad Z(t_{min}) = \min_{t\in[0,T]} Z(t) = \min_{t\in\mathbb{R}} Z(t). \qquad (\mathbf{2}.16)$$

Then we have

$$Z'(t_{max}) = x_1'(t_{max}) - x_2'(t_{max}) = 0, \quad Z''(t_{max}) \leq 0, \qquad (\mathbf{2}.17)$$

and

$$Z'(t_{min}) = x_1'(t_{min}) - x_2'(t_{min}) = 0, \quad Z''(t_{min}) \geq 0. \qquad (\mathbf{2}.18)$$

In view of (**2**.15)–(**2**.18), we get

$$\begin{aligned} & g_1(t_{max}, x_1(t_{max} - \tau_1(t_{max}))) - g_1(t_{max}, x_2(t_{max} - \tau_1(t_{max}))) \\ & +g_2(t_{max}, x_1(t_{max} - \tau_2(t_{max}))) - g_2(t_{max}, x_2(t_{max} - \tau_2(t_{max}))) \\ = \; & -Z''(t_{max}) - [f(t_{max}, x_1'(t_{max})) - f(t_{max}, x_2'(t_{max}))] \\ = \; & -Z''(t_{max}) \geq 0 \end{aligned} \qquad (\mathbf{2}.19)$$

and

$$\begin{aligned} & g_1(t_{min}, x_1(t_{min} - \tau_1(t_{min}))) - g_1(t_{min}, x_2(t_{min} - \tau_1(t_{min}))) \\ & +g_2(t_{min}, x_1(t_{min} - \tau_2(t_{min}))) - g_2(t_{min}, x_2(t_{min} - \tau_2(t_{min}))) \\ = \; & -Z''(t_{min}) - [f(t_{min}, x_1'(t_{min})) - f(t_{min}, x_2'(t_{min}))] \\ = \; & -Z''(t_{min}) \leq 0, \end{aligned} \qquad (\mathbf{2}.20)$$

which implies there exists a constant $t_0 \in \mathbb{R}$ such that

$$\begin{aligned} & g_1(t_0, x_1(t_0 - \tau_1(t_0))) - g_1(t_0, x_2(t_0 - \tau_1(t_0))) \\ & +g_2(t_0, x_1(t_0 - \tau_2(t_0))) - g_2(t_0, x_2(t_0 - \tau_2(t_0))) = 0. \end{aligned} \qquad (\mathbf{2}.21)$$

From (H_1) and (**2**.21), we have

$$\begin{aligned} & Z(t_0 - \tau_1(t_0))Z(t_0 - \tau_2(t_0)) \\ & = (x_1(t_0 - \tau_1(t_0)) - x_2(t_0 - \tau_1(t_0)))(x_1(t_0 - \tau_2(t_0)) - x_2(t_0 - \tau_2(t_0))) \\ & \leq 0, \end{aligned}$$

which implies there exists a constant $t_{00} \in \mathbb{R}$, such that

$$Z(t_{00}) = 0$$

Set $t_{00} = nT + \widetilde{t}_0$, where $\widetilde{t}_0 \in [0, T]$ and n is an integer. Noticing $Z(t + T) = Z(t)$, we get

$$Z(\widetilde{t}_0) = Z(nT + \widetilde{t}_0) = Z(t_{00}) = 0. \qquad (\mathbf{2}.22)$$

Hence, for any $t \in [\widetilde{t}_0, \widetilde{t}_0 + T]$, we obtain

$$|Z(t)| = \left| Z(\widetilde{t}_0) + \int_{\widetilde{t}_0}^{t} Z'(s)ds \right| \leq \int_{\widetilde{t}_0}^{t} |Z'(s)|ds$$

and

$$\begin{aligned} |Z(t)| &= \left| Z(\widetilde{t}_0 + T) + \int_{\widetilde{t}_0+T}^{t} Z'(s)ds \right| \\ &= \left| -\int_{t}^{\widetilde{t}_0+T} Z'(s)ds \right| \leq \int_{t}^{\widetilde{t}_0+T} |Z'(s)|ds. \end{aligned}$$

Combining above two inequalities, we get

$$|Z(t)| \leq \frac{1}{2}\int_0^T |Z'(s)|ds.$$

Using Schwarz inequality yields

$$|Z|_\infty = \max_{t\in[\widetilde{t}_0,\widetilde{t}_0+T]} |Z(t)| \leq \frac{1}{2}\int_0^T |Z'(s)|ds \leq \frac{1}{2}|1|_2|Z'|_2 = \frac{1}{2}\sqrt{T}|Z'|_2. \qquad (\mathbf{2}.23)$$

Multiplying $Z''(t)$ and (**2**.15) and then integrating it from 0 to T, by Lemma 2.1, (H_3), (H_4), (**2**.23) and Schwarz inequality, we obtain

$$\begin{aligned} |Z''|_2^2 &= -\int_0^T [f(t,x_1'(t)) - f(t,x_2'(t))]Z''(t)dt \\ &\quad -\int_0^T [g_1(t,x_1(t-\tau_1(t))) - g_1(t,x_2(t-\tau_1(t)))]Z''(t)dt \\ &\quad -\int_0^T [g_2(t,x_1(t-\tau_2(t))) - g_2(t,x_2(t-\tau_2(t)))]Z''(t)dt \\ &\leq \int_0^T |f(t,x_1'(t)) - f(t,x_2'(t))||Z''(t)|dt \\ &\quad +\int_0^T |g_1(t,x_1(t-\tau_1(t))) - g_1(t,x_2(t-\tau_1(t)))||Z''(t)|dt \\ &\quad +\int_0^T |g_2(t,x_1(t-\tau_2(t))) - g_2(t,x_2(t-\tau_2(t)))||Z''(t)|dt \\ &\leq \int_0^T C_0|x_1'(t) - x_2'(t)||Z''(t)|dt \\ &\quad +\int_0^T C_1|x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))||Z''(t)|dt \\ &\quad +\int_0^T C_2|x_1(t-\tau_2(t)) - x_2(t-\tau_2(t))||Z''(t)|dt \\ &\leq \int_0^T C_0|Z'(t)||Z''(t)|dt + \int_0^T C_1|Z(t-\tau_1(t))||Z''(t)|dt \\ &\quad +\int_0^T C_2|Z(t-\tau_2(t))||Z''(t)|dt \end{aligned}$$

$$
\begin{aligned}
&\leq C_0\left(\int_0^T |Z'(t)|^2dt\right)^{1/2}\left(\int_0^T |Z''(t)|^2dt\right)^{1/2}\\
&\quad +C_1\left(\int_0^T |Z(t-\tau_1(t))|^2dt\right)^{1/2}\left(\int_0^T |Z''(t)|^2dt\right)^{1/2}\\
&\quad +C_2\left(\int_0^T |Z(t-\tau_2(t))|^2dt\right)^{1/2}\left(\int_0^T |Z''(t)|^2dt\right)^{1/2}\\
&\leq C_0|Z'|_2|Z''|_2+(C_1+C_2)\sqrt{T}|Z|_\infty|Z''|_2\\
&\leq \left[C_0\frac{T}{2\pi}+(C_1+C_2)\frac{T^2}{4\pi}\right]|Z''|_2^2 \qquad (\mathbf{2}.24)
\end{aligned}
$$

Since $Z(t), Z'(t)$ and $Z''(t)$ are continuous T-periodic functions, by Lemma 2.1, (H_5) and ($\mathbf{2}$.23), we get

$$
Z(t)=Z'(t)=Z''(t)=0, \quad \text{for any} \quad t\in[\widetilde{t}_0,\widetilde{t}_0+T].
$$

Thus, $x_1(t)\equiv x_2(t)$, for all $t\in\mathbb{R}$. Hence, ($\mathbf{1}$.1) has at most one T-periodic solution. This completes the proof. □

Lemma 2.4. *Suppose* (H_1)–(H_5) *hold.*
Then the set of T-periodic solutions of ($\mathbf{2}$.1) *is bounded in* C_T^1.

Proof. Let $S\subset C_T^1$ be the set of T-periodic solutions of ($\mathbf{2}$.1). If $S=\emptyset$, the proof is ended. Suppose $S\neq\emptyset$, and let $x\in S$. Since $f(t,0)=0$ for all $t\in\mathbb{R}$, by Lemma 2.1, Lemma 2.2, (H_3), (H_4), ($\mathbf{2}$.1) and Schwarz inequality, we obtain

$$
\begin{aligned}
|x''|_2^2 &= -\lambda\int_0^T f(t,x'(t))x''(t)dt-\lambda\int_0^T g_1(t,x(t-\tau_1(t)))x''(t)dt\\
&\quad -\lambda\int_0^T g_2(t,x(t-\tau_2(t)))x''(t)dt+\lambda\int_0^T e(t)x''(t)dt\\
&= -\lambda\int_0^T [f(t,x'(t))-f(t,0)]x''(t)dt\\
&\quad -\lambda\int_0^T [g_1(t,x(t-\tau_1(t)))-g_1(t,0)+g_1(t,0)]x''(t)dt\\
&\quad -\lambda\int_0^T [g_2(t,x(t-\tau_2(t)))-g_2(t,0)+g_2(t,0)]x''(t)dt\\
&\quad +\lambda\int_0^T e(t)x''(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^T |f(t,x'(t))-f(t,0)||x''(t)|dt \\
&\quad + \int_0^T |g_1(t,x(t-\tau_1(t)))-g_1(t,0)||x''(t)|dt \\
&\quad + \int_0^T |g_1(t,0)||x''(t)|dt \\
&\quad + \int_0^T |g_2(t,x(t-\tau_2(t)))-g_2(t,0)||x''(t)|dt \\
&\quad + \int_0^T |g_2(t,0)||x''(t)|dt + \int_0^T |e(t)||x''(t)||dt \\
&\le \int_0^T C_0|x'(t)||x''(t)|dt + \int_0^T C_1|x(t-\tau_1(t))||x''(t)|dt \\
&\quad + G_1\int_0^T |x''(t)|dt + \int_0^T C_2|x(t-\tau_2(t))||x''(t)|dt \\
&\quad + G_2\int_0^T |x''(t)|dt + |e|_\infty \int_0^T |x''(t)|dt \\
&\le C_0\left(\int_0^T |x'(t)|^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2} \\
&\quad + C_1\left(\int_0^T |x(t-\tau_1(t))|^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2} \\
&\quad + C_2\left(\int_0^T |x(t-\tau_2(t))|^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2} \\
&\quad + (G_1+G_2+|e|_\infty)\left(\int_0^T 1^2dt\right)^{1/2}\left(\int_0^T |x''(t)|^2dt\right)^{1/2} \\
&\le C_0|x'|_2|x''|_2^2 + (C_1+C_2)\sqrt{T}|x|_\infty|x''|_2 \\
&\quad + (G_1+G_2+|e|_\infty)|1|_2|x''|_2 \\
&\le \frac{C_0}{2\pi}T|x''|_2^2 + (C_1+C_2)\sqrt{T}(D+\frac{1}{2}\sqrt{T}|x'|_2)|x''|_2 \\
&\quad + (G_1+G_2+|e|_\infty)\sqrt{T}|x''|_2 \\
&\le \left[C_0\frac{T}{2\pi} + (C_1+C_2)\frac{T^2}{4\pi}\right]|x''|_2^2 + G_3\sqrt{T}|x''|_2, \qquad (\mathbf{2}.25)
\end{aligned}
$$

where $G_1 = max\{|g_1(t,0)| : t \in [0,T]\}$, $G_2 = max\{|g_2(t,0)| : t \in [0,T]\}$ and $G_3 = G_1 + G_2 + |e|_\infty + (C_1+C_2)D$.
By (H_5) and (**2**.25), there exists a constant $M_0 > 0$ such that

$$|x''|_2 < M_0. \qquad (\mathbf{2}.26)$$

Since $x(0) = x(T)$, there exists a constant $\tilde{t} \in [0,T]$ such that $x'(\tilde{t}) = 0$. For any $t \in [\tilde{t}, \tilde{t}+T]$, by Schwarz inequality, we have

$$|x'(t)| = \left|x'(\tilde{t}) + \int_{\tilde{t}}^t x''(s)ds\right| \le \int_0^T |x''(s)|ds \le |1|_2|x''|_2 = \sqrt{T}|x''|_2,$$

which implies

$$|x'|_\infty = \max_{t\in[\bar{t},\bar{t}+T]} |x'(t)| \le \sqrt{T}|x''|_2. \tag{2.27}$$

By Lemma 2.1, Lemma 2.2, (**2**.26) and (**2**.27), there exists a constant $M > M_0$ such that

$$|x|_\infty < M \text{ and } |x'|_\infty < M.$$

This completes the proof. □

The following Mawhin's continuation is useful in obtaining the existence of T-periodic solutions of (**1**.1).

Let X and Y be real Banach spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and dim Ker$L$=dim(Y/ImL)$< +\infty$. Consider the supplementary subspaces X_1, Y_1 of X, Y respectively, such that $X = X_1 \bigoplus \text{Ker}L$ and $Y = Y_1 \bigoplus \text{Im}L$ and let $P : X \to \text{Ker}L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $\text{Ker}L \cap (D(L) \cap X_1)) = \{0\}$, thus the restriction $L_P := L_{D(L)\cap X_1}$ is invertible. Denote by L_P^{-1} the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \ne \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $L_P^{-1}(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.5. ([2, p. 40]) *Let X and Y be two Banach spaces. Suppose that* $L : D(L) \subset X \to Y$ *is a Fredholm operator with index zero and* $N : X \to Y$ *is L-compact on* $\overline{\Omega}$, *where* Ω *is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

(i) $Lx \ne \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(ii) $Nx \overline{\in} ImL$, *for all* $x \in \partial\Omega \cap KerL$;

(iii) the Brouwer degree $deg\{JQN, \Omega \cap KerL, 0\} \ne 0$,

where $J : ImQ \to KerL$ *is an isomorphism.*

Then equation $Lx = Nx$ *has at least one solution on* $\overline{\Omega} \cap D(L)$.

3. Main results

Now we are in the position to give our main results.

Theorem 1. *Suppose* (H_1)–(H_5) *hold. Then (**1**.1) has a unique T-periodic solution.*

Proof. By Lemma 2.4, there exists a constant $M > D$ such that, for any T-periodic solution $x(t)$ of (**2**.1)

$$|x|_\infty < M \text{ and } |x'|_\infty < M. \tag{3.1}$$

Set

$$\Omega = \{x : x \in C_T^1, |x|_\infty < M, |x'|_\infty < M\}. \tag{3.2}$$

Define a linear operator $L : D(L) \subset C_T^1 \to C_T$ by setting

$$D(L) = \{x : x \in C_T^1, x'' \in C(\mathbb{R}, \mathbb{R})\}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{3.3}$$

We also define a nonlinear operator $N : C_T^1 \to C_T$ by setting

$$Nx = -f(t, x'(t)) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + e(t), \tag{3.4}$$

Then, (**2**.1) is equivalent to the following operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1). \tag{3.5}$$

It is easy to see that

$$KerL = \mathbb{R} \text{ and } ImL = \left\{x : x \in C_T, \int_0^T x(s)ds = 0\right\},$$

Then, L is a Fredholm operator with index zero.
Also let projectors $P : C_T^1 \to KerL$ and $Q : C_T \to C_T/ImL$ defined by

$$Px = x(0) \quad \text{where } x \in C_T^1$$

and

$$Qx = \frac{1}{T}\int_0^T x(s)ds \quad \text{where } x \in C_T,$$

Hence, ImP=ImQ=Ker$L$=$\mathbb{R}$ and Ker$Q$=ImL.
Define the isomorphism as follows

$$J : ImQ \to KerL, \quad J(x) = x. \tag{\textbf{3}.6}$$

Let

$$L_P := L_{D(L)\cap Ker\, P} : D(L) \cap Ker\, P \to ImL,$$

Then, from [8], L_P has a continuous inverse L_P^{-1} on ImL defined by

$$(L_P^{-1}x)(t) = -\frac{t}{T}\int_0^T (t-s)x(s)ds + \int_0^t (t-s)x(s)ds, \tag{\textbf{3}.7}$$

In view of (**3**.2) and (**3**.7), N is L-compact on $\overline{\Omega}$. Note that (**2**.1) is equivalent to (**3**.5), by Lemma 2.4 and (**3**.5), the set of T-periodic solutions of (**3**.5) is bounded in C_T^1. Hence, there exists a sufficient large positive constant M such that, the operator equation $Lx \neq \lambda Nx$ ($where\ \lambda \in (0,1)$) for all $x \in \partial\Omega$, where M and Ω were defined by (**3**.1) and (**3**.2) respectively; which implies that the operator equation $Lx \neq \lambda Nx$ ($where\ \lambda \in (0,1)$) for all $x \in \partial\Omega \cap D(L)$, and the condition (i) of Lemma 2.5 is satisfied.

In view of (H_2)(1) and (H_2)(2), we will consider two cases as follows:
Case(i): If (H_2)(1) holds. Since

$$QNx = -\frac{1}{T}\int_0^T [f(t,x'(t)) + g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_2(t))) - e(t)]dt;$$

for any $x \in \partial\Omega \cap KerL$, $x = M$ or $x = -M$; and $f(t,0) = 0$ for all $t \in \mathbb{R}$, we obtain

$$QN(M) = -\frac{1}{T}\int_0^T [g_1(t,M) + g_2(t,M) - e(t)]dt < 0 \tag{\textbf{3}.8}$$

and

$$QN(-M) = -\frac{1}{T}\int_0^T [g_1(t,-M) + g_2(t,-M) - e(t)]dt > 0 \tag{\textbf{3}.9}$$

which implies the condition (ii) of Lemma 2.5 is satisfied.
Moreover, define

$$\begin{aligned} H(x,\mu) &= -\mu x + (1-\mu)QNx \\ &= -\mu x - (1-\mu)\frac{1}{T}\int_0^T [f(t,x'(t)) + g_1(t,x(t-\tau_1(t))) \\ &\quad + g_2(t,x(t-\tau_2(t))) - e(t)]dt, \end{aligned}$$

in view of (**3**.8) and (**3**.9), we get

$$xH(x,\mu) < 0, \quad \text{for all } x \in \partial\Omega \cap KerL \text{ and } \mu \in [0,1].$$

Hence, $H(x,\mu)$ is a homotopic transformation, together with (**3**.6) and by using homotopic invariance theorem, we have

$$\begin{aligned} deg\{JQN, \Omega \cap KerL, 0\} &= deg\{QN, \Omega \cap KerL, 0\} \\ &= deg\{-x, \Omega \cap KerL, 0\} \neq 0, \end{aligned}$$

so condition (iii) of Lemma 2.5 is satisfied.

Case(ii): If (H_2)(2) holds. Since

$$QNx = -\frac{1}{T}\int_0^T [f(t, x'(t)) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - e(t)]dt;$$

for any $x \in \partial\Omega \cap KerL$, $x = M$ or $x = -M$; and $f(t, 0) = 0$ for all $t \in \mathbb{R}$, we obtain

$$QN(M) = -\frac{1}{T}\int_0^T [g_1(t, M) + g_2(t, M) - e(t)]dt > 0 \qquad (\mathbf{3}.10)$$

and

$$QN(-M) = -\frac{1}{T}\int_0^T [g_1(t, -M) + g_2(t, -M) - e(t)]dt < 0 \qquad (\mathbf{3}.11)$$

which implies the condition (ii) of Lemma 2.5 is satisfied.
Moreover, define

$$\begin{aligned} H(x, \mu) &= \mu x + (1-\mu)QNx \\ &= \mu x - (1-\mu)\frac{1}{T}\int_0^T [f(t, x'(t)) + g_1(t, x(t-\tau_1(t))) \\ &\quad + g_2(t, x(t-\tau_2(t))) - e(t)]dt, \end{aligned}$$

in view of ($\mathbf{3}$.10) and ($\mathbf{3}$.11), we get

$$xH(x, \mu) > 0, \quad \text{for all } x \in \partial\Omega \cap KerL \text{ and } \mu \in [0, 1].$$

Hence, $H(x, \mu)$ is a homotopic transformation, together with ($\mathbf{3}$.6) and by using homotopic invariance theorem, we have

$$deg\{JQN, \Omega \cap KerL, 0\} = deg\{QN, \Omega \cap KerL, 0\} = deg\{x, \Omega \cap KerL, 0\} \neq 0,$$

so condition (iii) of Lemma 2.5 is satisfied. Therefore, it follows from Lemma 2.5 that there exists a T-periodic solution $x(t)$ of ($\mathbf{1}$.1). The uniqueness of this $x(t)$ is guaranteed by Lemma 2.3. This completes the proof. □

Remark 1. If $f(t, 0) \neq 0$, the problem of the existence and uniqueness of T-periodic solutions to ($\mathbf{1}$.1) can be converted to the following equation

$$x''(t) + f_1(t, x'(t)) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) = e_1(t), \qquad (\mathbf{3}.12)$$

where $f_1(t, x'(t)) = f(t, x'(t)) - f(t, 0), e_1(t) = e(t) - f(t, 0)$. As $f_1(t, 0) = 0$ for all $t \in \mathbb{R}$, ($\mathbf{3}$.12) can be studied by Theorem 1 in this paper.

Remark 2. To our knowledge, except [7,18], [1–3,5,6,8,9,11–16,19] only considered the existence of periodic solutions of many kinds of delay Rayleigh equations. In [7], Li and Huang discussed the existence and uniqueness of periodic solutions of a Rayleigh equation without any delays; in [18], Zhou et al. investigated the existence and uniqueness of periodic solutions of the following Rayleigh equation with a deviating argument :

$$x''(t) + f(x'(t)) + g(t, x(t-\tau(t))) = p(t), \qquad (\mathbf{3}.13)$$

and got some results as follows (for convenience of comparison, here we state their main results in a somewhat different form):

Theorem A. *Suppose* (A_1)(or (A_2)) *and* (A_3) *hold. Then* ($\mathbf{3}$.13) *has a unique T-periodic solution.*

Theorem B. *Suppose* (A_1)(or (A_2)) *and* (A_4) *hold. Then* ($\mathbf{3}$.13) *has a unique T-periodic solution.*

If $f(t, x') = f(x')$ and $g_2(t, x(t-\tau_2(t))) = 0$, we can see that ($\mathbf{1}$.1) reduces to ($\mathbf{3}$.13) and the condition ($H_3$) equals the condition ($A_0$); on the other hand, it is easy to see that the condition (H_5) is weaker than the inequality $C_1\frac{T}{2\pi} + b\frac{T^2}{2\pi} < 1$ in the condition (A_3), so

Theorem 1 improves Theorem A. If $f(t,x') \neq f(x')$ or $g_2(t,x(t-\tau_2(t))) \neq 0$, Theorem 1 have new meaning.

4. Example and remark

In this section, we apply the main results obtained in previous sections to an example.

Example 4.1. Consider the following Rayleigh equation with two deviating arguments

$$x''(t) + f(t,x'(t)) + g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_2(t))) = e(t), \qquad (\mathbf{4}.1)$$

where $T = 2\pi$, $\tau_1(t) = \sin t$, $\tau_2(t) = \cos t$, $f(t,x'(t)) = \frac{1}{20(1+\sin^2 t)}x'(t)$ $\arctan x'(t)$, $g_1(t,x(t-\tau_1(t))) = \frac{1}{30(1+\cos^2 t)}x(t-\sin t)$, $g_2(t,x(t-\tau_2(t))) = \frac{1}{20}(2+\sin t)\arctan(x(t-\cos t)+1)$ and $e(t) = \sin^2 t$.

Set $C_0 = \frac{\pi+1}{40}, C_1 = \frac{1}{30}, C_2 = \frac{3}{20}$, and let D be big enough. Then it is easy to check that all the conditions of Theorem 1 in this paper hold, which implies (**4**.1) has a unique 2π-periodic solution.

Remark 3. (**4**.1) is a very simple version of Rayleigh equation with two deviating arguments, all the results in [1–3, 5–9, 11–16, 18, 19] and the references therein cannot be applicable to (**4**.1) for securing the existence and uniqueness of 2π-periodic solutions, which implies the results in this paper are new and they complement the previously known results.

5. Acknowledgements

The authors are grateful to the referee for her or his careful reading of the manuscript and helpful suggestions on this work. This work was supported by the National Natural Science Foundation of China (No. 71901184), the Youth Science and Technology Innovation Team of Southwest Petroleum University (No. 2019CXTD03) and the Program of Science and Technology of Sichuan Province of China (No. 20YYJC0145).

REFERENCES

[1] Cheng, Z., Ren, J., Periodic solutions for a fourth-order Rayleigh type p-Laplacian delay equation, Nonlinear Anal., 2009, 70: 516–523.

[2] Gaines, R.E., Mawhin, J.L., Coincidence Degree, and Nonlinear Differential Equations, Lecture Notes in Mathematics, vol. 568, Springer-Verlag, Berlin, New York, 1977.

[3] Gao, F., Lu, S., Periodic solutions for a Rayleigh type equation with a variable coefficient ahead of the nonlinear term, Nonlinear Anal.: Real World Appl., 2009, 10: 254–258.

[4] Hardy, G.H., Littlewood, J.E., Polya, G., Inequalities, Cambridge Univ. Press, London, 1988.

[5] Huang, C., He, Y., et al., New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments, Math. Comput. Model., 2007, 46: 604–611.

[6] Invernizzi, S., Zanolin, F., Periodic solutions of a differential delay equation of Rayleigh type, Rend. Sere. Mat. Univ. Padova, 1979, 61: 115–124.

[7] Li, Y., Huang, L., New results of periodic solutions for forced Rayleigh-type equations, J. Comput. Appl. Math., 2008, 221: 98–105.

[8] Lu, S., Ge, W., et al., Periodic solutions for neutral differential equation with deviating arguments, Appl. Math. Comput., 2004, 152: 17–27.

[9] Liang, F., Guo, L., Lu, S., New results of periodic solutions for Rayleigh type p-Laplacian equation with a variable coefficient ahead of the nonlinear term, Nonlinear Anal., 2009, 70: 2072–2077.

[10] Mawhin, J., Periodic solutions of some vector retarded functional differential equations, J. Math. Anal. Appl., 1974, 45: 588–603.

[11] Peng, L., Liu, B., et al., Periodic solutions for a kind of Rayleigh equation with two deviating arguments, J. Franklin. I., 2006, 343: 676–687.

[12] Tang, M., Liu, X., Liu, X., New results on periodic solutions for a kind of Rayleigh equation, Appl. Math., 2009, 54: 79–85.

[13] Wang, Y., Zhang, L., Existence of asymptotically stable periodic solutions of a Rayleigh type equation, Nonlinear Analysis: Theory Methods & Applications, 2009, 71(5-6): 1728–1735.

[14] Wang, G., Yan, J., On existence of periodic solutions of the Rayleigh equation of retarded type, *Internat. J. Math. & Math. Sci.* **23**(1), (2000) 65–68.

[15] Xiao, B., Liu, B., Periodic solutions for Rayleigh type p-Laplacian equation with a deviating argument, Nonlinear Anal.: Real World Appl., 2009, 10: 16–22.

[16] Zanolin, F., Periodic solutions for differential systems of Rayleigh type, Rend. Istit. Mat. Univ., Trieste, 1980, 12(1/2): 69–77.

[17] Zhang, F., Li, Y., Existence and uniqueness of periodic solutions for a kind of duffing type p-Laplacian equation, Nonlinear Anal.: Real World Appl., 2008, 9: 985–989.

[18] Zhou, Q., Xiao, B., et al., Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with a deviating argument, J. Korean Math. Soc., 2007, 44: 673–682.

[19] Zhou, Y., Tang, X., On existence of periodic solutions of a kind of Rayleigh equation with a deviating argument, Nonlinear Anal., 2008, 69: 2355–2361.

[20] Zhang, L., Wang, Y., Li, H.X., Zhang, L., Periodic Solutions and Global Attraction for N-Dimension Discrete-Time Neural Networks with Time-Varying Delays, Bulletin of the Belgian Mathematical Society - Simon Stevin, 2011, 18(3): 483–491.

[21] Tian, J., Zhong, S., Wang, Y., Improved exponential stability criteria for neural networks with time-varying delays, Neurocomputing, 2012, 97: 164–173.

[22] Dai, X., Zhang, Z., Wang, Y., Li, J., Chen, L., Multi-parameter magnetoelectric response modeling of magnetostrictive/piezoelectric laminate composites considering shear strain, Journal of Applied Physics, 2014, 115(014104): 1:8, DOI: 10.1063/1.4860992.

[23] Wang, Y., Novel existence and uniqueness criteria for periodic solutions of a Duffing type p-Laplacian equation, Applied Mathematics Letters, 2010, 23(4): 436–439.

[24] Wang, Y., Yi, X.Y., Transient pressure behavior of a fractured vertical well with a finite-conductivity fracture in triple media carbonate reservoir, Journal of Porous Media, 2017, 20(8): 707–722.

[25] Wang, Y., Tao, Z.W., Chen, L., Ma, X., The nonlinear oilCwater two-phase flow behavior for a horizontal well in triple media carbonate reservoir, Acta Geophysica, 2017, 65(5): 977–989.

[26] Wang, Y., Zhang, C., Chen, T., Ma, X., Modeling the nonlinear flow for a multiple-fractured horizontal well with multiple finite-conductivity fractures in triple media carbonate reservoir, Journal of Porous Media, 2018, 21(12): 1283–1305.

[27] Wang, Y., Zhang, C., Huang, R., Cao, H.T., Ma, X., Modeling oil-water two-phase flow behavior of a fractured vertical well with a finite-conductivity fracture in triple media carbonate reservoir, Advances in Applied Mathematics and Mechanics, 2018, 10(3): 581–610.

[28] Wang, Y., Li, M.J., Ma, X., Gao, W.B., Xue, G.Q., Zhou, J., Modeling the nonlinear oil-water two-phase flow behavior for a multiple-fractured horizontal well in triple media carbonate reservoir, Advances in Applied Mathematics and Mechanics, 2018, 10(4): 998–1024.

[29] Wang, Y., Yi, X.Y., Flow modeling of well test analysis for a multiple-fractured horizontal well in triple media carbonate reservoir, International Journal of Nonlinear Sciences and Numerical Simulation, 2018, 19(5): 439–457.

[30] Wang, Y., Yi, X.Y., Some Results for Periodic Solutions of a Kind of Liénard Equation, Journal of Function Spaces, 2015, Article ID 519747, DOI: 10.1155/2015/519747.

[31] Wang, Y., Dai, X.Z., Xia, X.X., On the existence of a unique periodic solution to a Liénard type p-Laplacian non-autonomous equation, Nonlinear Analysis: Theory Methods & Applications, 2009, 71(1-2): 275–280.

[32] Wang, Y., Dai, X.Z., Existence and stability of periodic solutions of a Rayleigh type equation, BULLETIN of the Australian Mathematical Society, 2009, 79(3): 377–390.

[33] Wang, Y., Dai, X.Z., New Results on Existence of Asymptotically Stable Periodic Solutions of a Forced Liénard Type Equation, Results in Mathematics, 2009, 54(3-4): 359–375.

[34] Wang, Y., New results of periodic solutions for a class of delay Rayleigh equation, Bulletin of the Belgian Mathematical Society-Simon Stevin, 2009, 16(3): 409–420.

[35] Wang, Y., New results of periodic solutions for a Rayleigh equation with a deviating argument, Zeitschrift fur angewandte Mathematik und Physik, 2009, 60(4): 583–593.

[36] Wang, Y., Tian, J.K., Periodic solutions for a Liénard equation with two deviating arguments, Electronic Journal of Differential Equations, 2009, Article ID 140.

[37] Ma, X., Liu, Z.B., Wang, Y., Application of a novel nonlinear multivariate grey Bernoulli model to predict the tourist income of China, Journal of Computational and Applied Mathematics, 2019, 347: 84–94.

[38] Wu, W.Q., Ma, X., Zeng, B., Wang, Y., Cai, W., Application of the novel fractional grey model FAGMO(1,1,k) to predict China's nuclear energy consumption, Energy, 2018, 165: 223–234.

[39] Ma, X., Xie. M., Wu. W.Q., Zeng, B., Wang, Y., Wu, X.X., The novel fractional discrete multivariate grey system model and its applications, Applied Mathematical Modelling, 2019, 70: 402–424.

[40] Wu, W.Q., Ma, X., Zeng, B., Wang, Y., Cai, W., Forecasting short-term renewable energy consumption of China using a novel fractional nonlinear grey Bernoulli model, Renewable Energy, 2019, 140: 70–87.

[41] Wang, Y., Tian, D.H., Li, G.F., Zhang, C., Chen, T., Dynamic analysis of a fractured vertical well in triple media carbonate reservoir, Chemistry and Technology of Fuels and Oils, 2019, 55(1): 56–65.

[42] Ma, X., Wu, W.Q., Zeng, B., Wang, Y., Wu, X.X., The conformable fractional grey system model, ISA Transactions, 2019, doi:10.1016/j.isatra.2019.07.009.

[43] Ma, X., Xie, M., Wu, W.Q., Wu, X.X., Zeng, B., A novel fractional time delayed grey model with Grey Wolf Optimizer and its applications in forecasting the natural gas and coal consumption in Chongqing China, Energy, 2019, 178: 487–507.

[44] Ma, X., Liu, Z.B., The GMC (1, n) Model with Optimized Parameters and Its Application, Journal of Grey System, 2017, 29(4): 122–138.

[45] Xia, J., Ma, X., Wu, W.Q., Huang, B.L., Li, W.P., Application of a new information priority accumulated grey model with time power to predict short-term wind turbine capacity, Journal of Cleaner Production, 2020, 244, Article ID 118573, doi: 10.1016/j.jclepro.2019.118573.

[46] Zhao, Y.L., Shan, B.C., Zhang, L.H., Liu, Q.G., Seepage flow behaviors of multi-stage fractured horizontal wells in arbitrary shaped shale gas reservoirs, Journal of Geophysics and Engineering, 2016, 13(5): 1–10, DOI: 10.1088/1742-2132/13/5/674.

[47] Zhao, Y.L., Zhang L.H., Xiong, Y., Zhou, Y.H., Liu, Q.G., Chen, D., Pressure response and production performance for multi-fractured horizontal wells with complex seepage mechanism in box-shaped shale gas reservoir, Journal of Natural Gas Science and Engineering, 2016, 32: 66–80, DOI: 10.1016/j.jngse.2016.04.037.

[48] Zhao, Y.L., Zhang L.H., Feng, G.Q., Zhang, B.N., Kang, B., Performance Analysis of Fractured Wells with Stimulated Reservoir Volume in Coal Seam Reservoirs, Oil & Gas Science and Technology, 2016, 71(1): 8, DOI: 10.2516/ogst/2014026.

[49] Ren, J.J., Guo, P., A general analytical method for transient flow rate with the stress-sensitive effect, Journal of Hydrology, 2018, 565: 262–275.

[50] Ren, J.J., Guo, P., Analytical method for transient flow rate with the effect of the quadratic gradient term, Journal of Petroleum Science and Engineering, 2018, 162: 774–784.

[51] Ren, J.J., Guo, P., Nonlinear flow model of multiple fractured horizontal wells with stimulated reservoir volume including the quadratic gradient term, Journal of Hydrology, 2017, 554: 155–172.

[52] Ren, J.J, Guo, P., Performance of vertical fractured wells with multiple finite-conductivity fractures, Journal of Geophysics and Engineering, 2015, 12(6): 978–987.

[53] Ren, J.J, Guo, P., Anomalous diffusion performance of multiple fractured horizontal wells in shale gas reservoirs, Journal of Natural Gas Science and Engineering, 2015, 26: 642–651.

[54] Feng, Z.H., Li, F.Y., Liu, J.X., Notes on a Boundary Value Problem with a Periodic Nonlinearity, Optik - International Journal for Light and Electron Optics, 2018, 156: 439–446.

[55] Feng, Z.H., Li, F.Y., Lv, Y., Zhang, S.Q., A Note on Cauchy-Lipschitz-Picard Theorem, Journal of Inequalities and Applications, 2016, 271: 1–6, DOI: 10.1186/s13660-016-1214-x.

[56] Feng, Z.H., Wu, F., Li, H.X., Multiple solutions for a modified Kirchhoff-type equation in R^N, Mathematical Methods in the Applied Sciences, 2015, 38(4): 708–725.

[57] Feng, Z.H., Wu, X.X., Yang, L., Stability of a mathematical model with piecewise constant arguments for tumor-immune interaction under drug therapy, International Journal of Bifurcation and Chaos, 2019, 29(1), Article ID 1950009, DOI: 10.1142/S0218127419500093.

[58] Zhang L., Xing, Y.F., Extremal Solutions for Nonlinear First-Order Impulsive Integro-Differential Dynamic Equations, Mathematical Notes, 2019, 105(1): 124–132, DOI: 10.1134/S0001434619010139.

[59] Tian, D.H., Yang, B.W., Chen, J.H., Zhao, Y., A multi-experts and multi-criteria risk assessment model for safety risks in oil and gas industry integrating risk attitudes, Knowledge-Based Systems, 2018, 156: 62–73.

[60] Tian, D.H., Zhao, C.L., Wang, B., Zhou, M., A MEMCIF-IN method for safety risk assessment in oil and gas industry based on interval numbers and risk attitudes, Engineering Applications of Artificial Intelligence, 2019, 85: 269–283.