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# FILTERS THEORY OF SMARANDACHE RESIDUATED LATTICE

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*Residuated lattices play an important role in the study of fuzzy logic and filters are basic concepts in residuated lattices and other algebraic structures. The aim of this paper is to introduce a Smarandache $B_L$-residuated lattice. Then we define the notions of $B_L$-Smarandache deductive systems and $B_L$-Smarandache filters and investigate the relations among them. It is proved that every $B_L$-Smarandache filter is a $B_L$-Smarandache deductive system, but the converse may not be true, while filters and deductive systems are equivalent in the most of algebraic structures. Finally we introduce the concept of $B_L$-Smarandache implicative, positive implicative and fantastic filters and obtain some relationships between them.*

**Keywords:** $BL$-algebra, residuated lattice, Smarandache $B_L$-residuated lattice, $B_L$-Smarandache deductive system, $B_L$-Smarandache (implicative, positive implicative, fantastic) filter.

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## 1. Introduction and Preliminaries

It is well known that certain information processing, especially inferences based on certain information, is based on the classical logic (classical two-valued logic). Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. For this reason, various kinds of non-classical logic systems have been extensively proposed and researched. In fact, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. On the other hand, various logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, residuated lattices, $MV$-algebras, $BL$-algebras, lattice implication algebras, $MTL$-algebras and $NM$-algebras, etc. Among these logical algebras, residuated lattices are basic and important algebraic structures because the other logical algebras are all particular cases of residuated lattices. Residuated lattices, are introduced by Ward and Dilworth in [7].

A Smarandache structure on a set $A$ means a weak structure $W$ on $A$ such that there exists a proper subset $B$ of $A$ which is embedded with a strong structure $S$. In [6], Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. It will be very interesting to study the Smarandache structure in

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these algebraic structures. A. Borumand Saeid and et.al. defined the Smarandache structure in $BL$-algebras in [1].
It is clear that any $BL$-algebra is a residuated lattice. A $BL$-algebra is a weaker structure than residuated lattice, then we can consider in any residuated lattice a weaker structure as $BL$-algebra [2].
In the following, some preliminary theorems and definitions are stated from [3]. In sections 2, we introduce the notions of Smarandache $B_L$-residuated lattices, $B_L$-Smarandache filters and $B_L$-Smarandache deductive systems. By an example We show that $B_L$-Smarandache filters and $B_L$-Smarandache deductive systems are not equivalent. In sections 3, we define $B_L$-Smarandache implicative filters and $B_L$-Smarandache positive implicative filters. Then we determine relationships between these filters. In section 4 we introduce $B_L$-Smarandache fantastic filters and obtain some related results.

**Definition 1.1.** [3] *A residuated lattice is an algebra* $L$=$(L, \wedge, \vee, \odot, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *equipped with an order* $\leq$ *satisfying the following:*
*($LR_1$)* $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
*($LR_2$)* $(A, \odot, 1)$ *is a commutative ordered monoid,*
*($LR_3$)* $\odot$ *and* $\to$ *form an adjoint pair, i.e.* $c \leq a \to b \Leftrightarrow a \odot c \leq b$, *for all* $a, b, c \in A$.
*We denote* $a^* = a \to 0$, *for all* $a \in A$.

A $BL$-algebra is a residuated lattice $L$ satisfying the following identity, for all $a, b \in L$ :
($BL_1$) $(a \to b) \vee (b \to a) = 1$,
($BL_2$) $a \wedge b = a \odot (a \to b)$ .

**Theorem 1.1.** [3] *Let* $L$ *be a residuated lattice. Then the following properties hold for all* $x, y, z \in L$
*($lr_1$)* $x \to x = 1, 1 \to x = x$,
*($lr_2$)* $x \to y \leq (z \to x) \to (z \to y)$,
*($lr_3$)* $x \to y \leq (y \to z) \to (x \to z)$,
*($lr_4$)* $x \leq y \Leftrightarrow x \to y = 1$,
*($lr_5$)* $x \to (y \to z) = y \to (x \to z) = (x \odot y) \to z$,
*($lr_6$)* $x \odot (x \to y) \leq y$, $x \leq (y \to (x \odot y))$ *and* $y \leq (y \to x) \to x$,
*($lr_7$) If* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$.

The following example shows a residuated lattice that is not a $BL$-algebra

**Example 1.1.** *Let* $L = \{0, a, b, c, d, 1\}$ *with* $0 < a, b < c < d < 1$, *but* $a, b$ *are incomparable. Then* $L$ *becomes a residuated lattice relative to the following operations:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | $a$ | $a$ |
| $b$ | 0 | 0 | 0 | 0 | $b$ | $b$ |
| $c$ | 0 | 0 | 0 | 0 | $c$ | $c$ |
| $d$ | 0 | 0 | 0 | 0 | $d$ | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

| $\to$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $d$ | 1 | $d$ | 1 | 1 | 1 |
| $b$ | $d$ | $d$ | 1 | 1 | 1 | 1 |
| $c$ | $d$ | $d$ | $d$ | 1 | 1 | 1 |
| $d$ | 0 | $a$ | $b$ | $c$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

*It is easy to check that* $L$ *is not a* $BL$*-algebra, because* $(a \to b) \vee (b \to a) = d \neq 1$.

## 2. **Smarandache $B_L$-residuated lattices**

**Definition 2.1.** *A Smarandache $B_L$-residuated lattice is a residuated lattice $L$ in which there exists a proper subset $B$ of $L$ such that:*
*($SB_1$) $0, 1 \in B$ and $\mid B \mid > 2$,*
*($SB_2$) $B$ is a $BL$-algebra under the operations of $L$.*

**Example 2.1.** *Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but $a, b$ are incomparable. Then $L$ becomes a residuated lattice relative to the following operations:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | $a$ | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ | $b$ |
| $c$ | 0 | $a$ | $b$ | $c$ | $c$ |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | $b$ | 1 | 1 |
| $b$ | $a$ | $a$ | 1 | 1 | 1 |
| $c$ | 0 | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

*We can see that $B = \{0, c, 1\}$ is a $BL$-algebra which is properly contained in $L$, then $L$ is a Smarandache $B_L$-residuated lattice.*

**Example 2.2.** *Let $L = \{0, a, b, c, d, e, f, 1\}$ with $0 < d < c < b < a < 1$, $0 < d < e < f < a < 1$ but $b, f$ and $c, e$ are incomparable. Then $L$ becomes a residuated lattice relative to the following operations:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $c$ | $c$ | $c$ | 0 | $d$ | $d$ | $a$ |
| $b$ | 0 | $c$ | $c$ | $c$ | 0 | 0 | $d$ | $b$ |
| $c$ | 0 | $c$ | $c$ | $c$ | 0 | 0 | 0 | $c$ |
| $d$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d$ |
| $e$ | 0 | $d$ | 0 | 0 | 0 | $d$ | $d$ | $e$ |
| $f$ | 0 | $d$ | $d$ | 0 | 0 | $d$ | $d$ | $f$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

| $\to$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $d$ | 1 | $a$ | $a$ | $f$ | $f$ | $f$ | 1 |
| $b$ | $e$ | 1 | 1 | $a$ | $f$ | $f$ | $f$ | 1 |
| $c$ | $f$ | 1 | 1 | 1 | $f$ | $f$ | $f$ | 1 |
| $d$ | $a$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $e$ | $b$ | 1 | $a$ | $a$ | $a$ | 1 | 1 | 1 |
| $f$ | $c$ | 1 | $a$ | $a$ | $a$ | $a$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

*We can see that $B = \{0, 1\}$ is the only $BL$-algebra which is properly contained in $L$, then $L$ is not a Smarandache $B_L$-residuated lattice.*

From now on $L_B = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a Smarandache $B_L$-residuated lattice and B=$(B, \wedge, \vee, \odot, \to, 0, 1)$ is a $BL$-algebra unless otherwise specified.

**Definition 2.2.** *A nonempty subset $F$ of $L_B$ is called a Smarandache deductive system of $L_B$ related to $B$ (or briefly $B_L$-Smarandache deductive system of $L_B$ ) if it satisfies:*
*($DB_1$) $1 \in F$,*
*($DB_2$) if $x \in F$, $y \in B$ and $x \to y \in F$, then $y \in F$.*

**Example 2.3.** *In Example* 2.2, $F_1 = \{c, 1\}$, $F_2 = \{a, c, 1\}$, $F_3 = \{b, c, 1\}$, $F_4 = \{0, c, 1\}$, $F_5 = \{0, b, c, 1\}$, $F_6 = \{0, a, c, 1\}$, $F_7 = \{0, a, b, c, 1\}$ *and* $F_8 = \{1\}$ *are all $B_L$-Smarandache deductive system of $L_B$.*

**Theorem 2.1.** *Let $F$ be a nonempty subset of $L_B$ and $0 \in F$. Then $F$ is a $B_L$-Smarandache deductive system of $L_B$ if only and if $B \subseteq F$.*

*Proof.* Let $a \in B$. We have $0 \to a = 1 \in F$, then by hypothesis we obtain $a \in F$. Hence $B \subseteq F$.
Conversely, since $1 \in B \subseteq F$, then $1 \in F$. Now let $x$, $x \to y \in F$ and $y \in B$, since $B \subseteq F$, then $y \in F$. □

**Definition 2.3.** *A nonempty subset $F$ of $L_B$ is called a Smarandache filter of $L_B$ related to $B$ (or briefly $B_L$-Smarandache filter of $L_B$ ) if it satisfies:*
*($FB_1$) if $x, y \in F$, then $x \odot y \in F$,*
*($FB_2$) if $x \in F$, $y \in B$ and $x \leq y$, then $y \in F$.*

We can see that $F_1$-$F_8$ in Example 2.5 are $B_L$-Smarandache filters of $L$.

**Theorem 2.2.** *Let $F$ be a $B_L$-Smarandache filter of $L_B$, then $F$ is a $B_L$-Smarandache deductive system of $L_B$.*

Note that if $F$ is a $B_L$-Smarandache deductive system of $L_B$, then $F$ satisfies in ($FB_2$).

By the following example we show that if $F$ is a $B_L$-Smarandache deductive system of $L_B$, then ($FB_1$) may not hold.

**Example 2.4.** *Let $L = \{0, a, b, c, d, 1\}$ with $0 < b < a < 1$, $0 < d < c < a < 1$ and $c$ and $d$ are incomparable with $b$. Then $L$ becomes a residuated lattice relative to the following operations:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $b$ | $d$ | $d$ | $a$ |
| $b$ | $c$ | $b$ | $b$ | 0 | 0 | $b$ |
| $c$ | $b$ | $d$ | 0 | $d$ | $d$ | $c$ |
| $d$ | $b$ | $d$ | 0 | $d$ | $d$ | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

| $\to$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 0 | 1 | $b$ | $c$ | $c$ | 1 |
| $b$ | $c$ | 1 | 1 | $c$ | $c$ | 1 |
| $c$ | $b$ | 1 | $b$ | 1 | $a$ | 1 |
| $d$ | $b$ | 1 | $b$ | 1 | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

*We can check that $B = \{0, a, 1\}$ is a $BL$-algebra, which is properly contained in $L$, hence $L$ is a Smarandache $B_L$-residuated lattice. Also $F = \{c, a, 1\}$ is a $B_L$-Smarandache deductive system of $L_B$, while it does not satisfy ($FB_1$), since for $c, a \in F$, $c \odot a = d \notin F$ .*

**Theorem 2.3.** *Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. If $F \subseteq B$, then $F$ is a $B_L$-Smarandache filter of $L_B$.*

In Example 2.2, $F = \{a, c, 1\}$ is both a $B_L$-Smarandache deductive system and a $B_L$-Smarandache filter, while $F \not\subseteq B$. So the converse of the above theorem may not be true.
By the Definition 2.4, we have the following theorem.

**Theorem 2.4.** *Let $F = \{a, 1\}$ be a subset of $L_B$, where $B = \{0, a, 1\}$. Then*
(1) *If $a \neq a^*$, then $F$ is a $B_L$-Smarandache deductive system of $L_B$.*
(2) *If $a = a^*$, then $F$ is not a $B_L$-Smarandache deductive system of $L_B$.*

## 3. $B_L$-Smarandache (positive) implicative filters

**Definition 3.1.** *A nonempty subset $F$ of $L_B$ is called a Smarandache implicative filter of $L_B$ related to $B$ (or briefly $B_L$-Smarandache implicative filter of $L_B$ ) if it satisfies:*
*($FB_1$) $1 \in F$,*
*($FB_3$) if $z \in F$, $x, y \in B$ and $z \to ((x \to y) \to x) \in F$, then $x \in F$.*

$F_1 - F_7$ in Example 2.5 are $B_L$-Smarandache implicative filters of $L$,while $F_8$ is not a $B_L$-Smarandache implicative filter

**Theorem 3.1.** *If $F$ is a $B_L$-Smarandache implicative filter of $L_B$, then $F$ is a $B_L$-Smarandache deductive system of $L_B$.*

*Proof.* Let $F$ be a $B_L$-Smarandache implicative filter of $L_B$, $y \in B$ and $x, x \to y \in F$. Since $x \to ((y \to 1) \to y) = x \to y \in F$, then $x \to ((y \to 1) \to y) \in F$, so by $x \in F$, $1, y \in B$, we get that $y \in F$. Therefore $F$ is a $B_L$-Smarandache deductive system of $L_B$. □

By the following example we show that the converse of the above theorem may not hold.

**Example 3.1.** *Consider $L = \{0, a, b, c, 1\}$ with $0 < c < a < b < 1$ and the following operations:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | $a$ | 0 | $a$ |
| $b$ | 0 | $a$ | $b$ | $c$ | $b$ |
| $c$ | 0 | 0 | $c$ | 0 | $c$ |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $a$ | 1 | 1 | $a$ | 1 |
| $b$ | 0 | $a$ | 1 | $c$ | 1 |
| $c$ | $a$ | 1 | 1 | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

*Then $L = \{0, a, b, c, 1\}$ is a residuated lattice, $B = \{0, b, 1\}$ is a BL-algebra which is properly contained in L, hence L is a Smarandache $B_L$-residuated lattice. Also $F = \{1\}$ is a $B_L$-Smarandache deductive system of $L$ while it is not a $B_L$-Smarandache implicative filter of $L$, since $1 \to ((b \to 0) \to b) = 1 \in F$, $1 \in F$ and $b, 0 \in B$ while $b \notin F$.*

**Theorem 3.2.** *Let $F$ be a nonempty subset of $L_B$ and $0 \in F$. Then $F$ is a $B_L$-Smarandache implicative filter of $L_B$ if only and if $B \subseteq F$.*

*Proof.* Let $a \in B$. We have $0 \to ((a \to 1) \to a) = 1 \in F$, thus by hypothesis $a \in F$. Therefore $B \subseteq F$.

Conversely, since $1 \in B \subseteq F$, then $1 \in F$. Now let $y$, $y \to ((x \to z) \to x) \in F$ and $x, z \in B$. By $B \subseteq F$ we get that $x \in F$. □

**Theorem 3.3.** *Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then $F$ is a $B_L$-Smarandache implicative filter if and only if $(x \to y) \to x \in F$ implies $x \in F$, for all $x, y \in B$.*

**Theorem 3.4.** *Let $F$ be a $B_L$-Smarandache implicative filter of $L_B$. Then $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for all $x, y \in B$.*

*Proof.* Let $F$ be a $B_L$-Smarandache implicative filter and $(x \to y) \to y \in F$. From $x \leq (y \to x) \to x$ and Theorem 1.2 we can conclude that
$((y \to x) \to x) \to y \leq x \to y$.
so we have

$$\begin{aligned}(x \to y) \to y &\leq (y \to x) \to ((x \to y) \to x)\\ &= (x \to y) \to ((y \to x) \to x)\\ &\leq (((y \to x) \to x) \to y) \to ((y \to x) \to x)\end{aligned}$$

Thus by hypothesis we get that$(((y \to x) \to x) \to y) \to ((y \to x) \to x) \in F$ and so by Theorem 3.5 $(y \to x) \to x \in F$. □

By the following example we show that the converse of the above theorem may not hold.

**Example 3.2.** *Consider the residuated lattice in Example* 3.3. $B = \{0, a, 1\}$ *is a $BL$-algebra, which is properly contained in $L$, hence $L$ is a Smarandache $B_L$-residuated lattice. We can see that $F = \{1\}$ is not a $B_L$-Smarandache implicative filter of $L_B$, since $1 \to ((a \to 0) \to a) = 1 \in F$, $1 \in F$ and $a, 0 \in B$ but $a \notin F$. Also $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for $x, y \in B$.*

Similar to Theorem 3.15[5], we can see the following theorem. The proof is similar to this theorem and omitted.

**Theorem 3.5.** *If $F$ is a $B_L$-Smarandache implicative filter of $L_B$, then every $B_L$-Smarandache deductive system $G$ containing $F$ is also a $B_L$-Smarandache implicative filter.*

In the following example we show that the converse of Theorem 3.8 is not true in general.

**Example 3.3.** *In Example* 3.3, $\{1\} \subseteq \{b, 1\}$. *$\{b, 1\}$ is a $B_L$-Smarandache implicative filter of $L$ while $\{1\}$ is not a $B_L$-Smarandache implicative filter of $L$.*

**Corollary 3.1.** *In any Smarandache $B_L$-residuated lattice $L_B$, the following conditions are equivalent:*
*(a) $\{1\}$ is a $B_L$-Smarandache implicative filter,*
*(b) Any $B_L$-Smarandache deductive system of $L_B$ is a $B_L$-Smarandache implicative filter.*

**Definition 3.2.** *A nonempty subset $F$ of $L_B$ is called a Smarandache positive implicative filter of $L_B$ related to $B$ (or briefly $B_L$-Smarandache positive implicative filter of $L_B$) if it satisfies:*

*($FB_1$)* $1 \in F$,
*($FB_4$) if* $x, y, z \in B$ , $z \to (x \to y) \in F$ *and* $z \to x \in F$, *then* $z \to y \in F$.

**Example 3.4.** *In Example* 2.2, $F_1 = \{c, 1\}$, $F_2 = \{a, c, 1\}$, $F_3 = \{b, c, 1\}$, $F_4 = \{0, c, 1\}$, $F_5 = \{0, b, c, 1\}$, $F_6 = \{0, a, c, 1\}$, $F_7 = \{0, a, b, c, 1\}$, $F_8 = \{a, 1\}$, $F_9 = \{b, 1\}$ *and* $F_{10} = \{a, b, 1\}$ *are* $B_L$*-Smarandache positive implicative filters of* $L$.

**Theorem 3.6.** *If* $F$ *is a* $B_L$*-Smarandache implicative filter of* $L_B$*, then* $F$ *is a* $B_L$*-Smarandache positive implicative filter of* $L_B$.

*Proof.* Let for $x, y, z \in B$, $x \to (y \to z) \in F$ and $x \to y \in F$. Then by Theorem 1.2 we have:

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

Hence by hypothesis we have

$$(x \to y) \to (x \to (x \to z)) \in F \Rightarrow (x \to (x \to z)) \in F$$

Now by Theorem 1.2 we get

$$(x \to (x \to z)) \leq ((x \to z) \to z) \to (x \to z).$$

Then $((x \to z) \to z) \to (x \to z) \in F$. Therefore by hypothesis we get that $x \to z \in F$. □

By the following example, we show that the converse of Theorem 3.13 is not true in general.

**Example 3.5.** *In Example* 2.2, $F = \{b, 1\}$ *is a* $B_L$*-Smarandache positive implicative filter of* $L_B$ *while it is not a* $B_L$*-Smarandache implicative filter of* $L_B$*, since* $1 \to ((c \to 0) \to c) = 1 \in F$, $1 \in F$ *and* $c, 0 \in B$ *but* $c \notin F$.

**Theorem 3.7.** *If* $F$ *is a* $B_L$*-Smarandache positive implicative filter of* $L_B$ *which is contained in* $B$*, then* $F$ *is a* $B_L$*-Smarandache deductive system of* $L_B$.

*Proof.* Let $x$, $x \to y \in F$ and $y \in B$. We have $x \in F \subseteq B$, hence $x \in B$. Since $1 \to (x \to y) = x \to y \in F$ and $1 \to x = x \in F$, then by hypothesis $1 \to y = y \in F$. □

**Example 3.6.** *In Example* 2.2, $F = \{b, 1\}$ *is a* $B_L$*-Smarandache positive implicative filter of* $L$ *while it is not a* $B_L$*-Smarandache deductive system of* $L$*, since* $b \to c = 1 \in F$, $b \in F$ *and* $c \in B$ *but* $c \notin F$. *So the condition* $F \subseteq B$*, in the above theorem is necessary.*

In the following example we show that the converse of Theorem 3.15 is not true in general.

**Example 3.7.** *Consider* $B = \{0, a, 1\}$ *in Example* 3.7. *We can see that* $F = \{1\}$ *is a* $B_L$*-Smarandache deductive system of* $L_B$ *while it is not a* $B_L$*-Smarandache positive implicative filter of* $L_B$*, since* $a \to (a \to 0) = 1 \in F$ *and* $a \to a = 1 \in F$, *but* $a \to 0 = a \notin F$.

**Theorem 3.8.** *Let* $F$ *be a* $B_L$*-Smarandache deductive system of* $L_B$*. Then* $F$ *is a* $B_L$*-Smarandache positive implicative filter if and only if for any* $a \in B$, $A_a = \{x \in B \mid a \to x \in F\}$ *is a* $B_L$*-Smarandache deductive system of* $L_B$.

*Proof.* Let $F$ be a $B_L$-Smarandache positive implicative filter, and $a \in B$. Since $a \to 1 = 1 \in F$, then $1 \in A_a$. If $x, x \to y \in A_a$, then $a \to x$, $a \to (x \to y) \in F$.

Since $F$ is a $B_L$-Smarandache positive implicative filter, hence $a \to y \in F$. Then $y \in A_a$. Therefore $A_a$ is a $B_L$-Smarandache deductive system of $L_B$.
Conversely, let $x \to (y \to z)$ and $x \to y \in F$, for $x, y, z \in B$. Then $y \to z$, $y \in A_x$ and so $z \in A_x$ that is $x \to z \in F$. Hence $F$ is a $B_L$-Smarandache positive implicative filter of $L_B$. □

**Theorem 3.9.** *Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then the following are equivalent:*
*(a) $F$ is a $B_L$-Smarandache positive implicative filter,*
*(b) $y \to (y \to x) \in F$ implies $y \to x \in F$, for all $x, y \in B$,*
*(c) $z \to (y \to x) \in F$ implies $(z \to y) \to (z \to x) \in F$, for all $x, y, z \in B$,*
*(d) $z \to (y \to (y \to x)) \in F$ and $z \in F$ implies $(y \to x) \in F$, for all $x, y \in B$.*

*Proof.* $(a) \Rightarrow (b)$: Let $F$ be a $B_L$-Smarandache positive implicative filter, and $y \to (y \to x) \in F$, for $x, y \in B$. Then by $y \to y = 1 \in F$, we get that $y \to x \in F$.
$(b) \Rightarrow (c)$: Let $z \to (y \to x) \in F$. By Theorem 1.2, we have
$(z \to (z \to ((z \to y) \to x)) = (z \to ((z \to y) \to (z \to x)) \geq z \to (y \to x)$.Hence $(z \to (z \to ((z \to y) \to x)) \in F$ and so by (b) we conclude that $(z \to y) \to (z \to x) = z \to ((z \to y) \to x) \in F$, for $x, y, z \in B$.
$(c) \Rightarrow (d)$: Let $z$, $z \to (y \to (y \to x)) \in F$. Since $F$ is a $B_L$-Smarandache deductive system, then $y \to (y \to x) \in F$. Thus $y \to x = (y \to y) \to (y \to x) \in F$.
$(d) \Rightarrow (a)$: Let for $x, y, z \in B$ , $z \to (y \to x) \in F$ and $z \to y \in F$. By Theorem 1.2,we have
$z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x))$, so by hypothesis, $(z \to y) \to (z \to (z \to x)) \in F$. Therefor by $(d)$ we get that $z \to x \in F$. □

**Theorem 3.10.** *If $F$ is a $B_L$-Smarandache positive implicative filter of $L_B$, then every $B_L$-Smarandache deductive system $G$ containing $F$ is also a $B_L$-Smarandache positive implicative filter.*

*Proof.* The proof is similar to the proof of Theorem 3.6[5] □

In the following example we show that the converse of Theorem 3.20 is not true in general.

**Example 3.8.** *In Example 3.7, $\{1\} \subseteq \{0, a, b, 1\}$. $\{0, a, b, 1\}$ is a $B_L$-Smarandache positive implicative filter of $L_B$ while $\{1\}$ is not a $B_L$-Smarandache positive implicative filter of $L$.*

## 4. $B_L$-Smarandache fantastic filters

**Definition 4.1.** *A nonempty subset $F$ of $L_B$ is called a Smarandache fantastic filter of $L_B$ related to $B$ (or briefly $B_L$-Smarandache fantastic filter of $L_B$) if it satisfies:*
*$(FB_1)$ $1 \in F$,*
*$(FB_5)$ if $x, y \in B$ and $z$, $z \to (y \to x) \in F$, then $((x \to y) \to y) \to x \in F$.*

**Example 4.1.** *In Example 2.5, $F_1 - F_8$ are $B_L$-Smarandache fantastic filters of $L$.*

**Theorem 4.1.** *Every $B_L$-Smarandache fantastic filter of $L_B$ is a $B_L$-Smarandache deductive system.*

*Proof.* Let $z$, $z \to x \in F$ and $x \in B$ . Since $z \to x = z \to (1 \to x) \in F$, then $x = ((x \to 1) \to 1) \to x \in F$. □

In the following example we show that the converse of Theorem 4.3 is not true in general.

**Example 4.2.** *In Example* 3.3, $F = \{1\}$ *is a* $B_L$*-Smarandache deductive system of* $L$ *while it is not a* $B_L$*-Smarandache fantastic filter of* $L$*. Since* $1$, $1 \to (0 \to b) = 1 \in F$ *and* $0, b \in B$*, while* $((b \to 0) \to 0) \to b = b \notin F$*.*

**Theorem 4.2.** *Let* $F$ *be a* $B_L$*-Smarandache deductive system of* $L_B$*. Then* $F$ *is* $B_L$*-Smarandache fantastic filter if and only if* $y \to x \in F$ *implies* $((x \to y) \to y) \to x \in F$*, for all* $x, y \in B$*.*

**Theorem 4.3.** *If* $F$ *is a* $B_L$*-Smarandache fantastic filter of* $L_B$*, then every* $B_L$*-Smarandache deductive system* $G$ *containing* $F$ *is also a* $B_L$*-Smarandache fantastic filter.*

*Proof.* The proof is similar to the proof of Theorem 4.4[5] □

In the following example we show that the converse of Theorem 4.6 is not true in general.

**Example 4.3.** *In Example* 3.3, $\{1\} \subseteq \{b, 1\}$. $\{b, 1\}$ *is a* $B_L$*-Smarandache fantastic filter of* $L_B$ *while* $\{1\}$ *is not a* $B_L$*-Smarandache fantastic filter of* $L$*.*

**Theorem 4.4.** *Every* $B_L$*-Smarandache implicative filter of* $L$ *is a* $B_L$*-Smarandache fantastic filter.*

*Proof.* Let $x, y \in B$ and $y \to x \in F$. Since $x \leq ((x \to y) \to y) \to x$, then $(((x \to y) \to y) \to x) \to y \leq x \to y$
and also we have
$((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)$
$\geq (x \to y) \to (((x \to y) \to y) \to x)$
$= ((x \to y) \to y) \to ((x \to y) \to x)$
$\geq y \to x.$
By $y \to x \in F$, we have
$((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \in F,$
since $F$ is a $B_L$-Smarandache implicative filter, we conclude that
$(((x \to y) \to y) \to x) \in F.$ □

In the following example we show that the converse of Theorem 4.8 is not true in general.

**Example 4.4.** *Let* $L = \{0, a, b, c, d, e, f, g, 1\}$ *with* $0 < a < b < e < 1$, $0 < c < f < g < 1$, $a < d < g$, $c < d < e$*, but* $\{a, c\}$, $\{b, d\}$, $\{d, f\}$, $\{b, f\}$ *and respective* $\{e, g\}$ *are incomparable. Then* $L$ *becomes a residuated lattice relative to the following operations:*

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | $a$ | 0 | 0 | $a$ | 0 | 0 | $a$ |
| $b$ | 0 | $a$ | $b$ | 0 | $a$ | $b$ | 0 | $a$ | $b$ |
| $c$ | 0 | 0 | 0 | 0 | 0 | 0 | $c$ | $c$ | $c$ |
| $d$ | 0 | 0 | $a$ | 0 | 0 | $a$ | $c$ | $c$ | $d$ |
| $e$ | 0 | $a$ | $b$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $f$ | 0 | 0 | 0 | $c$ | $c$ | $c$ | $f$ | $f$ | $f$ |
| $g$ | 0 | 0 | $a$ | $c$ | $c$ | $d$ | $f$ | $f$ | $g$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | 1 |

| $\rightarrow$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $g$ | 1 | 1 | $g$ | 1 | 1 | $g$ | 1 | 1 |
| $b$ | $f$ | $g$ | 1 | $f$ | $g$ | 1 | $f$ | $g$ | 1 |
| $c$ | $e$ | $e$ | $e$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $d$ | $d$ | $e$ | $e$ | $g$ | 1 | 1 | $g$ | 1 | 1 |
| $e$ | $c$ | $d$ | $e$ | $f$ | $g$ | 1 | $f$ | $g$ | 1 |
| $f$ | $b$ | $b$ | $b$ | $e$ | $e$ | $e$ | 1 | 1 | 1 |
| $g$ | $a$ | $b$ | $b$ | $d$ | $e$ | $e$ | $g$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | 1 |

*It is easy to check that $B = \{0, d, 1\}$ is a $BL$-algebra which is properly contained in $L$, hence $L$ is a Smarandache $B_L$-residuated lattice. Also $F = \{g, 1\}$ is a $B_L$-Smarandache fantastic filter of $L$ while it is not a $B_L$-Smarandache implicative filter of $L$, since $g \rightarrow ((d \rightarrow 0) \rightarrow d) = 1 \in F$, $g \in F$ and $0, d \in B$ but $0 \notin F$.*

In the following example we show that any $B_L$-Smarandache fantastic filter of $L_B$ may not be a $B_L$-Smarandache positive implicative filter of $L_B$.

**Example 4.5.** *In Example* 3.17, *$F = \{1\}$ is a $B_L$-Smarandache fantastic filter of $L$, while it is not a $B_L$-Smarandache positive implicative filter of $L$.*

In the following example we show that any $B_L$-Smarandache positive implicative filter of $L_B$ may not a $B_L$-Smarandache fantastic filter of $L_B$.

**Example 4.6.** *In Example* 2.2, *$F = \{a, 1\}$ is a $B_L$-Smarandache positive implicative filter of $L$ while it is not a $B_L$-Smarandache fantastic filter of $L$, since $a \rightarrow (1 \rightarrow c) = 1 \in F$, $a \in F$ and $c, 1 \in B$, while $((c \rightarrow 1) \rightarrow 1) \rightarrow c = c \notin F$.*

By the following figure, we determine the relations between the different kinds of filters:

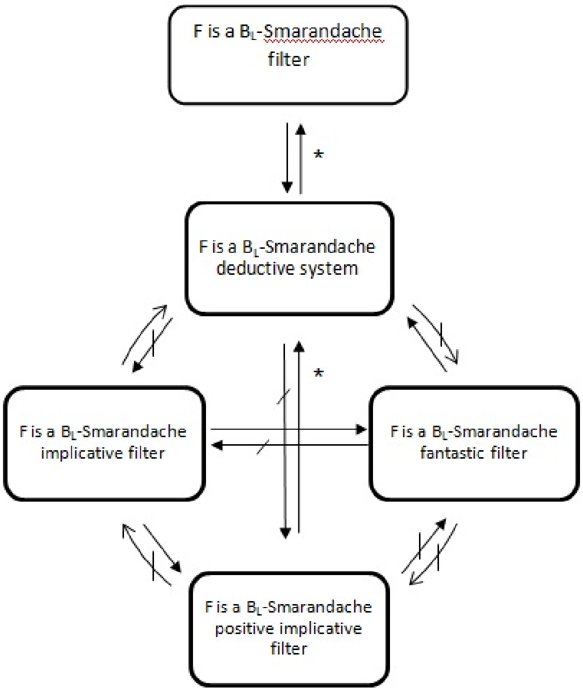


FIGURE 1. $*$: If $F$ contained in $B$

## 5. **Conclusion**

Smarandache structure occurs as a weak structure in any structure. In the present paper, by using this notion we have introduced the concept of Smarandache $B_L$-residuated lattice and investigated some of their useful properties. It is well known that the filters with special properties play an important role in the logic system. The aim of this article is to investigate some kinds of filters on the Smarandache structures and we obtain the related properties.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as lattices and Lie algebras etc. It is our hope that this work would other foundations for further study of the theory of residuated lattices, $BL$-algebras and $MV$-algebras.

In our future study of the Smarandache structure of these algebraic structures, the following topics might be considered:

(1) To get more results in Smarandache $B_L$-residuated lattice and applications;
(2) To get more connections between residuated lattices and $BL$-algebras;
(3) To define some Smarandache structures and some filters on these;
(4) To define fuzzy structure of Smarandache $B_L$-residuated lattice.

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