GLOBAL ATTRACTOR FOR A MATHEMATICAL MODEL OF 2D MAGNETO-VISCOELASTIC FLOWS

Aibo Liu[1], Changchun Liu[2]

We consider a mathematical model for magneto-viscoelastic flows in two dimensional bounded domains. The model couples the Navier–Stokes equations with evolutionary equations for the deformation gradient and the magnetization obtain from a special case of the micromagnetic energy. By the combination of the suitable dissipative estimates with the energy techniques, we establish the existence of a global attractor on a suitable phase-space and prove that the attractor has a regular compact absorbing set.

Keywords: Global attractor, weak solutions, magneto-viscoelastic flows.
MSC2010: 35B41, 76D05, 35Q35, 35D30.

1. Introduction

In this paper, we consider the following magneto-viscoelastic flows

$$\begin{cases} v_t + (v \cdot \nabla)v - \nu\Delta v + \nabla p = \nabla \cdot (FF^\top - \nabla M \odot \nabla M), \\ \nabla \cdot v = 0, \\ F_t + (v \cdot \nabla)F - \nabla v F = \kappa \Delta F, \\ M_t + (v \cdot \nabla)M = \mu, \\ \mu = \Delta M - f(M), \end{cases} \tag{1}$$

in the domain $Q_T := (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is a bounded regular domain with smooth boundary, and $T > 0$ is a given time. Here $v(x,t) : Q_T \to \mathbb{R}^2$ is the velocity field, $F : Q_T \to \mathbb{R}^{2\times 2}$ is the deformation gradient, $M : Q_T \to \mathbb{R}^3$ is the magnetization vector, $p(x,t)$ stands for the fluid pressure, $A^\top$ is the transpose of a matrix A, $(A \odot B)_{i,j} = \sum_{k=1}^m A_{ki}B_{kj}$, and ν, κ are positive constants. Throughout this paper, we use ∂_i to denote $\frac{\partial}{\partial x_i}$ and the Hamilton operator $\nabla = (\partial_1, \partial_2, \cdots, \partial_d)$, the Laplace operator $\Delta = \sum_{i=1}^d \partial_i^2$.

The system (1) is completed with Dirichlet boundary conditions for v, F and the Neumann boundary condition for M

$$v|_{\partial\Omega} = 0, \quad F|_{\partial\Omega} = 0, \quad \frac{\partial M}{\partial n}\Big|_{\partial\Omega} = 0, \tag{2}$$

and the initial conditions

$$\begin{aligned} &v(0,x) = v_0(x), \text{ with } \quad \nabla \cdot v_0 = 0 \quad \text{in } \Omega, \\ &F(0,x) = F_0(x) = I, \quad M(0,x) = M_0(x) \quad \text{in } \Omega, \end{aligned} \tag{3}$$

where I is the 2×2 identity matrix.

[1] Department of Mathematics, Liaoning Normal University, Dalian 116029, China

[2] Corresponding author, Department of Mathematics, Jilin University, Changchun 130012, China, e-mail: liucc@jlu.edu.cn

We set that $f:\mathbb{R}^2\to\mathbb{R}^2$ is a function and $G(r)=\int_0^r f(\xi)d\xi$ is a potential function as follows

$$f(M)=(|M|^2-1)M,\quad G(M)=\frac{1}{4}(|M|^2-1)^2,$$

and introduce the basic energy:

$$\mathcal{E}(t)=\frac{1}{2}\|v\|_{L^2}^2+\frac{1}{2}\|F\|_{L^2}^2+\frac{1}{2}\|\nabla M\|_{L^2}^2+\int_\Omega G(M)dx.$$

Magnetic materials are of great importance in technological applications. Therein, magnetoelastic materials are strongly susceptible to be the phenomenon of converting applied into changes of the magnetic field and vice versa. They can be regarded as smart materials. Magnetoelastic materials have been of interest for a variety of applications. For instance, they can be found in sensors to measure the torque of a force, and used in magnetic actuators and generators for ultrasonic sounds.

The system (1) is a mathematical model for magneto-viscoelastic flows of a typical magnetoelastic material. First derived in [4], the model couples the Navier–Stokes equations with evolutionary equations for the deformation gradient and magnetization obtained as a special case of the micromagnetic energy

$$\begin{cases} v_t+(v\cdot\nabla)v-\nu\Delta v+\nabla p=\nabla\cdot\Big(W'(F)F^\top-2A(\nabla M\odot\nabla M)\Big),\\ \nabla\cdot v=0,\\ F_t+(v\cdot\nabla)F-\nabla vF=\kappa\Delta F,\\ M_t+(v\cdot\nabla)M=2A\Delta M-\frac{1}{\gamma^2}(|M|^2-1)M,\end{cases}\tag{4}$$

where A,γ and κ are positive constants. The elastic density $W:\mathbb{R}^{d\times d}\longrightarrow\mathbb{R}_0^+$ is assumed to have the properties

$$\begin{aligned} &W\in C^2(\mathbb{R}^{d\times d};\mathbb{R}),\quad C_1|\Xi|^2\le W(\Xi)\le C_1(|\Xi|^2+1),\\ &|W'(\Xi)|\le C_2(1+|\Xi|),\quad W'(0)=0,\quad |W''(\Xi)|\le C_3,\\ &(W''(\Xi_1)\nabla\Xi_2)\vdots\nabla\Xi_2\ge a|\nabla\Xi_2|^2\quad a.e.\text{ in }\Omega,\end{aligned}$$

for some positive constants C_1,C_2,C_3,a, any $\Xi\in\mathbb{R}^{d\times d}$, and any $\Xi_1,\Xi_2\in\mathbf{H}^1(\mathbb{R}^{d\times d})$. The system (1) is derived by choosing $A=1/2$, $\mu=1$ and $W(F)=\frac{1}{2}|F|^2$ (which means $W'(F)$=F); obviously, this $W(F)$ satisfies the above conditions, existence of weak solutions has been proved in [4], and the uniqueness of a solution has been studied in [13] for $d=2$ and 3.

Forster [4] also derived another model for magneto-viscoelastic flows as follows:

$$\begin{cases} v_t+(v\cdot\nabla)v-\nu\Delta v+\nabla p=\nabla\cdot(FF^\top-\nabla M\odot\nabla M),\\ \nabla\cdot v=0,\\ F_t+(v\cdot\nabla)F-\nabla vF=\kappa\Delta F,\\ M_t+(v\cdot\nabla)M=-M\times\Delta M-M\times(M\times\Delta M),\end{cases}\tag{5}$$

it couples the Landau–Lifshitz–Gilbert equation (LLG equation) with elasticity in the small strain setting. The existence of weak solutions has been proved in [4] and [2]. More mathematical studies for this model are needed, and we discovered that the model has critical structure or so called self-similar solutions. That is, if (v,F,M) is a solution to the system, the scaling relation solution:

$$(v_\lambda,F_\lambda,M_\lambda):=(\lambda v(\lambda^2t,\lambda x),\lambda F(\lambda^2t,\lambda x),M(\lambda^2t,\lambda x))$$

is also a solution of the system, which means that we can study this system in critical Sobolev or Besov spaces.

We now analyze the structures of the system (1). If $M = 0, \kappa = 0$, the system (1) reduces to a model for incompressible viscoelastic flows, c.f.[6, 7, 8, 9]. If $F = 0$, it reduces to incompressible liquid crystal flows. Grasselli and Wu [5] proved that the system has a finite-dimensional global attractor in $\mathbb{R}^2$ with periodic boundary conditions. You and Li [15] gave the Pullback attractor of this system in two dimensions (see [10]). If $M = F = 0$, it becomes the Navier–Stokes equations, where one has uniqueness of weak solutions in two dimensions and weak-strong uniqueness in three spatial dimensions, also has unique global classical solutions for smallness data or largeness viscosity ν(compared to the initial data), and the global existence of classical solutions for general data in three spatial dimensions is still in open problem. Since the system (1) contains the Navier-Stokes equations as a subsystem, one cannot expect better results than those for Navier-Stokes equations.

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to deal with this problem for a dissipative system is to analyse the existence and structure of its attractor. Generally speaking, the attractor has a very complicated geometry that reflects the complexity of the long-time behavior of the system. During the past years, many authors have paid much attention to the attractor of higher order parabolic equations [11, 12, 17, 18]. In this paper, we study the existence of global attractor for the system (1). The main difficulties for treating the system (1) are caused by the strong coupling nonlinear terms and the Neumann boundary conditions. The results for incompressible liquid crystal flows generally account for the Dirichlet boundary conditions or period boundary conditions.

The plan of the paper is as follows. In section 2, we introduce the associated spaces and recall some useful lemmas for the proof of the global attractor. Section 3 is devoted to the proof of a number of dissipative estimates that entail the existence of compact absorbing sets in the phase space. In section 4, we prove the existence of the global attractor.

2. **Preliminaries**

We intorduce the spaces as follows

$$\mathcal{V} := \{v : v \in C_0^\infty,\ \mathrm{div} v = 0\},$$

$$\mathbf{H} := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega),$$

$$\mathbf{V} := \text{closure of } \mathcal{V} \text{ in } H^1(\Omega),$$

$$H_0^m(\Omega) := \{v \in H^m(\Omega), \quad v|_{\partial\Omega} = 0\},$$

$$H_n^m(\Omega) := \{v \in H^m(\Omega), \quad \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0\},$$

$$H^m(\Omega) := W^{m,2}(\Omega), \text{ the Sobolev spaces } W^{m,p}(\Omega) \text{ with } p = 2,$$

and the phase-space

$$\mathbb{Y} := \mathbf{H} \times L^2(\Omega) \times H_n^1(\Omega).$$

We denote the inner product on L^2 by $(\cdot, \cdot)$ and the associated norm by $\|\cdot\|_{L^2}$. The space $H^m(\Omega)$ will be shorthanded by H^m. Einstein summation convention is used, that is, summation sign is omitted and the sum is over all indices which appear twice. Such as: $a \cdot b = \Sigma_{i=1}^d a_i b_i := a_i b_i$, by this way, we denote $(a \otimes b)_{i,j} = a_i b_j$ for vector a and b. $A : B = A_{ij} B_{ij}$, $\nabla A \vdots \nabla B = \partial_k A_{ij} \partial_k B_{ij}$ and $(A \odot B)_{i,j} = A_{ki} B_{kj}$ for matrix A and B.

The following inequalities are suited for Neumann boundary conditions, and are very useful for the proof of our theorem.

Lemma 2.1. ([3]) *Let $\Omega \subset \mathbb{R}^3$ be a bounded regular open set. There exists a positive constant C such that for all $M \in H^2(\Omega)$ satisfying $\frac{\partial M}{\partial n} = 0$ on $\partial\Omega$,*

$$\|M\|_{H^2(\Omega)} \leq C\big(\|M\|^2_{L^2(\Omega)} + \|\Delta M\|^2_{L^2(\Omega)}\big)^{1/2},$$
$$\|\nabla M\|_{H^1(\Omega)} \leq C\big(\|\nabla M\|^2_{L^2(\Omega)} + \|\Delta M\|^2_{L^2(\Omega)}\big)^{1/2},$$
$$\|M\|_{L^\infty(\Omega)} \leq C\big(\|M\|^2_{L^2(\Omega)} + \|\Delta M\|^2_{L^2(\Omega)}\big)^{1/2}.$$

If $\Omega \subset \mathbb{R}^2$, then

$$\|\nabla M\|_{L^4(\Omega)} \leq C\|\nabla M\|^{1/2}_{L^2} \times (\|\nabla M\|^2_{L^2} + \|\Delta M\|^2_{L^2})^{1/4},$$
$$\|\Delta M\|_{L^4(\Omega)} \leq C\|\Delta M\|^{1/2}_{L^2} \times (\|\Delta M\|^2_{L^2} + \|\nabla \Delta M\|^2_{L^2})^{1/4}$$
$$\|\nabla^2 M\|_{L^4(\Omega)} \leq C(\|M\|^2_{L^2} + \|\Delta M\|^2_{L^2})^{1/2} + (\|M\|^2_{L^2} + \|\Delta M\|^2_{L^2})^{1/4}\|\nabla \Delta M\|^{1/2}_{L^2}.$$

The system (1) has a weak solution defined as follows:

Definition 2.1. ([4]) *The triple (v, F, M) is called a weak solution to the system (1) in Q_T, for $0 < T < +\infty$, provided that*

$$v \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}),$$
$$F \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{d\times d})) \cap L^2(0, T; H^1(\Omega; \mathbb{R}^{d\times d})),$$
$$M \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^3) \cap L^2(0, T; H^2(\Omega; \mathbb{R}^3)),$$

and if for test functions $\zeta \in W^{1,\infty}(0, T; \mathbb{R})$ with $\zeta(T) = 0$, $\xi \in \mathbf{V}$, $\Xi \in H^1_0(\Omega; \mathbb{R}^{d\times d})$, $\varphi \in H^1(\Omega; \mathbb{R}^3)$ together with the boundary conditions (2), it satisfies the equalities

$$\int_0^T \int_\Omega -v \cdot (\zeta'\xi) + (v \cdot \nabla)v \cdot (\zeta\xi) + \big(W'(F)F^\top - \nabla M \odot \nabla M\big) : (\zeta\nabla\xi)dxdt$$
$$- \int_\Omega v(0)(\zeta(0)\xi)dx = -\int_0^T \int_\Omega \nu\nabla v : (\zeta\nabla\xi)dxdt,$$
$$\int_0^T \int_\Omega -F : (\zeta'\Xi) + (v \cdot \nabla)F : (\zeta\Xi) - (\nabla v F) : (\zeta\Xi)dxdt$$
$$- \int_\Omega F(0)(\zeta(0)\Xi)dx = -\int_0^T \int_\Omega \kappa\nabla F \vdots (\zeta\nabla\Xi)dxdt$$
$$\int_0^T \int_\Omega -M \cdot (\zeta'\varphi) + (v \cdot \nabla)M \cdot (\zeta\varphi)dxdt - \int_\Omega M(0) \cdot (\zeta(0)\varphi)dx$$
$$= \int_0^T \int_\Omega -\nabla M : (\zeta\nabla\varphi) - \frac{1}{\mu^2}(|M|^2 - 1)M \cdot (\zeta\varphi)dxdt.$$

The existence and uniqueness of weak solutions of the system (1) for $(v_0, F_0, M_0) \in \mathbf{H} \times L^2(\Omega) \times H^1_n(\Omega)$ have been proved in [4] and [13]. We now introduce the energy estimates for weak solutions.

Lemma 2.2. *Let (v, F, M) be a weak solution to the system (1), the basic energy $\mathcal{E}(t)$ is introduced in section 1. Then we have*

$$\frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}\Big(\frac{1}{2}\|v\|^2_{L^2} + \frac{1}{2}\|F\|^2_{L^2} + \frac{1}{2}\|\nabla M\|^2_{L^2} + \int_\Omega G(M)\Big)$$
$$+ \int_\Omega \Big(\nu|\nabla v|^2 + \kappa|\nabla F|^2 + |\mu|^2\Big)dx = 0. \tag{6}$$

Proof. Multiplying equation $(1)_1$ by v, equation $(1)_3$ by F, equation $(1)_4$ by $-\mu$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2\\ &=\int_\Omega\Big(-\nu|\nabla v|^2-(v\cdot\nabla)v\cdot v+\Big[\nabla\cdot(FF^\top-\nabla M\odot\nabla M)\Big]\cdot v\Big)dx,\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\|F\|_{L^2}^2=\int_\Omega\Big(-\kappa|\nabla F|^2-(v\cdot\nabla)F:F+(\nabla vF):F\Big)dx,$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla M\|_{L^2}^2+G(M))\\ &=\int_\Omega\Big(-|\mu|^2+(v\cdot\nabla)M\cdot\Delta M-(v\cdot\nabla)M\cdot f(M)\Big)dx.\end{aligned}$$

Using Einstein summation convention (Einstein notation), for $d\times d$ matrix A and B, we define $AB=\sum_{i,j=1}^d A_{ik}B_{kj}=A_{ik}B_{kj}$, $A:B=\sum_{i,j=1}^d A_{ij}B_{ij}=A_{ij}B_{ij}$, the transpose $(A^\top)_{ij}=A_{ji}$ and $(A^\top B)_{ij}=A_{ki}B_{kj}$. Hence we can rewrite above equality as

$$\begin{aligned}\int_\Omega(\nabla\cdot FF^\top)\cdot v dx&=\int_\Omega\partial_j(F_{ik}F_{jk})v_i dx=-\int_\Omega F_{ik}F_{jk}\partial_j v_i dx\\ &=-\int_\Omega\underbrace{\partial_j v_i F_{jk}}_{ik}\underbrace{F_{ik}}_{ik}dx=-\int_\Omega(\nabla vF):F dx,\end{aligned}$$

$$\begin{aligned}\nabla\cdot(\nabla M\odot\nabla M)&=\partial_j(\partial_i M_k\partial_j M_k)=\partial_j\partial_i M_k\partial_j M_k+\partial_i M_k\partial_j^2 M_k\\ &=\frac{1}{2}\partial_i|\partial_j M_k|^2+\underbrace{\partial_i M_k}_{ki}\underbrace{\partial_j^2 M_k}_{kj}=\nabla\frac{|\nabla M|^2}{2}+\nabla^\top M\Delta M\end{aligned}$$

and

$$(\nabla^\top M\Delta M)\cdot v=\underbrace{\partial_i M_k\partial_j^2 M_k}_{ij}v_i=\underbrace{v_i\partial_i}_{(v\cdot\nabla)}M_k\partial_j^2 M_k=(v\cdot\nabla)M\cdot\Delta M.$$

Above discussion and $\nabla G(M)=f(M)\cdot\nabla M$, we can easy obtain the last equality.

$$\int_\Omega\Big(\nabla G(M)+\nabla\frac{|\nabla M|^2}{2}\Big)\cdot v dx=\int_\Omega(v\cdot\nabla)v\cdot v dx=\int_\Omega(v\cdot\nabla)F:F dx=0.$$

Using free divergence condition ($\nabla\cdot v=0$), we have

$$\int_\Omega\Big(\nabla G(M)+\nabla\frac{|\nabla M|^2}{2}\Big)\cdot v dx=-\int_\Omega\Big(G(M)+\frac{|\nabla M|^2}{2}\Big)\underbrace{(\nabla\cdot v)}_{=0}dx=0,$$

$$\int_\Omega(v\cdot\nabla)v\cdot v dx=\int_\Omega\nabla|v|^2\cdot v dx=-\int_\Omega|v|^2(\nabla\cdot v)dx=0,$$

and

$$\int_\Omega(v\cdot\nabla)F:F dx=\int_\Omega\nabla|F|^2\cdot v dx=-\int_\Omega|F|^2(\nabla\cdot v)dx=0.$$

Summing the above notations, we obtain (6). □

We now introduce a Gronwall inequality which is useful to prove dissipative estimates.

Lemma 2.3 ([16], Lemma 6.2.1). *Let T be given, $0 < T \leq \infty$. Suppose that $y(t)$ and $h(t)$ are nonnegative continuous functions defined on $[0, T]$, which satisfy the following conditions:*

$$\frac{dy}{dt} \leq c_1 y^2 + c_2 + h(t), \quad \int_0^T y(t)dt \leq c_3, \quad \int_0^T h(t)dt \leq c_4,$$

where c_i $(i = 1, 2, 3, 4)$ are given nonnegative constants. Then, for any $r \in (0, T)$ the following estimate holds:

$$y(t + r) \leq \big(\frac{c_3}{r} + c_2 r + c_4\big)e^{c_1 c_3}, \quad \forall t \in [0, T - r].$$

3. Dissipative estimates

We begin to prove the first basic dissipative inequality that is a direct consequence of the basic energy law (6).

Lemma 3.1. *There exist constants $C_0 > 0$, $\theta > 0$ independent of initial data (v_0, F_0, M_0), such that*

$$\frac{d}{dt}\mathcal{E}(t) + \kappa\mathcal{E}(t) \leq C_0, \quad \forall t \geq 0. \tag{7}$$

Proof. Taking the scalar product in $L^2(\Omega)$ of $(1)_5$ with M, we obtain

$$\begin{aligned}(\mu, M)_{L^2} =& (\Delta M, M)_{L^2} - ((|M|^2 - 1)M, M)_{L^2} \\ =& -\|\nabla M\|_{L^2}^2 - \|M\|_{L^4}^4 + \|M\|_{L^2}^2.\end{aligned} \tag{8}$$

On the other hand, by the Hölder and Young inequalities, we have

$$-(\mu, M)_{L^2} \leq \|\mu\|_{L^2}\|M\|_{L^2} \leq \frac{1}{2}\|\mu\|_{L^2}^2 + \frac{1}{2}\|M\|_{L^2}^2, \tag{9}$$

$$\|M\|_{L^2}^2 \leq \frac{1}{3}\|M\|_{L^4}^4 + \frac{3}{4}|\Omega|. \tag{10}$$

Combining (8) and energy equality (6), we get

$$\frac{d}{dt}\mathcal{E}(t) + \theta\mathcal{E}(t) = \Lambda(t),$$

where

$$\begin{aligned}\Lambda(t) :=& \frac{\theta}{2}\|v\|_{L^2}^2 + \frac{\theta}{2}\|F\|_{L^2}^2 + \frac{\theta}{2}\|\nabla M\|_{L^2}^2 + \theta\int_\Omega G(M)dx - \nu\|\nabla v\|_{L^2}^2 \\ &- \kappa\|\nabla F\|_{L^2}^2 - \|\mu\|_{L^2}^2 + (-\|\nabla M\|_{L^2}^2 - \|M\|_{L^4}^4 + \|M\|_{L^2}^2 - (\mu, M)_{L^2}).\end{aligned} \tag{11}$$

Moreover, we notice that

$$\begin{aligned}\theta\int_\Omega G(M)dx =& \theta\int_\Omega \frac{1}{4}(|M|^2 - 1)^2 dx \\ \leq& \frac{\theta}{4}\|M\|_{L^4}^4 + \frac{\theta}{2}\|M\|_{L^2}^2 + \frac{\theta|\Omega|}{4} \leq \frac{\theta}{2}\|M\|_{L^4}^4 + \frac{\theta|\Omega|}{2}.\end{aligned} \tag{12}$$

Applying (8)-(12) and the Poincaré's inequality for v and F, we deduce that

$$\begin{aligned}\Lambda(t) \leq& -(\nu - \frac{\theta}{2}C_\Omega)\|\nabla v\|_{L^2}^2 - (\kappa - \frac{\theta}{2}C_\Omega)\|\nabla F\|_{L^2}^2 - (1 - \frac{\theta}{2})\|\nabla M\|_{L^2}^2 \\ &- (\frac{1}{2} - \frac{\theta}{2})\|M\|_{L^4}^4 - \frac{1}{2}\|\mu\|_{L^2}^2 + \frac{|\Omega|}{2}\big(\frac{9}{4} + \theta\big),\end{aligned} \tag{13}$$

where C_Ω is the optimal Poincaré constant. We choose

$$\theta = \min\left\{1, \frac{2\nu}{C_\Omega}, \frac{2\kappa}{C_\Omega}\right\},$$

and set

$$C_0 = \frac{|\Omega|}{2}\big(\frac{9}{4} + \theta\big),$$

then we obtain

$$\frac{d}{dt}\mathcal{E}(t) + \theta\mathcal{E}(t) \leq C_0,\ t \geq 0.$$

□

Thanks to Lemma 3.1 and the uniqueness property of weak solutions, ([13]) we have the following proposition.

Proposition 3.1. *Let (v, F, M) be the unique weak solution to system (1). The system (1) defines a nonlinear strongly continuous semigroup*

$$\mathcal{S}(t) : \mathbb{Y} \to \mathbb{Y}, \tag{14}$$

by setting, for $t \geq 0$, $\mathcal{S}(t)(v_0, F_0, M_0) = (v(t), F(t), M(t))$.

Proposition 3.2. *Assume that (v, F, M) is a weak solution to the system (1). Then there exists a time t_0, and positive constants M_1, M_2 depending on C_0 and $\mathcal{E}_0$, such that*

$$\|v(t)\|^2_{L^2} + \|F(t)\|^2_{L^2} + \|M(t)\|^2_{H^1} \leq M_1, \quad \forall t \geq t_0 \tag{15}$$

and

$$\int_t^{t+1} (\|v(\tau)\|^2_{H^1} + \|F(\tau)\|^2_{H^1} + \|M(\tau)\|^2_{H^2}) d\tau \leq M_2, \quad \forall t \geq t_0. \tag{16}$$

Proof. Multiplying (7) by $e^{-\theta t}$ and integrating the relation from 0 to t, we have

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\theta t} + \frac{C_0}{\theta}, \quad \forall t \geq t_0.$$

Taking

$$t_0 = \frac{1}{\theta}\left|\ln\frac{\theta}{C_0\mathcal{E}(0)}\right|,$$

we have

$$\mathcal{E}(t) \leq \frac{2C_0}{\theta}, \quad \forall t \geq t_0.$$

From this and (10), we obtain

$$\|M\|^2_{H^1} \leq C(\|\nabla M\|^2_{L^2} + \|M\|^2_{L^2}) \leq C(\|\nabla M\|^2_{L^2} + \|M\|^4_{L^4} + 1) \leq C\mathcal{E}(0),$$

which implies the constant M_1 depending on C_0 and $\mathcal{E}(0)$, such that (15) holds.

Integrating (6) from t to $t+1$, we obtain

$$\mathcal{E}(t+1) + \int_t^{t+1} (\nu\|\nabla v(\tau)\|^2_{L^2} + \kappa\|\nabla F(\tau)\|^2_{L^2} + \|\mu(\tau)\|^2_{L^2}) d\tau = \mathcal{E}(t).$$

This implies

$$\int_t^{t+1} (\nu\|\nabla v(\tau)\|^2_{L^2} + \kappa\|\nabla F(\tau)\|^2_{L^2} + \|\mu(\tau)\|^2_{L^2}) d\tau$$
$$\leq \mathcal{E}(t) \leq \mathcal{E}(0) \leq \frac{2C_0}{\kappa}, \quad \forall t \geq t_0.$$

Using (15), the interpolation theorem and the Sobolev imbedding theorem [1], with $\mu = \Delta M - (|M|^2-1)M$, we deduce

$$\begin{aligned}
&\int_t^{t+1}\|M(\tau)\|_{H^2}^2 d\tau \le C_1\int_t^{t+1}(\|\Delta M(\tau)\|_{L^2}^2+\|M(\tau)\|_{L^2}^2)d\tau\\
\le& C_1\int_t^{t+1}(\|(\Delta M-(|M|^2-1)M)(\tau)\|_{L^2}^2)d\tau\\
&+C_1\int_t^{t+1}(\|(|M|^2-1)M(\tau)\|_{L^2}^2+\|M(\tau)\|_{L^2}^2)d\tau\\
\le& C_2\int_t^{t+1}(\|\mu(\tau)\|_{L^2}^2+\|M(\tau)\|_{L^6}^3+\|M(\tau)\|_{L^2}^2)d\tau\\
\le& C_3\int_t^{t+1}(\|\mu(\tau)\|_{L^2}^2+\|M(\tau)\|_{H^1}^3+\|M(\tau)\|_{H^1}^2)d\tau\\
\le& C_3\big(\frac{2C_0}{\theta}+M_1^{3/2}+M_1\big).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\int_t^{t+1}(\|v(\tau)\|_{H^1}^2+\|F(\tau)\|_{H^1}^2+\|M(\tau)\|_{H^2}^2)d\tau\\
&\le \frac{2C_0}{\nu\theta}+\frac{2C_0}{\kappa\theta}+C_3\big(\frac{2C_0}{\theta}+M_1^{3/2}+M_1\big):=M_2, \quad \forall t\ge t_0.
\end{aligned}$$

Therefore, (16) holds. □

Now, we prove the flowing proposition.

Proposition 3.3. *Assume that $(v_0,F_0,M_0)\in \mathbf{H}\times L^2(\Omega)\times H_n^1(\Omega)$ and (v,F,M) is a weak solution to the system (1). Then there exists positive constants M_3, M_4, such that for $t_1=t_0+1$, the following uniform estimate hold,*

$$\|v(t)\|_{H^1}^2+\|F(t)\|_{H^1}^2+\|M(t)\|_{H^2}^2\le M_3, \quad \forall t\ge t_1, \tag{17}$$

$$\int_t^{t+1}(\|v(\tau)\|_{H^2}^2+\|F(\tau)\|_{H^2}^2+\|M(\tau)\|_{H^3}^2)d\tau\le M_4, \quad \forall t\ge t_1. \tag{18}$$

Proof. We take the inner product of $(1)_1$ in $L^2(\Omega)$ with $-2\Delta v$ (we can do that within a suitable Galerkin discretization scheme), and obtain

$$\begin{aligned}
&\frac{d}{dt}\|\nabla v\|_{L^2}^2+2\nu\|\Delta v\|_{L^2}^2=2\int_\Omega(v\cdot\nabla v)\cdot\Delta v dx\\
&-2\int_\Omega\nabla\cdot(FF^\top)\cdot\Delta v dx+2\int_\Omega\nabla\cdot(\nabla M\odot\nabla M)\cdot\Delta v dx.
\end{aligned}\tag{19}$$

We take the inner product of $(1)_3$ in $L^2(\Omega)$ with $-2\Delta F$, and have

$$\frac{d}{dt}\|\nabla F\|_{L^2}^2+2\kappa\|\Delta F\|_{L^2}^2=2\int_\Omega(v\cdot\nabla F):\Delta F dx-2\int_\Omega\nabla vF:\Delta F dx. \tag{20}$$

We differentiate in $(1)_4$ by ∇ and take the inner product in $L^2(\Omega)$ with $2\nabla\Delta M$, to get

$$\begin{aligned}
&\frac{d}{dt}\|\Delta M\|_{L^2}^2+2\|\nabla\Delta M\|_{L^2}^2\\
&=-2\int_\Omega\nabla(v\cdot\nabla M):\nabla\Delta M dx-2\int_\Omega\nabla f(M):\nabla\Delta M dx.
\end{aligned}\tag{21}$$

Adding up the relationships (19), (20) and (21), we have that

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla v\|_{L^2}^2+\|\nabla F\|_{L^2}^2+\|\Delta M\|_{L^2}^2\Big)+2\Big(\nu\|\Delta v\|_{L^2}^2+\kappa\|\Delta F\|_{L^2}^2+\|\nabla\Delta M\|_{L^2}^2\Big)\\
&=2\int_\Omega(v\cdot\nabla v)\cdot\Delta v dx-2\int_\Omega\nabla\cdot(FF^\top)\cdot\Delta v dx\\
&+2\int_\Omega\nabla\cdot(\nabla M\odot\nabla M)\cdot\Delta v dx+2\int_\Omega(v\cdot\nabla F):\Delta F dx-2\int_\Omega\nabla vF:\Delta F dx\\
&-2\int_\Omega\nabla(v\cdot\nabla M):\nabla\Delta M dx-2\int_\Omega\nabla f(M):\nabla\Delta M dx=\sum_{i=1}^7 I_i(t). \qquad (22)
\end{aligned}$$

We now estimate I_i term by term. For I_1, by the interpolation inequality and Young's inequality, we have

$$\begin{aligned}
&I_1(t)=2\int_\Omega(v\cdot\nabla v)\cdot\Delta v dx\le 2\|v\|_{L^4}\|\nabla u\|_{L^4}\|\Delta u\|_{L^2}\\
&\le c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}\\
&=c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{3/2}\\
&\le c_\nu\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\frac{\nu}{4}\|\Delta u\|_{L^2}^2\\
&\le c_\nu\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\frac{\nu}{6}\|\Delta u\|_{L^2}^2,
\end{aligned}$$

where we use the Young's inequality coefficient 4 and 4/3, $(1/4+3/4=1)$ and the GagliardoCNirenberg inequality [1] for two dimensions as follows

$$\|u\|_{L^4}\le c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2},\quad \|\nabla u\|_{L^4}\le c\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}.$$

For I_2, using Einstein summation convention, we know

$$\big(\nabla\cdot(FF^\top)\big)_{i,j}=\partial_j(F_{ik}F_{jk})=(\partial_jF_{ik})F_{jk}+F_{ik}(\partial_jF_{jk}),$$

which means

$$\begin{aligned}
&I_2=-2\int_\Omega\nabla\cdot(FF^\top)\cdot\Delta v dx\le c\|F\|_{L^4}\|\nabla F\|_{L^4}\|\Delta v\|_{L^2}\\
&\le c\|F\|_{L^2}^{1/2}\|\nabla F\|_{L^2}^{1/2}\|\nabla F\|_{L^2}^{1/2}\|\Delta F\|_{L^2}^{1/2}\|\Delta v\|_{L^2}\\
&\le c\|\nabla F\|_{L^2}\|\Delta F\|_{L^2}^{1/2}\|\Delta v\|_{L^2}\le c_\nu\|\nabla F\|_{L^2}^2\|\Delta F\|_{L^2}+\frac{\nu}{4}\|\Delta v\|_{L^2}^2\\
&\le c_{\nu,\kappa}\|\nabla F\|_{L^2}^4+\frac{\kappa}{3}\|\Delta F\|_{L^2}^2+\frac{\nu}{6}\|\Delta v\|_{L^2}^2.
\end{aligned}$$

For I_3, recall that

$$\big(\nabla\cdot(\nabla M\odot\nabla M)\big)_{i,j}=\nabla_j(\nabla_iM_k\nabla_jM_k)=\nabla\frac{|\nabla M|^2}{2}+\nabla^\top M\Delta M$$

and (notice $\nabla\cdot v=0$)

$$(\nabla^\top M\Delta M)\cdot\Delta v=(\Delta v\cdot\nabla)M\cdot\Delta M,\qquad \int_\Omega\nabla\frac{|\nabla M|^2}{2}\cdot\Delta v dx=0,$$

then (notice $\|M\|_{H^1} \le C$ for $t \ge t_0$) for Young's inequality and Lemma 2.1

$$\begin{aligned}
I_3 &= 2\int_\Omega \nabla\cdot(\nabla M\odot\nabla M)\cdot\Delta v dx = 2\int_\Omega(\Delta v\cdot\nabla)M\cdot\Delta M dx\\
&\le c\|\Delta v\|_{L^2}\|\nabla M\|_{L^4}\|\Delta M\|_{L^4}\\
&\le c\|\nabla M\|_{L^2}^{1/2}(\|\nabla M\|_{L^2}^2+\|\Delta M\|_{L^2}^2)^{1/4}\\
&\qquad\cdot\|\Delta M\|_{L^2}^{1/2}(\|\Delta M\|_{L^2}^2+\|\nabla\Delta M\|_{L^2}^2)^{1/4}\|\Delta v\|_{L^2}\\
&\le c\Big(\|\Delta M\|_{L^2}+\|\Delta M\|_{L^2}^{3/2}+\|\Delta M\|_{L^2}^{1/2}\|\nabla\Delta M\|_{L^2}^{1/2}\\
&\qquad+\|\Delta M\|_{L^2}\|\nabla\Delta M\|_{L^2}^{1/2}\Big)\|\Delta v\|_{L^2}\\
&\le c_\nu\|\Delta M\|_{L^2}^4+\frac14\|\nabla\Delta M\|_{L^2}^2+\frac{\nu}{4}\|\Delta v\|_{L^2}^2+c.
\end{aligned}$$

For I_4 and I_5,

$$\begin{aligned}
I_4+I_5 &= 2\int_\Omega(v\cdot\nabla F):\Delta F dx-2\int_\Omega\nabla vF:\Delta F dx\\
&\le c\|v\|_{L^4}\|\nabla F\|_{L^4}\|\Delta F\|_{L^2}+c\|\nabla v\|_{L^4}\|F\|_{L^4}\|\Delta F\|_{L^2}\\
&\le c\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}\|\nabla F\|_{L^2}^{1/2}\|\Delta F\|_{L^2}^{3/2}\\
&\quad+\|\nabla v\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{1/2}\|F\|_{L^2}^{1/2}\|\nabla F\|_{L^2}^{1/2}\|\Delta F\|_{L^2}\\
&\le c\|\nabla v\|_{L^2}^2\|\nabla F\|_{L^2}^2+\frac{\kappa}{3}\|\Delta F\|_{L^2}^2+\frac{\nu}{6}\|\Delta v\|_{L^2}^2,
\end{aligned}$$

where we use the Young's inequality coefficients 4 and 4/3, $(1/4+3/4=1)$ for the first part, the coefficient 2 and 2, $(1/2+1/2=1)$ for the second part.

For I_6, we have

$$\big(\nabla(v\cdot\nabla M)\big)_{i,j}=\partial_j(v_k\partial_k M_i)=\partial_j v_k\partial_k M_i+v_k\partial_k\partial_j M_i,$$

which means

$$\begin{aligned}
&-2\int_\Omega\nabla(v\cdot\nabla M):\nabla\Delta M dx\\
&=-2\int_\Omega\nabla v\nabla^\top M:\nabla\Delta M dx-2\int_\Omega v\cdot\nabla\nabla M:\nabla\Delta M dx,
\end{aligned}$$

where $\nabla\nabla M=\nabla^2 M$ stands for $\partial_k\partial_j M_i$. Lemma 2.1 implies

$$\begin{aligned}
&\|\nabla^2 M\|_{L^4(\Omega)}\le C(\|M\|_{L^2}^2+\|\Delta M\|_{L^2}^2)^{1/2}+(\|M\|_{L^2}^2+\|\Delta M\|_{L^2}^2)^{1/4}\|\nabla\Delta M\|_{L^2}^{1/2},\\
&\|\nabla M\|_{L^4(\Omega)}\le C\|\nabla M\|_{L^2}^{1/2}\times(\|\nabla M\|_{L^2}^2+\|\Delta M\|_{L^2}^2)^{1/4},
\end{aligned}$$

then, we obtain (analogous estimate method as above terms)

$$\begin{aligned}
I_6 &= -2\int_\Omega(\nabla v\nabla^\top M):\nabla\Delta M dx-2\int_\Omega v\cdot\nabla\nabla M:\nabla\Delta M dx\\
&\le c(\|\nabla v\|_{L^4}\|\nabla M\|_{L^4}+\|v\|_{L^4}\|\nabla^2 M\|_{L^4})\|\nabla\Delta M\|_{L^2}\\
&\le c\Big(\|\nabla v\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{1/2}\|\nabla M\|_{L^4}+\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}\|\nabla^2 M\|_{L^4}\Big)\|\nabla\Delta M\|_{L^2}\\
&\le c_\nu+\frac14\|\nabla\Delta M\|_{L^2}^2+\frac{\nu}{6}\|\Delta v\|_{L^2}^2+c\|\nabla v\|^2\|\Delta M\|_{L^2}^2.
\end{aligned}$$

We now estimate the last term. First, recalling the embedding inequality

$$\|M\|_{L^\infty(\Omega)}\le c\|M\|_{H^2(\Omega)},$$

when combining it with Lemma 2.1, we have

$$\begin{aligned} I_7 &= -2\int_\Omega \nabla f(M) : \nabla\Delta M dx = -2\int_\Omega \nabla((|M|^2-1)M) : \nabla\Delta M dx \\ &= -2\int_\Omega M\otimes(M\nabla M) : \nabla\Delta M dx + \int_\Omega |M|^2\nabla M : \nabla\Delta M dx \\ &\le c\|M\|_{L^\infty}^2\|\nabla M\|_{L^2}\|\nabla\Delta M\|_{L^2} \\ &\le c(\|\Delta M\|_{L^2}^2+\|M\|_{L^2}^2)\|\nabla M\|_{L^2}\|\nabla\Delta M\|_{L^2} \\ &\le c + c\|\Delta M\|_{L^2}^4 + \frac{1}{4}\|\nabla\Delta M\|_{L^2}^2. \end{aligned}$$

Now we set

$$A(t) = \Big(\|\nabla v\|_{L^2}^2 + \|\nabla F\|_{L^2}^2 + \|\Delta M\|_{L^2}^2\Big).$$

Summarizing the estimates of $I_1 \sim I_7$, we have

$$\frac{d}{dt}A(t) + \Big(\nu\|\Delta v\|_{L^2}^2 + \kappa\|\Delta F\|_{L^2}^2 + \|\nabla\Delta M\|_{L^2}^2\Big) \le C_1 A(t)^2 + C_2.$$

By (16), we get

$$\int_t^{t+1} A(\tau)d\tau \le M_2.$$

The uniform Gronwall inequality (Lemma 2.3) implies that

$$A(t+1) \le (M_2+C_2)e^{C_1M_2} := M_3,\ t \ge t_0,$$

which means $A(t) \le (M_2+C_2)e^{C_1M_2} := M_3,\ t \ge t_1 := t_0+1$. The same way, we can obtain (18), from $\|M\|_{H^3(\Omega)}^2 \le C\big(\|M\|_{L^2(\Omega)}^2 + \|\nabla\Delta M\|_{L^2(\Omega)}^2\big)$. The proof is completed. □

4. **Global attractor**

First of all, we recall the definition and a lemma about global attractors. ([14])

Definition 4.1. *An attractor is a set $\mathcal{A}$ which belongs to a metric space H and enjoys the following properties:*

(1) *$\mathcal{A}$ is an invariant set under ($\mathcal{S}(t)\mathcal{A} = \mathcal{A}$), for all $t \ge 0$.*

(2) *$\mathcal{A}$ possesses an open neighborhood $\mathfrak{u}$ that for every u_0 in $\mathfrak{u}$, $\mathcal{S}(t)u_0$ converges to $\mathcal{A}$ as $t \to \infty$, $dist(\mathcal{S}(t)u_0, \mathcal{A}) \to 0$, as $t \to \infty$. Here dist is understood as the distance of a point to a set; $d(x,\mathcal{A}) = \inf_{y\in\mathcal{A}} d(x,y)$.*

Definition 4.2. *If $\mathcal{A}$ is an attractor, the largest open set $\mathfrak{u}$ that satisfies (2) of the Definition 4.1 is called the basin of attraction of $\mathcal{A}$. We say that $\mathcal{A}$ uniformly attracts a set $\mathcal{B} \subset \mathfrak{u}$ if $dist(\mathcal{S}(t)\mathcal{B}, \mathcal{A}) \to 0$, as $t \to \infty$, where $d(\mathcal{B}_0, \mathcal{B}_1) = \sup_{x\in\mathcal{B}_0}\inf_{y\in\mathcal{B}_1} d(x,y)$.*

Definition 4.3. *We say that $\mathcal{A} \subset H$ is a global attractor for the semigroup $\{\mathcal{S}(t)\}_{t\ge0}$ if $\mathcal{A}$ is a compact attractor that attracts the bounded sets of H.*

We now present the generate lemma of global attractors.

Lemma 4.1. *Let the closed semigroup $\{\mathcal{S}(t)\}_{t\ge0}$ have a connect compact attracting set $\mathcal{B}$. Assume also that $\mathcal{S}(t)\mathcal{B} \subset \mathcal{B}$ for every t sufficiently large. Then $\mathcal{S}(t)$ has a connected global attractor $\mathcal{A}$.*

The main result of the paper is contained in the following theorem.

Theorem 4.1. *The dynamical system $(\mathbb{Y}, \mathcal{S}(t))$ of (1) possesses a connected global attractor $\mathcal{A}$ which is bounded in $\mathbf{H} \times L^2(\Omega) \times H_n^1(\Omega)$.*

Proof. By Proposition 3.2, we know that the dynamical system $(\mathbb{Y}, \mathcal{S}(t))$ has a bounded absorbing set in $\mathbf{H}\times L^2(\Omega)\times H_n^1(\Omega)$. Proposition 3.3 implies that it has a compact absorbing set which is contained in $\mathbf{H} \times L^2(\Omega) \times H_n^1(\Omega)$. On the basis of Lemma 4.1, we know that the dynamical system $(\mathbb{Y}, \mathcal{S}(t))$ has a global attractor $\mathcal{A}$. The proof of Theorem 4.1 is now complete. □

5. **Conclusions**

The dynamic properties of the equation (1), such as the global asymptotical behaviors of solutions and existence of global attractors are important. The main difficulties for treating the system (1) are caused by the strong coupling nonlinear terms and the Neumann boundary conditions. Based on the combination of the suitable dissipative estimates with the energy techniques, we established the existence of global attractor $\mathcal{A}$ on a suitable phase-space and proved that the attractor have a regular compact absorbing set.

Acknowledgment

The authors would like to express their deep thanks to the referee's valuable suggestions for the revision and improvement of the manuscript.

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