VARIOUS NOTIONS OF AMENABILITY FOR CERTAIN BANACH ALGEBRAS RELATED TO THE MULTIPLIERS

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In this paper, for the Banach algebra $\mathcal{A}_T$, we study the various notions of amenability like pseudo amenability, Johnson pseudo-contractibility and module amenability, where $\mathcal{A}$ is a Banach algebra and T is a left multiplier on $\mathcal{A}$. For a dual Banach algebra $\mathcal{A}$, under some conditions, we show that if $\mathcal{A}_T$ is Connes amenable (resp. Connes biprojective), then $\mathcal{A}$ is Connes amenable (resp. Connes biprojective). For a non-zero multiplicative linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, we study the relationship between φ-amenability of $\mathcal{A}$ and φR-amenability of $\mathcal{A}_T$, where (T, R) be a double centralizer of $\mathcal{A}$.

Keywords: Multiplier, pseudo amenability, Johnson pseudo-contractibility, Connes amenability, Character amenability.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be a Banach algebra and T a bounded multiplier on $\mathcal{A}$. Laali in [7] introduced a new Banach algebra $\mathcal{A}_T$ related to $\mathcal{A}$ and T. Indeed a new multiplication on $\mathcal{A}$ was defined by $a * b = aT(b)$ for every $a, b \in \mathcal{A}$. With the norm

$$\|a\|_T = \|T\|\|a\| \quad (a \in \mathcal{A}).$$

$\mathcal{A}_T$ becomes a Banach algebra, provided that $T \in Mul_l(\mathcal{A})$, see [7, Theorem 2.1(i)]. Laali also studied approximate amenability of $\mathcal{A}_T$, see [7, Theorem 3.1].

Motivated by these considerations, we investigate the other various notions of amenability for this new multiplication on Banach algebras. For a dual Banach algebra $\mathcal{A}$, under certain conditions, we show that if $\mathcal{A}_T$ is Connes amenable (resp. Connes biprojective), then $\mathcal{A}$ is Connes amenable (resp. Connes biprojective). Also we show that if $\mathcal{A}$ is Johnson pseudo-contractible, then $\mathcal{A}_T$ is Johnson pseudo-contractible with additional conditions. We study Module amenability of $\mathcal{A}_T$. Finally we give some examples of Banach algebras which are not φ-biprojective (resp. φ-biflat), but with this new multiplication they are φ-biprojective (resp. φ-biflat).

Suppose that T is a left multiplier and R is a right multiplier of $\mathcal{A}$, respectively. Then the pair (T, R) is a double centralizer of $\mathcal{A}$, if $aT(b) = R(a)b$ for every $a, b \in \mathcal{A}$. The set of all left (right) multipliers of $\mathcal{A}$ is denoted by $Mul_l(\mathcal{A})(Mul_r(\mathcal{A}))$, respectively. It is easy to see that $Mul_l(\mathcal{A})$ and $Mul_r(\mathcal{A})$ are unital Banach subalgebras of $B(\mathcal{A})$. For further information see [2].

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Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. Then a bounded linear map $D : \mathcal{A} \to E$ is called derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, for every $a, b \in \mathcal{A}$. Suppose that E is a Banach $\mathcal{A}$-bimodule. So E^* is also a Banach $\mathcal{A}$-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \qquad (f \cdot a)(x) = f(a \cdot x) \quad (a \in \mathcal{A}, x \in E, f \in E^*).$$

A Banach algebra $\mathcal{A}$ is called amenable if for every Banach $\mathcal{A}$-bimodule E, each derivation $D : \mathcal{A} \longrightarrow E^*$ is inner, that is, D has the form

$$D(a) = a \cdot f - f \cdot a, \quad (a \in \mathcal{A}),$$

for some $f \in E^*$. A derivation $D : \mathcal{A} \longrightarrow E^*$ is approximately inner if there exists a net (f_α) in E^* such that $D(a) = ||\cdot|| - \lim_\alpha a \cdot f_\alpha - f_\alpha \cdot a$ for every $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is called approximately amenable if every derivation $D : \mathcal{A} \longrightarrow E^*$ is approximately inner for all Banach $\mathcal{A}$-bimodule E, see [3].

2. Amenability and amenable-like properties of $\mathcal{A}_T$

The class of dual Banach algebras was introduced by Runde [8]. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. An $\mathcal{A}$-bimodule E is called dual, if there exists a closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. A dual Banach $\mathcal{A}$-bimodule E is normal, if for each $x \in E$, the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk^* continuous.

A dual Banach algebra $\mathcal{A}$ is called Connes amenable if every wk^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is inner, for all normal dual Banach $\mathcal{A}$-bimodule E, see [8].

For a Banach algebra $\mathcal{A}$ and for every $a \in \mathcal{A}$, consider the operators L_a and R_a defined by

$$L_a(b) = ab, \quad R_a(b) = ba \quad (b \in \mathcal{A}).$$

Then $L_a \in Mul_l(\mathcal{A})$, $R_a \in Mul_r(\mathcal{A})$ and also (L_a, R_a) is the double centralizer of $\mathcal{A}$. Let $\mathcal{A}$ be a dual Banach algebra. Since the multiplication in $\mathcal{A}$ is separately wk^*-continuous [8, Proposition 2.1], L_a and R_a are wk^*-continuous.

Theorem 2.1. *Let $\mathcal{A}$ be a dual Banach algebra. Suppose that $T \in Mul_l(\mathcal{A})$ and T is a wk^*-continuous map. Then $\mathcal{A}_T$ is a dual Banach algebra.*

Proof. Since the underlying space of $\mathcal{A}$ and $\mathcal{A}_T$ are the same and $\|\cdot\|_T \simeq \|\cdot\|$, the predual of them are the same. We only show that the multiplication in $\mathcal{A}_T$ is separately wk^*-continuous. To see this, suppose that $a_0 \in \mathcal{A}_T$. Let (a_α) be a net in $\mathcal{A}_T$ such that $a_\alpha \xrightarrow{wk^*} a_0$. Since T is wk^*-continuous and the multiplication in $\mathcal{A}$ is separately wk^*-continuous, $bT(a_\alpha) \xrightarrow{wk^*} bT(a_0)$. So $b * a_\alpha \xrightarrow{wk^*} b * a_0$. Similarly we can show that $a_\alpha * b \xrightarrow{wk^*} a_0 * b$. Thus $\mathcal{A}_T$ is a dual Banach algebra. □

For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, we denote by $\sigma wc(E)$, the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk-continuous. Note that $\sigma wc(E)$ is a closed submodule of E.

A dual Banach algebra $\mathcal{A}$ is called Connes biprojective, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$, see [14].

Using argument as in the proof of [7, Theorem 3.1], we have the following theorem.

Theorem 2.2. *Let $\mathcal{A}$ be a dual Banach algebra and let $T \in Mul_l(\mathcal{A})$. Suppose that T is surjective and wk^*-continuous. Then the following statements hold:*

(i) *If $\mathcal{A}_T$ is Connes amenable, then $\mathcal{A}$ is Connes amenable.*
(ii) *If $\mathcal{A}_T$ is Connes biprojective, then $\mathcal{A}$ is Connes biprojective.*

Proof. (i) Define $\theta : \mathcal{A}_T \longrightarrow \mathcal{A}$ by $\theta(a) = T(a)$ for every $a \in \mathcal{A}_T$. So

$$\theta(a * b) = \theta(aT(b)) = T(aT(b)) = T(a)T(b) = \theta(a)\theta(b) \quad (a, b \in \mathcal{A}_T).$$

Since T is surjective and wk^*-continuous, θ is a wk^*-continuous epimorphism. Applying [8, Proposition 4.2(ii)], $\mathcal{A}$ is Connes amenable.
(ii) Suppose that $\mathcal{A}_T$ is Connes biprojective. Let θ be as above. Then θ is a wk^*-continuous epimorphism. Using [14, Theorem 2.7(ii)] follows that $\mathcal{A}$ is Connes biprojective. □

For a Banach algebra $\mathcal{A}$ and a sequence (S_n) in $Mul_l(\mathcal{A})$, define

$$T_0 = I, \quad T_n = \prod_{k=1}^{n} S_k \quad (n \geq 1).$$

It is clear that $T_n \in Mul_l(\mathcal{A})$ for every $n \in \mathbb{N}$. Laali defined a new Banach algebra $\mathcal{A}_n = (\mathcal{A}, *_n)$ by the following multiplication

$$a *_n b = aT_n(b) \quad (n \geq 0).$$

We remind that if $\mathcal{A}$ is a dual Banach algebra and S_n is wk^*-continuous for every $n \in \mathbb{N}$, then $\mathcal{A}_n$ is a dual Banach algebra.

Inspired by arguments as in [7, Theorem 3.2], we have the following theorem:

Theorem 2.3. *Let (S_n) be a sequence of elements in $Mul_l(\mathcal{A})$ such that S_n is wk^*-continuous for every $n \in \mathbb{N}$. Then the following statements hold:*

(i) *Suppose that S_n is surjective for all n and $\mathcal{A}_k$ is Connes amenable for some k. Then $\mathcal{A}$ is Connes amenable.*
(ii) *Suppose that S_n is surjective for all n and $\mathcal{A}_k$ is Connes biprojective for some k. Then $\mathcal{A}$ is Connes biprojective.*

Proof. (i) Suppose that S_n is surjective for all n and $\mathcal{A}_k$ is Connes amenable for some k. If $k = 1$, then Theorem 2.2(i) implies that $\mathcal{A}$ is Connes amenable. Now suppose that $k > 1$ and $\mathcal{A}_k$ is Connes amenable. Define $\varphi : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by $\varphi(a) = S_k(a)$ for every $a \in \mathcal{A}_k$. Since S_k is wk^*-continuous, φ is wk^*-continuous. Also we have

$$\begin{aligned}\varphi(a *_k b) &= S_k(aT_k(b)) = S_k(a)T_k(b) = S_k(a)T_{k-1}S_k(b)\\ &= S_k(a) *_{k-1} S_k(b) = \varphi(a) *_{k-1} \varphi(b),\end{aligned}$$

for every $a, b \in \mathcal{A}_k$. Since S_k is surjective, φ is an epimorphism. Using [8, Proposition 4.2(ii)], $\mathcal{A}_{k-1}$ is Connes amenable. Repeating this method on k implies that $\mathcal{A}$ is Connes amenable.
(ii) Suppose that S_n is surjective for all n and $\mathcal{A}_k$ is Connes biprojective for some k. Let $\varphi : \mathcal{A}_k \to \mathcal{A}_{k-1}$ be as in (i). Clearly φ is a wk^*-continuous epimorphism. Applying [14, Theorem 2.7(ii)] follows that $\mathcal{A}_{k-1}$ is Connes biprojective. Repeating this method on k, gives that $\mathcal{A}$ is Connes biprojective. □

A Banach algebra $\mathcal{A}$ is called pseudo-amenable if there exists a not necessarily bounded net (u_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a \cdot u_\alpha - u_\alpha \cdot a \to 0$ and $\pi_{\mathcal{A}}(u_\alpha)a \to a$ for every $a \in \mathcal{A}$. For more details, see [4].

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and let $T \in Mul_l(\mathcal{A})$. Then the following statements hold:*

(i) *Suppose that T is surjective and $\mathcal{A}_T$ is pseudo-amenable. Then $\mathcal{A}$ is pseudo-amenable.*
(ii) *Suppose that T is bijective and $\mathcal{A}$ is pseudo-amenable. Then $\mathcal{A}_T$ is pseudo-amenable.*

Proof. (i) Define $\theta : \mathcal{A}_T \longrightarrow \mathcal{A}$ by $\theta(a) = T(a)$ for every $a \in \mathcal{A}_T$. So

$$\theta(a * b) = \theta(aT(b)) = T(aT(b)) = T(a)T(b) = \theta(a)\theta(b) \quad (a, b \in \mathcal{A}_T).$$

Since T is surjective and continuous, θ is a continuous epimorphism. Applying [4, Proposition 2.2] gives that $\mathcal{A}$ is pseudo amenable.
(ii) Since T is bijective, it is invertible. By open mapping theorem T^{-1} is a continuous and [7, Lemma 2.3] implies that $T^{-1} \in Mul_l(\mathcal{A})$. Define $\theta : \mathcal{A} \to \mathcal{A}_T$ by $\theta(a) = T^{-1}(a)$. It is easy to see that θ is continuous epimorphism. Applying [4, Proposition 2.2], $\mathcal{A}$ is pseudo-amenable. □

Theorem 2.5. *Let (S_n) be a sequence of elements in $Mul_l(\mathcal{A})$. Then the following statements hold:*

(i) *Let S_n be surjective for all n. Suppose that $\mathcal{A}_k$ is pseudo-amenable for some k. Then $\mathcal{A}$ is pseudo-amenable.*
(ii) *Suppose that S_n is invertible for every n. Let $\mathcal{A}$ be a pseudo-amenable Banach algebra. Then $\mathcal{A}_n$ is pseudo-amenable for every n.*

Proof. (i) Let S_n be surjective for all n. Suppose that $\mathcal{A}_k$ is pseudo-amenable for some k. If $k = 1$, then Theorem 2.4(i) implies that $\mathcal{A}$ is pseudo-amenable. Now suppose that $k > 1$ and $\mathcal{A}_k$ is pseudo amenable. Define $\varphi : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by $\varphi(a) = S_k(a)$ for every $a \in \mathcal{A}_k$. By similar arguments as in the proof of Theorem 2.3(i) and by [4, Proposition 2.2], $\mathcal{A}_{k-1}$ is pseudo amenable. It follows that $\mathcal{A}$ is pseudo-amenable.
(ii) By induction, we show that $\mathcal{A}_n$ is pseudo-amenable for every n. If $n = 1$, Theorem 2.4(ii) implies that $\mathcal{A}_1$ is pseudo amenable. Now suppose that $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are pseudo-amenable. Define $\theta : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by $\theta(a) = S_{n+1}^{-1}(a)$ for every $a \in \mathcal{A}$.

$$\begin{aligned}\theta(a *_n b) &= S_{n+1}^{-1}(aT_n(b)) = S_{n+1}^{-1}(aT_{n+1}S_{n+1}^{-1}(b)) \\ &= S_{n+1}^{-1}(a)T_{n+1}(S_{n+1}^{-1}(b)) = \theta(a) *_{n+1} \theta(b).\end{aligned}$$

Then θ is a continuous epimorphism. Using [4, Proposition 2.2], $\mathcal{A}_{n+1}$ is pseudo-amenable. □

A Banach algebra $\mathcal{A}$ is called biprojective, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ such that ρ is a right inverse for $\pi_{\mathcal{A}}$, see [2].

Theorem 2.6. *Let (T, R) be a double centralizer of $\mathcal{A}$ such that T is bijective. If $\mathcal{A}$ is biprojective, then $\mathcal{A}_T$ is biprojective.*

Proof. Since $\mathcal{A}$ is biprojective, there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho(a) = a$ for every $a \in \mathcal{A}$. Since T is bijective, it is invertible. By open mapping theorem T^{-1} is continuous. Now [7, Lemma 2.3] implies that $T^{-1} \in Mul_l(\mathcal{A})$. Consider the bounded maps $\psi : \mathcal{A} \to \mathcal{A}_T; a \mapsto T^{-1}(a)$ for every $a \in \mathcal{A}$. Define the map $\rho_T : \mathcal{A}_T \to \mathcal{A}_T\hat{\otimes}\mathcal{A}_T$ by $\rho_T(a) = (i \otimes \psi) \circ \rho(a)$, where $i : \mathcal{A} \to \mathcal{A}_T; a \mapsto a$ for every $a \in \mathcal{A}$. It is easy to see that ρ_T is a bounded map. We show that ρ_T is a bounded $\mathcal{A}_T$-bimodule morphism. To see this, take an arbitrary element $a \in \mathcal{A}$. Without loss of generality, suppose that $\rho(a) = b \otimes c$, for some $b, c \in \mathcal{A}$. So we have

$$\begin{aligned}\rho_T(\lambda * a) &= \rho_T(\lambda T(a)) = \rho_T(R(\lambda)a) = (i \otimes \psi) \circ \rho(R(\lambda)a) \\ &= (i \otimes \psi)(R(\lambda)b \otimes c) = R(\lambda)b \otimes T^{-1}(c) \\ &= \lambda T(b) \otimes T^{-1}(c) = \lambda * b \otimes T^{-1}(c) \\ &= \lambda \cdot \rho_T(a) \qquad (\lambda \in \mathcal{A}_T),\end{aligned}$$

and

$$\begin{aligned}\rho_T(a * \lambda) &= \rho_T(aT(\lambda)) = (i \otimes \psi) \circ \rho(aT(\lambda)) = (i \otimes \psi)(b \otimes cT(\lambda)) \\ &= b \otimes T^{-1}(cT(\lambda)) = b \otimes T^{-1}(c)T(\lambda) = b \otimes T^{-1}(c) * \lambda \\ &= \rho_T(a) \cdot \lambda \qquad (\lambda \in \mathcal{A}_T).\end{aligned}$$

Also we have

$$\begin{aligned}\pi_T \rho_T(a) &= \pi_T(i \otimes \psi)(b \otimes c) = \pi_T(b \otimes T^{-1}(c)) \\ &= b * T^{-1}(c) = bc = \pi_{\mathcal{A}}(b \otimes c) = \pi_{\mathcal{A}}\rho(a) \\ &= a.\end{aligned}$$

It gives that $\mathcal{A}_T$ is biprojective. □

The notion of φ-amenability for a Banach algebra was introduced by Kaniuth, Lau and Pym [6], where φ is a non-zero multiplicative linear functional on $\mathcal{A}$. Indeed $\mathcal{A}$ is φ-amenable if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for every $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. They characterized φ-amenability through the existence of a bounded net (u_α) in $\mathcal{A}$ such that $\|au_\alpha - \varphi(a)u_\alpha\| \to 0$ and $\varphi(u_\alpha) = 1$ for every α and $a \in \mathcal{A}$, see [6, Theorem 1.4]. The set of all non-zero characters on $\mathcal{A}$ is denoted by $\Delta(\mathcal{A})$.

Let (T, R) be a double centralizer of $\mathcal{A}$. One can see that φR and φT are non-zero characters on $\mathcal{A}_T$, where $\varphi \in \Delta(\mathcal{A})$.

Theorem 2.7. *Let (T, R) be a double centralizer of $\mathcal{A}$ and let $\varphi \in \Delta(\mathcal{A})$ such that $Im(R) \nsubseteq \ker(\varphi)$. If $\mathcal{A}$ is φ-amenable, then $\mathcal{A}_T$ is φR-amenable.*

Proof. Since $\mathcal{A}$ is φ-amenable, by [6, Theorem 1.4] there exists a bounded net (u_α) in $\mathcal{A}$ such that $au_\alpha - \varphi(a)u_\alpha \to 0$ and $\varphi(u_\alpha) = 1$ for every α and $a \in \mathcal{A}$. So we have

$$a * u_\alpha - \varphi R(a)u_\alpha = aT(u_\alpha) - \varphi(R(a))u_\alpha = R(a)u_\alpha - \varphi(R(a))u_\alpha \to 0.$$

Since $Im(R) \nsubseteq \ker(\varphi)$, we have $\varphi R(u_\alpha) \neq 0$. Let $\tilde{u}_\alpha = \frac{u_\alpha}{\varphi R(u_\alpha)}$. Then $a * \tilde{u}_\alpha - \varphi R(a)\tilde{u}_\alpha \to 0$ and $\varphi R(\tilde{u}_\alpha) = 1$ for every α and for every $a \in \mathcal{A}_T$. It implies that $\mathcal{A}_T$ is φR-amenable. □

A Banach algebra $\mathcal{A}$ is left φ-contractible, if there exists $m \in \mathcal{A}$ such that $am = \varphi(a)m$ and $\varphi(m) = 1$, for every $a \in \mathcal{A}$ [5].

Theorem 2.8. *Let (T, R) be a double centralizer of $\mathcal{A}$ and let $\varphi \in \Delta(\mathcal{A})$ such that $Im(R) \nsubseteq \ker(\varphi)$. If $\mathcal{A}$ is left φ-contractible, then $\mathcal{A}_T$ is left φR-contractible.*

Proof. It is similar to the arguments as in the proof of Theorem 2.7. □

3. Johnson pseudo-contractibility of $\mathcal{A}_T$

The notion of Johnson pseudo-contractibility for a Banach algebra was introduced by Sahami *et. al.*. Although it is weaker than amenability and pseudo-contractibility, it is stronger than pseudo-amenability, see [9]. A Banach algebra $\mathcal{A}$ is called Johnson pseudo-contractible, if there exists a not necessarily bounded net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_{\mathcal{A}}^{**}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and let $T \in Mul_l(\mathcal{A})$. Then the following statements hold:*

(i) *If T is surjective and $\mathcal{A}_T$ is Johnson pseudo-contractible, then $\mathcal{A}$ is Johnson pseudo-contractible.*

(ii) *If T is bijective and $\mathcal{A}$ is Johnson pseudo-contractible, then $\mathcal{A}_T$ is Johnson pseudo-contractible.*

Proof. (i) Use the similar argument as in the proof of Theorem 2.2(i) and apply [9, Proposition 2.9] which finishes the proof.
(ii) Since T is bijective, it is invertible. By open mapping theorem T^{-1} is continuous and [7, Lemma 2.3] implies that $T^{-1} \in Mul_l(\mathcal{A})$. Define $\theta : \mathcal{A} \to \mathcal{A}_T$ by $\theta(a) = T^{-1}(a)$. It is easy to see that θ is a continuous epimorphism. By applying [9, Proposition 2.9], $\mathcal{A}_T$ is Johnson pseudo-contractible. □

Theorem 3.2. *Let (S_n) be a sequence of elements in $Mul_l(\mathcal{A})$. Then the following statements hold:*

(i) *Suppose that S_n is surjective for all n. If $\mathcal{A}_k$ is Johnson pseudo-contractible for some k, then $\mathcal{A}$ is Johnson pseudo-contractible.*
(ii) *Suppose that S_n is invertible for every n. If $\mathcal{A}$ is Johnson pseudo-contractible, then $\mathcal{A}_n$ is Johnson pseudo-contractible for every n.*

Proof. (i) Suppose that $\mathcal{A}_k$ is Johnson pseudo-contractible for some k. If $k = 1$, then Theorem 3.1(i) implies that $\mathcal{A}$ is Johnson pseudo-contractible. Now suppose that $k > 1$ and $\mathcal{A}_k$ is Johnson pseudo-contractible. Define $\varphi : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by $\varphi(a) = S_k(a)$ for every $a \in \mathcal{A}_k$. By similar argument as in the proof of Theorem 2.3 and using [9, Proposition 2.9], $\mathcal{A}_{k-1}$ is Johnson pseudo-contractible. It gives that $\mathcal{A}$ is Johnson pseudo-contractible.
(ii) By induction we show that $\mathcal{A}_n$ is Johnson pseudo-contractible for every n. If $n = 1$, Theorem 3.1 (ii) implies that $\mathcal{A}_1$ is Johnson pseudo-contractible. Now suppose that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are Johnson pseudo-contractible. Define $\theta : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by $\theta(a) = S^{-1}_{n+1}(a)$ for every $a \in \mathcal{A}$.

$$\begin{aligned}\theta(a *_n b) &= S^{-1}_{n+1}(aT_n(b)) = S^{-1}_{n+1}(aT_{n+1}S^{-1}_{n+1}(b))\\ &= S^{-1}_{n+1}(a)T_{n+1}(S^{-1}_{n+1}(b)) = \theta(a) *_{n+1} \theta(b).\end{aligned}$$

Then θ is a continuous epimorphism. Using [9, Proposition 2.9], $\mathcal{A}_{n+1}$ is Johnson pseudo-contractible. It finishes the proof. □

Let $\mathcal{A}$ be a Banach algebra and $(T_\alpha)_\alpha \in I$ be a family of invertible elements of $Mul_l(\mathcal{A})$. So we have the family $(\mathcal{A}_{T_\alpha})_{\alpha\in I}$ of Banach algebras. We define

$$\ell^\infty\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha} = \big\{(x_\alpha)_{\alpha\in I} \mid \quad \forall\alpha; \quad x_\alpha \in \mathcal{A}_{T_\alpha} \text{ and } \sup_\alpha \|x_\alpha\|_{T_\alpha} < \infty\big\}.$$

Consider the subalgebra $c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$ of $\ell^\infty\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$ which is defined by

$$c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha} = \big\{(x_\alpha)_{\alpha\in I} \mid \quad (x_\alpha)_{\alpha\in I} \in \ell^\infty\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha} \text{ and } \lim_\alpha \|x_\alpha\|_{T_\alpha} = 0\big\}.$$

Theorem 3.3. *With above notations, if $\mathcal{A}$ is Johnson pseudo-contractible, then $c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$ is Johnson pseudo-contractible.*

Proof. Since $\mathcal{A}$ is Johnson pseudo-contractible, Theorem 3.1(ii) implies that for every α, $\mathcal{A}_{T_\alpha}$ is Johnson pseudo-contractible. Let $\epsilon > 0$ and F be an arbitrary finite subset $c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$. Then there exists a finite set J of I such that $\|a - P_J(a)\| < \epsilon/2$, where $a \in F$ and P_J is the projection function from $c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$ onto $c_0\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_{T_\alpha}$. Since every $\mathcal{A}_{T_\alpha}$ is Johnson pseudo-contractible, there exists a net (m^α_β) in $(\mathcal{A}_{T_\alpha}\hat{\otimes}\mathcal{A}_{T_\alpha})^{**}$ such that $x \cdot m^\alpha_\beta = m^\alpha_\beta \cdot x$ and $\lim_\beta \pi^{**}_{\mathcal{A}_{T_\alpha}}(m^\alpha_\beta) \cdot x = x$ for every $x \in \mathcal{A}_{T_\alpha}$. We know that $\mathcal{A}_{T_\alpha}$ is complemented in $c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$, for every α. Thus we have an embedding $L_\alpha : \mathcal{A}_{T_\alpha}\hat{\otimes}\mathcal{A}_{T_\alpha} \to (c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha})\hat{\otimes}(c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha})$. Follow the arguments as in the proof of [9, Theorem 2.11], let $n_\beta = \sum_{\alpha\in J} L_\alpha^{**}(m_\beta^\alpha)$. So for each $a \in F$,

$$\begin{aligned} a \cdot n_\beta &= \sum_{\alpha\in J} a \cdot L_\alpha^{**}(m_\beta^\alpha) = \sum_{\alpha\in J} L_\alpha^{**}(a \cdot m_\beta^\alpha) = \sum_{\alpha\in J} L_\alpha^{**}(P_\alpha(a) \cdot m_\beta^\alpha) \\ &= \sum_{\alpha\in J} L_\alpha^{**}(m_\beta^\alpha \cdot P_\alpha(a)) = \sum_{\alpha\in J} L_\alpha^{**}(m_\beta^\alpha \cdot a) = \sum_{\alpha\in J} L_\alpha^{**}(m_\beta^\alpha) \cdot a \\ &= n_\beta \cdot a, \end{aligned}$$

and also for a large enough β we have

$$\sum_{\alpha\in J} \|\pi_{\mathcal{A}_{T_\alpha}}^{**}(m_\beta^\alpha) \cdot P_\alpha(a) - P_\alpha(a)\| < \epsilon/2.$$

Hence for each $a \in F$

$$\begin{aligned} \|\pi_{\mathcal{A}}^{**}(n_\beta) \cdot a - a\| &= \|\pi_{\mathcal{A}}^{**}(n_\beta \cdot a) - a\| = \|\pi_{\mathcal{A}}^{**}(n_\beta \cdot P_J(a)) - a\| \\ &= \|\pi_{\mathcal{A}}^{**}(n_\beta \cdot P_J(a)) - P_J(a) + P_J(a) - a\| \\ &\le \|\pi_{\mathcal{A}}^{**}(n_\beta \cdot P_J(a)) - P_J(a)\| + \|P_J(a) - a\| \\ &\le \sum_{\alpha\in J} \|\pi_{\mathcal{A}_{T_\alpha}}^{**}(m_\beta^\alpha) \cdot P_\alpha(a) - P_\alpha(a)\| + \|P_J(a) - a\| \\ &\le \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore $c_0\text{-} \bigoplus_{\alpha\in I} \mathcal{A}_{T_\alpha}$ is Johnson pseudo-contractible. □

A dual Banach algebra $\mathcal{A}$ is called Johnson pseudo-Connes amenable, if there exists a not necessarily bounded net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a \cdot m_\alpha\rangle = \langle T, m_\alpha \cdot a\rangle$ and $i_{\mathcal{A}_*}^*\pi_{\mathcal{A}}^{**}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$, where $i_{\mathcal{A}_*} : \mathcal{A}_* \hookrightarrow \mathcal{A}^*$ is the canonical embedding. For more details see [13].

Theorem 3.4. *Let $\mathcal{A}$ be a dual Banach algebra. Then the following statements hold:*

(i) *Let $T \in Mul_l(\mathcal{A})$ be surjective and wk^*-continuous. If $\mathcal{A}_T$ is Johnson pseudo-Connes amenable, then $\mathcal{A}$ is Johnson pseudo-Connes amenable.*

(ii) *Let (S_n) be a sequence of elements in $Mul_l(\mathcal{A})$ such that S_n is wk^*-continuous for every $n \in \mathbb{N}$. If S_n is surjective for all n and $\mathcal{A}_k$ is Johnson pseudo-Connes amenable for some k, then $\mathcal{A}$ is Johnson pseudo-Connes amenable.*

Proof. (i) Suppose that $\mathcal{A}_T$ is Johnson pseudo-Connes amenable. Define $\theta : \mathcal{A}_T \longrightarrow \mathcal{A}$ by $\theta(a) = T(a)$ for every $a \in \mathcal{A}_T$. So

$$\theta(a * b) = \theta(aT(b)) = T(aT(b)) = T(a)T(b) = \theta(a)\theta(b) \quad (a, b \in \mathcal{A}_T).$$

Since T is surjective and wk^*-continuous, θ is a wk^*-continuous epimorphism. Applying [13, Proposition 2.8] follows that $\mathcal{A}$ is Johnson pseudo-Connes amenable.
(ii) Suppose that $\mathcal{A}_k$ is Johnson pseudo-Connes amenable for some k. If $k = 1$, then (i) implies that $\mathcal{A}$ is Johnson pseudo-Connes amenable. Now suppose that $k > 1$ and $\mathcal{A}_k$ is Johnson pseudo-Connes amenable. Define $\varphi : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by $\varphi(a) = S_k(a)$ for every $a \in \mathcal{A}_k$. Similar arguments as in th eproof of Theorem 2.3, and using [13, Proposition 2.8] follows that $\mathcal{A}_{k-1}$ is Johnson pseudo-Connes amenable. By repeating this method on k, $\mathcal{A}$ is Johnson pseudo-Connes amenable. □

4. Module amenability of $\mathcal{A}_T$

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with the compatible actions:

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha),$$

for every $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$ and $x \in X$. Similarly the right action is defined. In this case we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for every $a \in \mathcal{A}$, $x \in X$, then X is called a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. If X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as follows:

$$\langle \alpha \cdot f, x\rangle = \langle f, x \cdot \alpha\rangle, \quad \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad (\alpha \in \mathfrak{A}, a \in \mathcal{A}, x \in X, f \in X^*).$$

Similarly we can define the right action. Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D : \mathcal{A} \to X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

If X is commutative, then each $x \in X$ defines a module derivation as follows:

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}),$$

These are called inner module derivations. $\mathcal{A}$ is called module amenable as an $\mathfrak{A}$-module, if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, every module derivation $D : \mathcal{A} \to X^*$ is inner [1, Definition 2.1]. Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and let $T \in Mul_l(\mathcal{A})$ satisfying $T(a \cdot \alpha) = T(a) \cdot \alpha$ for every $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$. Then $\mathcal{A}_T$ be a Banach $\mathfrak{A}$-bimodule with the following actions:

$$\alpha \cdot (a * b) = (\alpha \cdot a) * b, \quad (a * b) \cdot \alpha = a * (b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}),$$

where $\cdot$ is the module action of $\mathfrak{A}$ on $\mathcal{A}$.

Theorem 4.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and let $T \in Mul_l(\mathcal{A})$ satisfying $T(a \cdot \alpha) = T(a) \cdot \alpha$ for every $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$. Then following statements hold:*

(i) *If T is surjective and $\mathcal{A}_T$ is module $\mathfrak{A}$-amenable, then $\mathcal{A}$ is module $\mathfrak{A}$-amenable.*
(ii) *If T is bijective and $\mathcal{A}$ is module $\mathfrak{A}$-amenable, then $\mathcal{A}_T$ is module $\mathfrak{A}$-amenable.*

Proof. (i) Define $\psi : \mathcal{A}_T \longrightarrow \mathcal{A}$ by $\psi(a) = T(a)$ for every $a \in \mathcal{A}_T$. So

$$\psi(a * b) = \psi(aT(b)) = T(aT(b)) = T(a)T(b) = \psi(a)\psi(b) \quad (a, b \in \mathcal{A}_T).$$

Since T is surjective and continuous, ψ is a continuous epimorphism. Applying [1, Proposition 2.5], $\mathcal{A}$ is module $\mathfrak{A}$-amenable.
(ii) Since T is bijective, it is invertible. By open mapping theorem T^{-1} is continuous and [7, Lemma 2.3] implies that $T^{-1} \in Mul_l(\mathcal{A})$. Define $\theta : \mathcal{A} \to \mathcal{A}_T$ by $\theta(a) = T^{-1}(a)$. It is easy to see that θ is continuous epimorphism. applying [1, Proposition 2.5], $\mathcal{A}_T$ is module $\mathfrak{A}$-amenable. □

Theorem 4.2. *Let (S_n) be a sequence of elements in $Mul_l(\mathcal{A})$ such that for every n, $S_n(a \cdot \alpha) = S_n(a) \cdot \alpha$, where $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$. Then the following statements hold:*

(i) *Suppose that S_n is surjective for all n. If $\mathcal{A}_k$ is module $\mathfrak{A}$-amenable for some k, then so is $\mathcal{A}$.*

(ii) *If S_n is invertible for every n and $\mathcal{A}$ is module $\mathfrak{A}$-amenable, then $\mathcal{A}_n$ is module $\mathfrak{A}$-amenable for every n.*

Proof. (i) Suppose that $\mathcal{A}_k$ is module $\mathfrak{A}$-amenable for some k. If $k = 1$, then Theorem 4.1(i) implies that $\mathcal{A}$ is module $\mathfrak{A}$-amenable. Now suppose that $k > 1$ and $\mathcal{A}_k$ is module $\mathfrak{A}$-amenable. Define $\varphi : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by $\varphi(a) = S_k(a)$ for every $a \in \mathcal{A}_k$. By similar argument as in the proof of Theorem 2.3 and using [1, Proposition 2.5], follows that $\mathcal{A}_{k-1}$ is module $\mathfrak{A}$-amenable. By repeating this method on k, gives that $\mathcal{A}$ is module $\mathfrak{A}$-amenable.
(ii) By induction, we show that $\mathcal{A}_n$ is module $\mathfrak{A}$-amenable for every n. If $n = 1$, then Theorem 4.1 (ii) implies that $\mathcal{A}_1$ is module $\mathfrak{A}$-amenable. Now suppose that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are module $\mathfrak{A}$-amenable. Define $\theta : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by $\theta(a) = S_{n+1}^{-1}(a)$ for every $a \in \mathcal{A}$.

$$\begin{aligned}\theta(a *_n b) &= S_{n+1}^{-1}(aT_n(b)) = S_{n+1}^{-1}(aT_{n+1}S_{n+1}^{-1}(b)) \\ &= S_{n+1}^{-1}(a)T_{n+1}(S_{n+1}^{-1}(b)) = \theta(a) *_{n+1} \theta(b).\end{aligned}$$

Then θ is a continuous epimorphism. Using [1, Proposition 2.5], $\mathcal{A}_{n+1}$ is module $\mathfrak{A}$-amenable. It finishes the proof. □

5. ϕ-homological properties of some matrix algebras

Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is called φ-biprojective, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\varphi \circ \pi_A \circ \rho = \varphi$. Also $\mathcal{A}$ is called φ-biflat if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\tilde{\varphi} \circ \pi_A^{**} \circ \rho = \varphi$, where $\tilde{\varphi}$ is a character on $\mathcal{A}^{**}$ defined by $\tilde{\varphi}(F) = F(\varphi)$ for every $F \in \mathcal{A}^{**}$. Obviously every φ-biprojective Banach algebra is φ-biflat. For more details see [12].

Example 5.1. *Let $\mathcal{A} = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. With the usual matrix multiplication and ℓ^1-norm, $\mathcal{A}$ is a Banach algebra. Consider a character φ on $\mathcal{A}$ defined by $\varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = c$. We show that $\mathcal{A}$ is not ϕ-biflat but $\mathcal{A}_T$ is ϕ-biflat, for some $T \in Mul_l(\mathcal{A})$ (it follows that $\mathcal{A}$ is not φ-biprojective). Suppose towards a contradiction that $\mathcal{A}$ is φ-biflat. Since $\mathcal{A}$ has an identity,* [11, Theorem 2.1] *implies that $\mathcal{A}$ is left ϕ-amenable. Consider the closed ideal $I = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ of $\mathcal{A}$. Since $\varphi|_I \neq 0$, by* [6, Lemma 3.1], *I is left φ-amenable. So there exists a net $(\begin{pmatrix} 0 & u_\alpha \\ 0 & v_\alpha \end{pmatrix})$ in I such that*

$$\begin{pmatrix} 0 & v_\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & u_\alpha \\ 0 & v_\alpha \end{pmatrix} - \varphi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 0 & u_\alpha \\ 0 & v_\alpha \end{pmatrix} \to 0,$$

and

$$\varphi(\begin{pmatrix} 0 & u_\alpha \\ 0 & v_\alpha \end{pmatrix}) = v_\alpha = 1,$$

which is a contradiction with $v_\alpha \to 0$. Define

$$T : \mathcal{A} \longrightarrow \mathcal{A}$$
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

It is easy to see that $T \in Mul_l(\mathcal{A})$. So $\mathcal{A}_T$ is a Banach algebra with the multiplication $*$ [7, Theorem 2.1(i)]. *Define $\rho : \mathcal{A}_T \to \mathcal{A}_T\hat{\otimes}\mathcal{A}_T$ by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.*

One can see that ρ is a bounded $\mathcal{A}_T$-bimodule morphism and also $\varphi\circ\pi_{\mathcal{A}_T}\circ\rho(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})$. *So $\mathcal{A}_T$ is φ-biprojective. It follows that $\mathcal{A}_T$ is φ-biflat.*

Now consider a character φ on $\mathcal{A}$ such that $\varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = a$. *We show that $\mathcal{A}$ is not φ-biflat but $\mathcal{A}_T$ is ϕ-biflat, for some $T \in Mul_l(\mathcal{A})$. It gives that $\mathcal{A}$ is not φ-biprojective.*

To see this assume in a contradiction that $\mathcal{A}$ is φ-biflat. Since $\mathcal{A}$ has an identity, [11, Theorem 2.1] *implies that $\mathcal{A}$ is right ϕ-amenable. Consider the closed ideal* $J = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix}$ *of $\mathcal{A}$. Since $\varphi|_J \neq 0$, similar to* [6, Lemma 3.1], *J is right φ-amenable. Then there exists a net (a_α) in J such that $a_\alpha X - \varphi(X)a_\alpha \to 0$ for every $X \in J$ and $\varphi(a_\alpha) = 1$. Pick an element $X \in J$ such that $\varphi(X) = 1$. So*

$$X - \varphi(X)a_\alpha = \varphi(a_\alpha)X - \varphi(X)a_\alpha = a_\alpha X - \varphi(X)a_\alpha \to 0.$$

Thus $a_\alpha \to X$. Hence for each $Y \in J$, we have $Y - \varphi(Y)a_\alpha = a_\alpha Y - \varphi(Y)a_\alpha \to 0$. It gives that $\varphi(Y)X = Y$. Therefore $\dim(J) = 1$, *which is a contradiction. Define*

$$T : \mathcal{A} \longrightarrow \mathcal{A}$$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $T \in Mul_l(\mathcal{A})$. So $\mathcal{A}_T$ is a Banach algebra with the multiplication $*$ [7, Theorem 2.1(i)]. *Define* $\rho : \mathcal{A}_T \to \mathcal{A}_T\hat{\otimes}\mathcal{A}_T$ *by* $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. *One can see that ρ is a bounded $\mathcal{A}_T$-bimodule morphism and also* $\varphi\circ\pi_{\mathcal{A}_T}\circ\rho\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})$. *So $\mathcal{A}_T$ is φ-biprojective. It follows that $\mathcal{A}_T$ is φ-biflat.*

Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is called approximate left (right) φ-biprojective if there exists a net of bounded linear maps from $\mathcal{A}$ into $\mathcal{A}\hat{\otimes}\mathcal{A}$, say $(\rho_\alpha)_{\alpha\in I}$, such that

(i) $\rho_\alpha(ab) - a\cdot\rho_\alpha(b) \to 0 \qquad (\rho_\alpha(ab) - \rho_\alpha(a)\cdot b \to 0)$,
(ii) $\rho_\alpha(ab) - \varphi(b)\rho_\alpha(a) \to 0 \qquad (\rho_\alpha(ab) - \varphi(a)\rho_\alpha(b) \to 0)$,
(iii) $\varphi\circ\pi_{\mathcal{A}}\circ\rho_\alpha(a) - \varphi(a) \to 0$.

for every $a, b \in \mathcal{A}$, respectively. For more details see [10].

Example 5.2. *The Banach algebra of $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{C}$, with finite ℓ^1-norm and matrix multiplication is denoted by*

$$UP(\mathbb{N},\mathbb{C}) = \Big\{ \left[\ a_{i,j}\ \right]_{i,j\in\mathbb{N}} ; a_{i,j} \in \mathbb{C} \text{ and } a_{i,j} = 0 \text{ for every } i > j \Big\}.$$

Consider a character φ on $UP(\mathbb{N},\mathbb{C})$ such that $\varphi(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ 0 & a_{22} & a_{23} & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}) = a_{11}$. *We claim that $UP(\mathbb{N},\mathbb{C})$ is not φ-biflat (so $UP(\mathbb{N},\mathbb{C})$ is not φ-biprojective). Suppose in contradiction that $UP(\mathbb{N},\mathbb{C})$ is φ-biflat. Since $UP(\mathbb{N},\mathbb{C})$ has a right approximate identity* [11, Lemma 5.1], *$UP(\mathbb{N},\mathbb{C})$ is right ϕ-amenable* [11, Theorem 2.1]. *Consider the closed ideal*

$$J = \Big\{ \left[\ a_{i,j}\ \right]_{i,j\in\mathbb{N}} \in UP(\mathbb{N},\mathbb{C}) \mid \quad a_{i,j} = 0 \text{ for every } i \neq 1 \Big\},$$

of $UP(\mathbb{N},\mathbb{C})$. *We know that* J *has at least two columns, thus* $\dim(J) \geq 2$. *Since* $\varphi|_J \neq 0$, *by similar to* [6, Lemma 3.1], J *is right* φ*-amenable. So there exists a net* (a_α) *in* J *such that* $a_\alpha X - \varphi(X)a_\alpha \to 0$ *and* $\varphi(a_\alpha) = 1$, *for every* $X \in J$. *Pick an element* $X \in J$ *such that* $\varphi(X) = 1$. *So*

$$X - \varphi(X)a_\alpha = \varphi(a_\alpha)X - \varphi(X)a_\alpha = a_\alpha X - \varphi(X)a_\alpha \to 0.$$

Then $a_\alpha \to X$. *We have*

$$Y - \varphi(Y)a_\alpha = a_\alpha Y - \varphi(Y)a_\alpha \to 0 \quad (Y \in J).$$

So $\varphi(Y)X = Y$. *Therefore* $\dim(J) = 1$, *which is a contradiction.*

We show that $UP(\mathbb{N},\mathbb{C})_T$ *is approximate right* φ*-biprojective. Define* $T : UP(\mathbb{N},\mathbb{C}) \longrightarrow UP(\mathbb{N},\mathbb{C})$ *by* $T\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & a_{22} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$. *It is easy to see that* $T \in Mul_l(UP(\mathbb{N},\mathbb{C}))$. *So* $UP(\mathbb{N},\mathbb{C})_T$ *is a Banach algebra with the multiplication* $*$ [7, Theorem 2.1(i)]. *For each* $n \in \mathbb{N}$, *define* $\rho_n : UP(\mathbb{N},\mathbb{C})_T \to UP(\mathbb{N},\mathbb{C})_T \hat{\otimes} UP(\mathbb{N},\mathbb{C})_T$ *by*

$$\rho_n \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & a_{22} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix} = \begin{pmatrix} \overbrace{1 \;\; \cdots \;\; 1}^{n} & 0 & \cdots \\ 0 \;\; \cdots \;\; 0 & 0 & \cdots \\ \vdots \;\;\; \vdots \;\;\; \vdots & \vdots & \cdots \end{pmatrix} \otimes \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

One can see that ρ_n *is a bounded linear map for every* $n \in \mathbb{N}$. *Also we have*

(i) $\rho_n(XY) - \rho_n(X) \cdot Y \to 0$,

(ii) $\rho_n(XY) - \varphi(X)\rho_n(Y) \to 0$,

(iii) $\varphi \circ \pi_{UP(\mathbb{N},\mathbb{C})_T} \circ \rho_n(X) - \varphi(X) \to 0$,

for every X, Y *in* $UP(\mathbb{N},\mathbb{C})_T$. *So* $UP(\mathbb{N},\mathbb{C})_T$ *is approximate right* φ*-biprojective.*

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