NEW ITERATION PROCESS AND NUMERICAL RECKONING FIXED POINTS IN BANACH SPACES

Kifayat Ullah[1], Muhammad Arshad[2]

In this paper we propose a new iteration process, called M^* iteration process, for approximation of fixed points. Some weak and strong convergence theorems for fixed point of Suzuki generalized nonexpansive mappings are proved in the setting of uniformly convex Banach spaces. A numerical examples is given to show the efficiency of M^* iteration process. Our results are the extension, improvement and generalization of many known results in the literature of iterations in fixed point theory.

Keywords: Suzuki generalized nonexpansive mapping; Uniformly convex Banach space; Iteration process; Weak convergence; Strong convergence.

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1. Introduction

Numerical reckoning fixed points for nonlinear operators is nowadays an active research direction of nonlinear analysis. This because they found applications to: variational inequalities, equilibrium problems, computer simulation, image encoding and much more. Classical iterations such as Picard, Mann and Ishikawa represent pioneers research work in this regard; please, see Mann [12] and Ishikawa [9]. Nowadays, this research direction is developed by: Agarwal et al. [2], Noor [13], Abbas et al. [1], Phuengrattana et al. [16], Karahan et al. [10], Chugh et al. [4], Sahu et al. [17], Khan [11], Gursoy et al. [8], Thakur et al. [21, 22, 23, 24, 25, 26] and Yao et al. [27, 28, 29, 30].

Motivated by above, in this paper, we introduce a new iteration process namely M^* iteration process for numerical reckoning fixed points of nonlinear mappings. An example of Suzuki generalized nonexpansive mapping is given which is not nonexpansive. Since we found that the speed of convergence of Picard-S iteration process [8] and Thakur New iteration process [22] are almost same so numerically we compare the speed of convergence of our new M^* iteration process with two-step Agarwal iteration process and leading three-step Picard-S iteration process for given example. Finally we prove some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings, which is the generalization of nonexpansive as well as contraction mappings, in the setting of uniformly convex Banach spaces.

[1]Department of Mathematics, International Islamic University, H-10 Islamabad, Pakistan, E-mail: kifayatmath@yahoo.com

[2] Department of Mathematics, International Islamic University, H-10 Islamabad, Pakistan, E-mail: marshadzia@iiu.edu.pk

2. **Preliminaries**

A Banach space X is called uniformly convex [7] if for each $\varepsilon \in (0,2]$ there is a $\delta > 0$ such that for $x, y \in X$,

$$\left.\begin{matrix} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x-y\| > \varepsilon \end{matrix}\right\} \Longrightarrow \left\| \frac{x+y}{2} \right\| \leq \delta.$$

A Banach space X is said to satisfy the Opial property [14] if for each sequence $\{x_n\}$ in X, converging weakly to $x \in X$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in X$ such that $y \neq x$.

A point p is called fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of mapping T. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called contraction if there exists $\theta \in (0,1)$ such that $\|Tx - Ty\| \leq \theta \|x - y\|$, for all $x, y \in C$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have $\|Tx - p\| \leq \|x - p\|$. In 2008, Suzuki [20] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|.$$

Suzuki [20] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The mapping satisfy condition (C) is called Suzuki generalized nonexpansive mapping. Suzuki also obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mapping. Recently, fixed point theorems for Suzuki generalized nonexpansive mapping have been studied by a number of authors see e.g. [22] and references therein. We now list some properties of Suzuki generalized nonexpansive mapping.

Proposition 2.1. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be any mapping. Then*

(i). [20, Proposition 1] *If T is a nonexpansive then T is a Suzuki generalized nonexpansive mapping.*

(ii). [20, Proposition 2] *If T is a Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(iii). [20, Lemma 7] *If T is an Suzuki generalized nonexpansive mapping, then $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$ for all $x, y \in C$.*

Lemma 2.1. [20, Proposition 3] *Let T be a mapping on a subset C of a Banach space X with the Opial property. Assume that T is a Suzuki generalized nonexpansive mapping. If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Lemma 2.2. [20, Theorem 5] *Let C be a weakly compact convex subset of a uniformly convex Banach space X. Let T be a mapping on C. Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.3. [18, Lemma 1.3] *Suppose that X is a uniformly convex Banach space and $\{t_n\}$ is any real sequence such that $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(C, \{x_n\})$ consists of exactly one point.

3. M^* iteration Process and Convergence Results

Through out this section, we have $n \geq 0$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$, C be any non-empty subset of Banach Space X.

Agarwal et al. [2] introduced the following iteration process with the claim that it converges at a rate that is the same as that of the Picard iteration process and faster than the Mann iteration process for contraction mappings;

$$\begin{cases} x_0 \in C \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n \end{cases} \tag{1}$$

The iteration process (1) is also known as S iteration process.

Recently Gursoy and Karakaya in [8] introduced new iteration process called Picard-S iteration process, as follow

$$\begin{cases} x_0 \in C \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \\ y_n = (1-\alpha_n)T x_n + \alpha_n T z_n \\ x_{n+1} = T y_n \end{cases} \tag{2}$$

They proved that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings. Also, by providing an example, it is shown that the Picard-S iteration process converge faster than all Mann [12], Ishikawa [9], Agarwal [2], Noor [13], Abbas [1], SP [16], S* [10], CR [4], Normal-S [17], and Picard Mann [11] iteration processes.

After this in 2015, Thakur et. al. [22] used the following new iteration process, we will call it Thakur-New iteration process,

$$\begin{cases} x_0 \in C \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n) \\ x_{n+1} = T y_n. \end{cases} \tag{3}$$

With the help of numerical example they proved that (3) is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration process for a Suzuki generalized nonexpansive mappings. We note that the speed of convergence of iteration process (2) and (3) is almost same.

Inspired from above, we introduce the following new iteration process known as "M^* iteration Process"

$$\begin{cases} x_0 \in C \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \\ y_n = T((1-\alpha_n)x_n + \alpha_n T z_n) \\ x_{n+1} = T y_n \end{cases} \tag{4}$$

First we prove weak and strong convergence results for Suzuki generalized nonexpansive mappings and for the sequence generated by iteration process (4). After this we will show that our new iteration process (4) is more efficient comparatively to other iteration processes.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4), then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T is a Suzuki generalized nonexpansive mapping,

$$\frac{1}{2}\|p - Tp\| = 0 \leq \|p - z\| \text{ implies that } \|Tp - Tz\| \leq \|p - z\|.$$

So by Proposition 2.1(ii), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{5}$$

Using (5), we get

$$\begin{aligned} \|y_n - p\| &= \|T((1-\alpha_n)x_n + \alpha_n Tz_n) - p\| \\ &\leq \|(1-\alpha_n)x_n + \alpha_n Tz_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|Tz_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|z_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{6}$$

Similarly, by using (6) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{7}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} \|x_n - p\|$ exists, as required. □

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|x_n - p\| = r. \tag{8}$$

From (5) and (8), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r. \tag{9}$$

From Proposition 2.1(ii), we get

$$\limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r. \tag{10}$$

On the other hand by using (5), we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|Ty_n-p\| \\
&\leq \|y_n-p\| \\
&= \|T((1-\alpha_n)x_n+\alpha_nTz_n)-Tp\| \\
&\leq \|(1-\alpha_n)x_n+\alpha_nTz_n-p\| \\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|Tz_n-Tp\| \\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|z_n-p\| \\
&= \|x_n-p\|-\alpha_n\|x_n-p\|+\alpha_n\|z_n-p\|.
\end{aligned}$$

This implies that

$$\frac{\|x_{n+1}-p\|-\|x_n-p\|}{\alpha_n} \leq \|z_n-p\|-\|x_n-p\|.$$

So

$$\|x_{n+1}-p\|-\|x_n-p\| \leq \frac{\|x_{n+1}-p\|-\|x_n-p\|}{\alpha_n} \leq \|z_n-p\|-\|x_n-p\|,$$

implies that

$$\|x_{n+1}-p\| \leq \|z_n-p\|.$$

Therefore

$$r \leq \liminf_{n\to\infty}\|z_n-p\|. \tag{11}$$

Using (9) and (11), we have

$$\begin{aligned}
r &= \lim_{n\to\infty}\|z_n-p\| \\
&= \lim_{n\to\infty}\|(1-\beta_n)x_n+\beta_nTx_n-p\| \\
&= \lim_{n\to\infty}\|\beta_n(Tx_n-p)+(1-\beta_n)(x_n-p)\|.
\end{aligned} \tag{12}$$

From (8), (10), (12) and Lemma 2.3, we have that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Let $p\in A(C,\{x_n\})$. By Proposition 2.1(*iii*), we have

$$\begin{aligned}
r(Tp,\{x_n\}) &= \limsup_{n\to\infty}\|x_n-Tp\| \\
&\leq \limsup_{n\to\infty}(3\|Tx_n-x_n\|+\|x_n-p\|) \\
&\leq \limsup_{n\to\infty}\|x_n-p\| \\
&= r(p,\{x_n\}).
\end{aligned}$$

This implies that $Tp\in A(C,\{x_n\})$. Since X is uniformly convex, $A(C,\{x_n\})$ is a singleton set and hence we have $Tp=p$. Hence $F(T)\neq\emptyset$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property, and let $T: C\to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0\in C$, let the sequence $\{x_n\}$ be generated by (4) for all $n\geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a,b]$ for some a,b with $0<a\leq b<1$ such that $F(T)\neq\emptyset$. Then $\{x_n\}$ converges weakly to a fixed point of T .*

Proof. $F(T)\neq\emptyset$ implies that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Since X is uniformly convex and hence reflexive, by Eberlin's theorem there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $q_1\in X$. Since C is closed and convex, by Mazur's theorem $q_1\in C$. By Lemma 2.1, $q_1\in F(T)$. Now, we show that $\{x_n\}$ converges weakly to q_1. In fact, if this is not true, then there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_2 \in C$ and $q_2 \neq q_1$. By Lemma 2.1, $q_2 \in F(T)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(T)$. By Theorem 3.1 and Opial's property, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{j\to\infty} \left\|x_{n_j} - q_1\right\| \\
&< \lim_{j\to\infty} \left\|x_{n_j} - q_2\right\| \\
&= \lim_{n\to\infty} \|x_n - q_2\| \\
&= \lim_{k\to\infty} \|x_{n_k} - q_2\| \\
&< \lim_{k\to\infty} \|x_{n_k} - q_1\| \\
&= \lim_{n\to\infty} \|x_n - q_1\|,
\end{aligned}$$

which is a contradiction. So $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to a fixed point of T. □

Next we prove the strong convergence theorem.

Theorem 3.3. *Let C be a nonempty compact convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.2, we have that $F(T) \neq \emptyset$ and so by Theorem 3.1 we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.1(*iii*), we have

$$\|x_{n_k} - Tp\| \leq 3\,\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|, \text{ for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, *i.e.*, $p \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(T)$ and so x_n converge strongly to p. □

Senter and Dotson [19] introduced the notion of a mappings satisfying condition (I) as.

A mapping $T : C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p\in F(T)} \|x - p\|$.

Now we prove the strong convergence theorem using condition (I).

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(T)$ and $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Suppose $r > 0$. From the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\|. \tag{13}$$

Since $F(T) \neq \emptyset$, by Theorem 3.2, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. So (13) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{14}$$

Since f is nondecreasing function, so from (14) we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$\|x_{n_k} - y_k\| < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

So Using (7), we get

$$\left\|x_{n_{k+1}} - y_k\right\| \le \|x_{n_k} - y_k\| < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} \|y_{k+1} - y_k\| &\le \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - y_k\| \\ &\le \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have that $x_n \to p \in F(T)$. □

4. **Numerical example**

For numerical interpretations first we construct an example of a Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 4.1. *Define a function $T : [0,1] \to [0,1]$ by*

$$Tx = \begin{cases} 1 - x \text{ if } x \in \left[0, \frac{1}{9}\right) \\ \frac{x+8}{9} \text{ if } x \in \left[\frac{1}{9}, 1\right]. \end{cases}$$

We need to prove that T is Suzuki generalized nonexpansive but not nonexpansive.

If $x = \frac{11}{100}$, $y = \frac{1}{9}$, then we have

$$\begin{aligned} \|Tx - Ty\| &= |Tx - Ty| \\ &= \left|1 - \frac{11}{100} - \frac{73}{81}\right| \\ &= \frac{91}{8100} \\ &> \frac{1}{900} \\ &= \|x - y\|. \end{aligned}$$

Hence T is not a nonexpansive mapping.

To verify that T is a Suzuki generalized nonexpansive mapping, consider the following cases:

***Case I**: Let $x \in \left[0, \frac{1}{9}\right)$. Then $\frac{1}{2}\|x - Tx\| = \frac{1-2x}{2} \in \left(\frac{7}{18}, \frac{1}{2}\right]$. For $\frac{1}{2}\|x - Tx\| \le \|x - y\|$ we must have $\frac{1-2x}{2} \le y - x$, i.e., $\frac{1}{2} \le y$ and hence $y \in \left[\frac{1}{2}, 1\right]$. We have*

$$\|Tx - Ty\| = \left|\frac{y+8}{9} - (1-x)\right| = \left|\frac{y + 9x - 1}{9}\right| < \frac{1}{9},$$

and

$$\|x - y\| = |x - y| > \left|\frac{1}{9} - \frac{1}{2}\right| = \frac{7}{18}.$$

Hence $\frac{1}{2}\|x - Tx\| \le \|x - y\| \Longrightarrow \|Tx - Ty\| \le \|x - y\|$.

Table 1. Sequences generated by M^*, Picard-S and S iteration processes.

	M^*	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.998765432098765	0.998765432098765	0.988888888888889
x_2	0.999991770471429	0.999986900558966	0.998938945276278
x_3	0.999999976046138	0.999999871532179	0.999906346958218
x_4	1	0.999999998800067	0.999992127236521
x_5	1	0.999999999989159	0.999999359857410
x_6	1	0.999999999999904	0.999999949193996
x_7	1	0.999999999999999	0.999999996041233
x_8	1	1.	0.999999999695968
x_9	1	1.	0.999999999976922
x_{10}	1	1.	0.999999999998265

Case II: *Let* $x \in \left[\frac{1}{9}, 1\right]$. *Then* $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\left|\frac{x+8}{9} - x\right| = \frac{8-8x}{18} \in \left[0, \frac{55}{162}\right]$. *For* $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ *we must have* $\frac{8-8x}{18} \leq |y - x|$, *which gives two possibilities:*

(a). *Let* $x < y$. *Then* $\frac{8-8x}{18} \leq y - x \Longrightarrow y \geq \frac{8+10x}{18} \Longrightarrow y \in \left[\frac{73}{162}, 1\right] \subset \left[\frac{1}{9}, 1\right]$. *So*

$$\|Tx - Ty\| = \left|\frac{x+8}{9} - \frac{y+8}{9}\right| = \frac{1}{9}\|x - y\| \leq \|x - y\|.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\|$.

(b). *Let* $x > y$. *Then* $\frac{8-8x}{18} \leq x - y \Longrightarrow y \leq x - \frac{8-8x}{18} = \frac{26x-8}{18} \Longrightarrow y \in \left[-\frac{37}{162}, 1\right]$. *Since* $y \in [0, 1]$, $y \leq \frac{26x-8}{18} \Longrightarrow x \geq \frac{18y+8}{26} \Longrightarrow x \in \left[\frac{4}{13}, 1\right]$. *So the case is* $x \in \left[\frac{4}{13}, 1\right]$ *an;8d* $y \in [0, 1]$.

Now $x \in \left[\frac{4}{13}, 1\right]$ *and* $y \in \left[\frac{1}{9}, 1\right]$ *is already included in case (a). So let* $x \in \left[\frac{4}{13}, 1\right]$ *and* $y \in \left[0, \frac{1}{9}\right)$. *Then*

$$\begin{aligned} \|Tx - Ty\| &= \left|\frac{x+8}{9} - (1 - y)\right| \\ &= \left|\frac{x + 9y - 1}{9}\right|. \end{aligned}$$

For convenience, first we consider $x \in \left[\frac{4}{13}, \frac{1}{2}\right]$ *and* $y \in \left[0, \frac{1}{9}\right)$. *Then* $\|Tx - Ty\| \leq \frac{1}{18}$ *and* $\|x - y\| \geq \frac{23}{117}$. *Hence* $\|Tx - Ty\| \leq \|x - y\|$.

Next consider $x \in \left[\frac{1}{2}, 1\right]$ *and* $y \in \left[0, \frac{1}{9}\right)$, *then* $\|Tx - Ty\| \leq \frac{1}{9}$ *and* $\|x - y\| \geq \frac{7}{18}$. *Hence* $\|Tx - Ty\| \leq \|x - y\|$. *So* $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\|$.

Hence T *is a Suzuki generalized nonexpansive mapping.*

In Table 1 we can see some of the first terms of a sequence generated by M^*, Picard-S and S iteration processes for $\alpha_n = \frac{2n}{\sqrt{7n+9}}$, $\beta_n = \frac{1}{\sqrt{3n+7}}$, where $x_0 = 0.9$ and operator T is that of Example 4.1. Set the stop parameter to $\|x_n - 1\| \leq 10^{-15}$, where "1" is the fixed point of T. Graphic representation in given in Figure 1. We can easily see that the new M^* iterations are the first converging one than the S iterations and the Picard-S iterations.

In order to see how far from exactly "1" the value of x_n is for a certainly value of n, we resort to arbitrary precision calculations and get Figure 2.

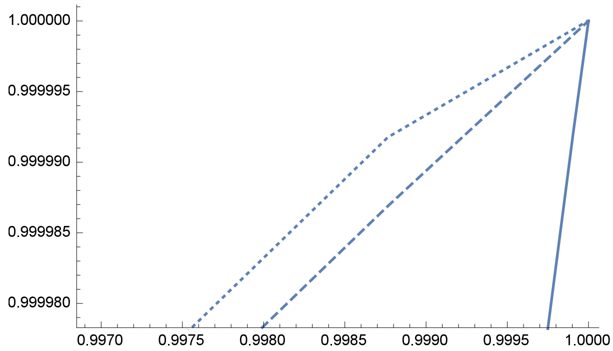

FIGURE 1. Convergence of iterative sequences generated by M^* (dots), Picard-S (dashes) and S (line) iteration processes to the fixed point 1 of mapping T defined in Example 4.1.

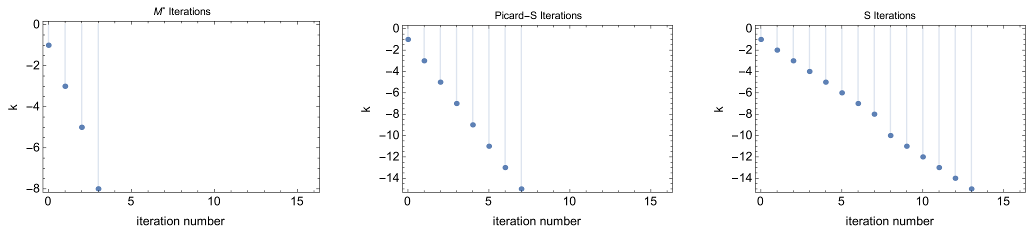

FIGURE 2. Graphs for M^*, Picard-S and S iteration processes where the value of k indicates that the value of the recursion after a certain number of steps is only 10^k units away from fixed point 1 of mapping T defined in Example 4.1.

5. Conclusions

New iteration process (4) namely M^* iteration process is introduced for approximating fixed points of Suzuki generalized nonexpansive mappings. Strong and weak convergence of M^* iteration process to the fixed point of Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces are proved. It is shown that our new iteration process is moving faster than the leading S iteration process (1) and Picard-S iteration process (2) using newly introduced Example 4.1. Our new iteration process is now available for the engineers, computer scientists, physicists as well as mathematicians to solve different problems more efficiently.

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