G-DUAL FRAMES IN HILBERT SPACES

Mohammad Ali Dehghan[1], Mohammad Ali Hasankhani Fard[2]

The duals of frames have an essential role in reconstruction of vectors (or signals) in terms of the frame elements. In this paper, g-duals of a frame in a separable Hilbert space $\mathcal{H}$ are introduced and characterized. By applying g-duals as well, (which also includes usual duals) we can achieve more reconstruction formulas to obtain signals. Also, we show that the set of approximately duals of a frame is a proper subset of the set of its g-duals and some examples of g-dual frames are discussed. Finally, application of g-duals in Gabor frames and perturbation of g-dual frames are given as well.

Keywords: Frame, dual frame, g-dual frame, Gabor frame.
MSC2010: Primary 46L 99; Secondary 47A 05, 42C 15, 46H 25.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [8] in the study of nonharmonic Fourier series in 1952. Frames have very important and interesting properties which make them very useful in the characterization of function spaces, signal processing and many other fields. A frame is a family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [6]. Given a separable Hilbert space $\mathcal{H}$ with inner product $< .,. >$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < C_1, C_2 < \infty$ such that for all $f \in \mathcal{H}$,

$$C_1\|f\|^2 \leq \sum_{k=1}^{\infty} | < f, f_k > |^2 \leq C_2\|f\|^2, \tag{1}$$

where C_1, C_2 are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. If $C_1 = C_2$, then $\{f_k\}_{k=1}^{\infty}$ is called a tight frame. The bounded linear operator T defined by

$$T : \ell_2(\mathbb{N}) \to \mathcal{H}, \quad T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

[1]Corresponding author, Department of Mathematics Vali-e-Asr University, Rafsanjan, Iran, P.O.Box 518, e-mail: dehghan@vru.ac.ir

[2]Ph.D student, Department of Mathematics Vali-e-Asr University, Rafsanjan, Iran, P.O.Box 518

is called the pre frame operator of $\{f_k\}_{k=1}^{\infty}$. Also the bounded linear operator S defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k=1}^{\infty} < f, f_k > f_k$$

is called the frame operator of $\{f_k\}_{k=1}^{\infty}$. It is easy to show that $S = TT^*$, where T^* is the adjoint operator of T. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ae_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $A \in B(\mathcal{H})$ is an invertible operators. Every Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$. For more information concerning frames refer to [2, 3, 4, 5, 9].

Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{H}$ if

$$f = \sum_{k=1}^{\infty} < f, f_k > g_k = \sum_{k=1}^{\infty} < f, g_k > f_k, \; \forall f \in \mathcal{H}.$$

Dual frames are important to reconstruction of vectors (or signals) in terms of the frame elements. Unfortunately, it is usually complicated to calculate a dual frame. Now we are going to extend this concept. We have more reconstruction formulas of vectors (or signals) in terms of the frame elements with use of the g-dual frames. In section 2 we introduce generalized dual frames and find some properties. In section 3 we characterize all generalized dual frames for a given frame. In Section 4 a comparison between approximately dual frames and g-dual frames as well as applications in Gabor frames are given. In Section 5 the perturbation theory of g-dual frames is discussed.

2. g-dual frames

Definition 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. A frame $\{g_k\}_{k=1}^{\infty}$ is called a generalized dual frame or g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ if there exists an invertible operator $A \in B(\mathcal{H})$ such that for all $f \in \mathcal{H}$,*

$$f = \sum_{k=1}^{\infty} < Af, g_k > f_k. \tag{2}$$

If V is a closed subspace of $\mathcal{H}$, a frame $\{g_k\}_{k=1}^{\infty}$ is called a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for V if (2) satisfies for all $f \in V$ and for some invertible operator $A \in B(V)$.

When $A = I$, $\{g_k\}_{k=1}^{\infty}$ is an ordinary dual frame of $\{f_k\}_{k=1}^{\infty}$. If S is the frame operator of the frame $\{f_k\}_{k=1}^{\infty}$, then for all $f \in \mathcal{H}$ we have

$$f = \sum_{k=1}^{\infty} < f, S^{-1} f_k > f_k = \sum_{k=1}^{\infty} < S^{-1} f, f_k > f_k, \tag{3}$$

and hence each frame is a g-dual frame for itself. The operator A in (2) is unique, since for all $f \in \mathcal{H}$,

$$A^{-1} f = \sum_{k=1}^{\infty} < f, g_k > f_k,$$

and hence $A^{-1} = TU^*$, where T and U are the pre-frame operators of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. Also, we say the frame $\{g_k\}_{k=1}^{\infty}$ is a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ with corresponding invertible operator (or with invertible operator) A.

Example 2.1. *Let m be a positive integer number, $\{g_k^i\}_{k=1}^{\infty}$ be a dual frame of frame $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ for $i = 1, 2, ..., m$, $\{c_i\}_{i=1}^{m}$ be a finite sequence of complex numbers such that $\sum_{i=1}^{m} c_i \neq 0$ and $h_k = \sum_{i=1}^{m} c_i g_k^i$. Then $Af = \frac{1}{\sum_{i=1}^{m} c_i} f$ defines a bounded invertible operator on $\mathcal{H}$ and*

$$f = \sum_{k=1}^{\infty} < Af, f_k > h_k, \forall f \in \mathcal{H}.$$

Hence the sequence $\{h_k\}_{k=1}^{\infty}$ is a g-dual frame of $\{f_k\}_{k=1}^{\infty}$.

By (3) the g-duality relation is reflexive whereas ordinary duality relation is not. In the following lemma and example, we show that g-duality relation is symmetric while it is not transitive.

Lemma 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be Bessel sequences in $\mathcal{H}$. Then the following are equivalent*
(a) There exists an invertible operator $A \in B(\mathcal{H})$ such that
$f = \sum_{k=1}^{\infty} < Af, f_k > g_k, \forall f \in \mathcal{H}$
(b) There exists an invertible operator $A \in B(\mathcal{H})$ such that
$f = \sum_{k=1}^{\infty} < A^* f, g_k > f_k, \forall f \in \mathcal{H}$
(c) There exists an invertible operator $A \in B(\mathcal{H})$ such that
$< f, g >= \sum_{k=1}^{\infty} < Af, f_k >< g_k, g >, \forall f, g \in \mathcal{H}.$
In case one of the equivalent conditions are satisfied, $\{g_k\}_{k=1}^{\infty}$ is a g-dual of $\{f_k\}_{k=1}^{\infty}$ and vice versa that are called g-dual frames.

Proof. Let (a) be satisfied and $f \in \mathcal{H}$. Then there exists $g \in \mathcal{H}$, such that $f = Ag$ and $g = \sum_{k=1}^{\infty} < Ag, f_k > g_k$. Therefore $f = Ag = \sum_{k=1}^{\infty} < f, f_k > Ag_k$. Since $\{Ag_k\}_{k=1}^{\infty}$ is Bessel sequence, by [6, Lemma 5.6.2] we have

$$\begin{aligned} f &= \sum_{k=1}^{\infty} < f, f_k > Ag_k \\ &= \sum_{k=1}^{\infty} < f, Ag_k > f_k \\ &= \sum_{k=1}^{\infty} < A^* f, g_k > f_k, \end{aligned}$$

and hence (b) holds. A similar argument shows that (b) derives (a). The rest of proof is as the proof of [6, Lemma 5.6.2]. If the conditions are satisfied for $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, then

$$\begin{aligned} \|f\|^4 &= |\sum_{k=1}^{\infty} < Af, f_k >< g_k, f > |^2 \\ &\leq \sum_{k=1}^{\infty} | < Af, f_k > |^2 \sum_{k=1}^{\infty} | < f, g_k > |^2 \\ &\leq C_2 \|A\|^2 \|f\|^2 \sum_{k=1}^{\infty} | < f, g_k > |^2, \end{aligned}$$

where C_2 is the upper frame bound for $\{f_k\}_{k=1}^{\infty}$. Then $\{g_k\}_{k=1}^{\infty}$ is a frame. Since (a) and (b) are equivalent, $\{f_k\}_{k=1}^{\infty}$ is also a frame. □

Example 2.2. *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Set*
$\{f_k\}_{k=1}^{\infty} = \{e_1, 0, e_2, 0, e_3, 0, ...\}$,
$\{g_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, ...\}$ *and*
$\{h_k\}_{k=1}^{\infty} = \{0, e_1, 0, e_2, 0, e_3, ...\}$.
Then $\{g_k\}_{k=1}^{\infty}$ is a g-dual frame (dual frame) of $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ is a g-dual frame (dual frame) of $\{g_k\}_{k=1}^{\infty}$.
Now we have $< g, f_k > h_k = 0$ for $k = 1, 2, ...$ and for all $g \in \mathcal{H}$. Therefore $\sum_{k=1}^{\infty} < Af, f_k > h_k = 0$, for all $f \in \mathcal{H}$ and for all invertible operator $A \in B(\mathcal{H})$. Hence $\{h_k\}_{k=1}^{\infty}$ is not a g-dual frame of $\{f_k\}_{k=1}^{\infty}$.

An orthonormal basis can not be a dual frame of another orthonormal basis. Now, it is shown that, not only orthonormal bases, but also Riesz bases are g-dual frames.

Proposition 2.1. *Every two Riesz bases are g-dual frames.*

Proof. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two Riesz bases for $\mathcal{H}$.There exist an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and bounded invertible operators A_1 and A_2 on $\mathcal{H}$ such that $f_k = A_1 e_k$ and $g_k = A_2 e_k$. Since A_1 and A_2 are invertible, there exists a bounded invertible operator A on $\mathcal{H}$ such that $A_2 A_1^* A = I$ and hence for all $f \in \mathcal{H}$ we have

$$\begin{aligned} f = A_2 A_1^* A f &= A_2 (\sum_{k=1}^{\infty} < A_1^* A f, e_k > e_k) \\ &= \sum_{k=1}^{\infty} < Af, A_1 e_k > A_2 e_k \\ &= \sum_{k=1}^{\infty} < Af, f_k > g_k. \end{aligned}$$

□

In section 3, we characterize all g-dual frames of a given frame. Now we are going to give a simple way for construction of infinitely many g-dual frames of a given frame (whit common invertible operator).

Proposition 2.2. *Assume that $\{g_k\}_{k=1}^{\infty}$ is a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with invertible operator $A \in B(\mathcal{H})$ and let α be a complex number. Then the sequence $\{h_k\}_{k=1}^{\infty}$ defined by $h_k = \alpha g_k + (1-\alpha)(A^{-1})^* S^{-1} f_k$, is a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with invertible operator A, where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$.*

Proof. For all $f \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} < Af, h_k > f_k &= \sum_{k=1}^{\infty} < Af, \alpha g_k + (1-\alpha)(A^{-1})^* S^{-1} f_k > f_k \\ &= \bar{\alpha} \sum_{k=1}^{\infty} < Af, g_k > f_k + (1-\bar{\alpha}) \sum_{k=1}^{\infty} < f, S^{-1} f_k > f_k \\ &= \bar{\alpha} f + (1-\bar{\alpha}) f = f \end{aligned}$$

□

In the next proposition we obtain a g-dual frame for $\mathcal{H}$ from a g-dual frame for a subspace of $\mathcal{H}$.

Proposition 2.3. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S and $\{g_k\}_{k=1}^{\infty}$ be a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $V := \overline{span}\{g_k\}_{k=1}^{\infty}$ with invertible operator $B \in B(V)$. Then the sequence $\{h_k\}_{k=1}^{\infty}$ defined by $h_k = B^*g_k + S^{-1}f_k$ is a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$.*

Proof. The operator B can be extended to operator B_1 on $\mathcal{H}$ defined by $B_1 := BP + Q$, where P and Q are the orthogonal projection onto V and $V^{\perp}$, respectively. Therefore $B_1(V^{\perp}) \subset V^{\perp}$ and $B_1 = B$ on V. Also $B_1^* = B^*$ on V, since for all $f \in V$ and $g \in \mathcal{H}$ we have

$$\begin{aligned} < B_1^*f, g > &=< f, B_1g >=< f, BPg + Qg > \\ &=< f, BPg >=< B^*f, Pg > \\ &=< B^*f, Pg + Qg >=< B^*f, g > . \end{aligned}$$

Let $A := I - \frac{1}{2}P$, where I denote the identity operator on $\mathcal{H}$. Since $\|I - A\| < 1$, the operator A is invertible. Let $f \in \mathcal{H}$, then there exist unique vectors $g \in V$ and $h \in V^{\perp}$ such that $f = g + h$. Therefore

$$\begin{aligned} \sum_{k=1}^{\infty} < Af, h_k > f_k &= \sum_{k=1}^{\infty} < \frac{1}{2}g + h, B^*g_k + S^{-1}f_k > f_k \\ &= \sum_{k=1}^{\infty} < \frac{1}{2}g + h, B_1^*g_k + S^{-1}f_k > f_k \\ &= \frac{1}{2}\sum_{k=1}^{\infty} < Bg, g_k > f_k + \sum_{k=1}^{\infty} < \frac{1}{2}g + h, S^{-1}f_k > f_k \\ &+ \sum_{k=1}^{\infty} < B_1h, g_k > f_k \\ &= \frac{1}{2}g + \frac{1}{2}g + h + 0 = f. \end{aligned}$$

□

Corollary 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S and $\{g_k\}_{k=1}^{\infty}$ be a dual frame of $\{f_k\}_{k=1}^{\infty}$ for $V := \overline{span}\{g_k\}_{k=1}^{\infty}$. Then the sequence $\{h_k\}_{k=1}^{\infty}$ defined by $h_k = g_k + S^{-1}f_k$ is a g-dual frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$.*

Example 2.1 shows that the sum of two dual frame is g-dual. In the following proposition a necessary condition for g-duality of the sum of two g-dual frames is given.

Proposition 2.4. *Let $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ be two g-dual frames of $\{f_k\}_{k=1}^{\infty}$ with corresponding invertible operators A and B, respectively. If $A^{-1} + B^{-1}$ is an invertible operator, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k + h_k\}_{k=1}^{\infty}$ are g-dual frames.*

Proof. Let $T \in B(\mathcal{H})$ be the inverse operator of $(A^{-1}+B^{-1})$. For all $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} < Tf, g_k + h_k > f_k &= \sum_{k=1}^{\infty} < Tf, g_k > f_k + \sum_{k=1}^{\infty} < Tf, h_k > f_k \\ &= A^{-1}Tf + B^{-1}Tf \\ &= (A^{-1}+B^{-1})Tf = f.\end{aligned}$$

□

Invertible operators preserve the g-duality property of frames.

Proposition 2.5. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two sequences in $\mathcal{H}$ and $U, V \in B(\mathcal{H})$ be two invertible operator on $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$ if and only if $\{Uf_k\}_{k=1}^{\infty}$ and $\{Vg_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$.*

Proof. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be g-dual frames for $\mathcal{H}$. Then there exists an invertible operator $A \in B(\mathcal{H})$ such that $f = \sum_{k=1}^{\infty} < Af, f_k > g_k$ for all $f \in \mathcal{H}$ and hance

$$\begin{aligned}f = VV^{-1}f &= V(\sum_{k=1}^{\infty} < AV^{-1}f, f_k > g_k) \\ &= \sum_{k=1}^{\infty} < (U^{-1})^* AV^{-1}f, Uf_k > Vg_k.\end{aligned}$$

The converse is obtained by acting the operators U^{-1} and V^{-1}. □

3. **Characterization of g-dual frames**

As we have seen, every frame in $\mathcal{H}$ is a g-dual of itself and hence all frames have at least one g-dual. Now, we characterize all g-dual frames of a given frame. The results for the case of dual frames and a similar proof of the following lemmas and theorem are given in [6].

Lemma 3.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with pre-frame operator T and $\{\delta_k\}_{k=1}^{\infty}$ be the canonical orthonormal basis for $\ell_2(\mathbb{N})$. The g-dual frames for $\{f_k\}_{k=1}^{\infty}$ are precisely the families $\{g_k\}_{k=1}^{\infty} = \{V\delta_k\}_{k=1}^{\infty}$, whereV $: \ell_2(\mathbb{N}) \to \mathcal{H}$ is a bounded left inverse of T^*A, for some invertible operator $A \in B(\mathcal{H})$.*

Lemma 3.2. *Let $A \in B(\mathcal{H})$ be an invertible operator and $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with pre-frame operator T and frame operator S. The bounded left inverses of T^*A, are precisely the operators of the form $A^{-1}S^{-1}T + W(I - T^*S^{-1}T)$, where $W : \ell_2(\mathbb{N}) \to \mathcal{H}$ is a bounded operator and I denotes the identity operator on $\ell_2(\mathbb{N})$.*

Theorem 3.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S. The g-dual frames of $\{f_k\}_{k=1}^{\infty}$ are precisely the families*

$$\{g_k\}_{k=1}^{\infty} = \{Af_k + h_k - \sum_{j=1}^{\infty} < S^{-1}f_k, f_j > h_j\}_{k=1}^{\infty}, \tag{4}$$

where $\{h_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$ and $A \in B(\mathcal{H})$ is an invertible operator.

In the ordinary case, every Riesz basis has a unique dual frame, whereas a Riesz basis can have infinitely many g-dual frames that they also are Riesz basis.

Corollary 3.1. *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis in $\mathcal{H}$, then the g-dual frames of $\{f_k\}_{k=1}^{\infty}$ having the form $\{g_k\}_{k=1}^{\infty} = \{Af_k\}_{k=1}^{\infty}$, where $A \in B(\mathcal{H})$ is an invertible operator. In particular $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis.*

Proof. Since $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, $\{f_k\}_{k=1}^{\infty}$ and $\{S^{-1}f_k\}_{k=1}^{\infty}$ are bi-orthogonal sequences, where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$. Hence by Theorem 3.1

$$\{g_k\}_{k=1}^{\infty} = \{Af_k + h_k - \sum_{j=1}^{\infty} < S^{-1}f_k, f_j > h_j\}_{k=1}^{\infty} = \{Af_k\}_{k=1}^{\infty},$$

where A is an invertible operator in $B(\mathcal{H})$. It remains to show that $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis. By the definition of a Riesz basis, there exist an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and invertible operator $U \in B(\mathcal{H})$ such that $f_k = Ue_k$. We conclude that $\{g_k\}_{k=1}^{\infty} = \{Af_k\}_{k=1}^{\infty} = \{AUe_k\}_{k=1}^{\infty}$, i.e., $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis. □

4. g-dual frames, approximately dual frames and Gabor frames

Approximately dual frames are defined by Christensen in [7]. Two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ with pre-frame operator T and U, respectively, are approximately dual frames for $\mathcal{H}$ if $\|I - TU^*\| < 1$ or $\|I - UT^*\| < 1$. In what follows, we study the relation between approximately dual frames and g-dual frames.

Proposition 4.1. *If two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are approximately dual frames for $\mathcal{H}$, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames.*

Proof. Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are approximately dual frames, $\|I - TU^*\| < 1$ or $\|I - UT^*\| < 1$, where T and U are pre-frame operators of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. Hence UT^* or TU^* are invertible. If UT^* is invertible, then for all $f \in \mathcal{H}$ we have

$$f = (UT^*)(UT^*)^{-1}f = \sum_{k=1}^{\infty} < (UT^*)^{-1}f, f_k > g_k,$$

and if TU^* is invertible, then for all $f \in \mathcal{H}$ we have

$$f = (TU^*)(TU^*)^{-1}f = \sum_{k=1}^{\infty} < (TU^*)^{-1}f, g_k > f_k.$$

Therefore $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames. □

The following example illustrates that the set of approximately duals of a frame is a proper subset of the set of its g-duals.

Example 4.1. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be dual frames for $\mathcal{H}$. Then $\{3f_k\}_{k=1}^{\infty}$ is a g-dual frame of $\{g_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with corresponding operator $Af = \frac{1}{3}f$. But*

$$\|f - \sum_{k=1}^{\infty} < f, g_k > 3f_k\| = \|f - \sum_{k=1}^{\infty} < f, 3f_k > g_k\| = 2\|f\|, \quad \forall f \in \mathcal{H},$$

and hence $\{3f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are not approximately dual frames.

If $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames with invertible operator A such that $\|I - A^{-1}\| < 1$, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are approximately dual frames.

Theorem 4.1. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two Bessel sequences in $\mathcal{H}$ with pre-frame operators T and U, respectively. If there exist constants $\lambda, \mu \in [0, 1)$ such that*

$$\|f - UT^*f\| \leq \lambda\|UT^*f\| + \mu\|f\|, \quad \forall f \in \mathcal{H},$$

then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$.

Proof. By [6, lemma (15.1.3)] UT^* is invertible and hence $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$. □

In the rest of this section, we consider g-dual Gabor frames with special corresponding invertible operator. For $a, b \in \mathbb{R}$, $c > 0$ consider the translation, modulation and dilation operator on $L_2(\mathbb{R})$, which are defined as

$$(T_a g)(x) = g(x - a), \quad (E_b g)(x) = e^{2\pi ibx} g(x), \quad (D_c g)(x) = \frac{1}{\sqrt{c}} g(\frac{x}{c}), \quad \forall x \in \mathbb{R},$$

respectively.

A Gabor frame is a frame for $L_2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ where $a, b > 0$, g is a fixed function in $L_2(\mathbb{R})$. The duality condition for a pair of Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is presented by Janssen as follows [1]:

Lemma 4.1. *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L_2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)} h(x - ka) = b\delta_{n,0}, \quad a.e. \ \ x \in [0, a].$$

In the following theorem a sufficient and necessary condition for the g-duality of two Bessel sequences is given.

Theorem 4.2. *The Bessel sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of Bessel sequence $\{E_{mbc}T_{na/c}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with operator D_c if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(xc - n/b - ka)} h(x - ka/c) = (b/\sqrt{c})\delta_{n,0}, \quad a.e. \ \ x \in [0, a/c].$$

Proof. The Bessel sequence$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of Bessel sequence $\{E_{mbc}T_{na/c}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with operator D_c if and only if $\{D_c^* E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mbc}T_{na/c}h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L_2(\mathbb{R})$. Since $D_c^* = D_{1/c}$, and

$$D_{1/c}E_{mb}T_{na} = E_{mbc}D_{1/c}T_{na} = E_{mbc}T_{na/c}D_{1/c},$$

by lemma 4.1, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mbc}T_{na/c}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with operator D_c if and only if

$$\sum_{k\in\mathbb{Z}} \overline{D_{1/c}g(x - n/bc - ka/c)} h(x - ka/c) = b\delta_{n,0}, \quad a.e. \ \ x \in [0, a/c].$$

The proof is completed by the equivalency of

$$\sum_{k\in\mathbb{Z}} \overline{D_{1/c}g(x - n/bc - ka/c)} h(x - ka/c) = b\delta_{n,0}, \quad a.e. \ \ x \in [0, a/c],$$

and

$$\sum_{k\in\mathbb{Z}} \overline{g(xc - n/b - ka)}h(x - ka/c) = (b/\sqrt{c})\delta_{n,0}, \quad a.e. \ \ x \in [0, a/c].$$

□

For complex number $\lambda \neq 0$, the operator A_λ defined by $A_\lambda f = \lambda f$, for all $f \in L_2(\mathbb{R})$ is a bounded invertible operator on $L_2(\mathbb{R})$. It is easy to show that, the Bessel sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of Bessel sequence $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with operator A_λ if and only if

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)}h(x - ka) = (b/\lambda)\delta_{n,0}, \quad a.e. \ \ x \in [0, a]. \tag{5}$$

A necessary and sufficient condition for two Gaber frames to be g-duals with invertible operator $A \in B(L_2(\mathbb{R}))$, in the case that A commutes with $E_{\pm b}$ and $T_{\pm a}$ is given in the following theorem.

Theorem 4.3. *Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are Bessel sequences in $L_2(\mathbb{R})$ and that A is a bounded invertible operator on $L_2(\mathbb{R})$ for which it commutes with $E_{\pm b}$ and $T_{\pm a}$. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with A if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{A^*g(x - n/b - ka)}h(x - ka) = b\delta_{n,0}, \quad a.e. \ \ x \in [0, a].$$

Proof. Because A commutes with $E_{\pm b}$ and $T_{\pm a}$, so does A^* . Therefore A^* commute with E_{mb} and T_{na} for all $m, n \in \mathbb{Z}$.
Thus $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with A if and only if $\{E_{mb}T_{na}A^*g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are dual frames for $L_2(\mathbb{R})$. The proof is completed by applying Lemma 4.1. □

Corollary 4.1. *If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are Bessel sequences in $L_2(\mathbb{R})$ and $j \in \mathbb{Z}$, then*
(a)$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with $T_{\frac{j}{b}}$ if and only if

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)}h(x - ka) = b\delta_{n,-j}, \quad a.e. \ \ x \in [0, a],$$

(b)$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with $E_{\frac{j}{a}}$ if and only if

$$\sum_{k\in\mathbb{Z}} \overline{E_{\frac{-j}{a}}g(x - n/b - ka)}h(x - ka) = b\delta_{n,0}, \quad a.e. \ \ x \in [0, a].$$

Proof. It is easy to show that two invertible operators $T_{\frac{j}{b}}$ and $E_{\frac{j}{a}}$ commute with operators T_{na} and E_{mb}. Now the results are obtained from Theorem 4.3. □

5. **Perturbation of g-dual frames**

In this section, the perturbation theory of g-dual frames is discussed. That is, if $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$ and $\{f_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{H}$ for which it is in some sense "close" to $\{h_k\}_{k=1}^{\infty}$, does it follows that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$? In the following theorem a sufficient condition that it makes $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ and also $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ to be g-dual frames for $\mathcal{H}$, is given.

Theorem 5.1. *Let $\{g_k\}_{k=1}^{\infty}$ be a g-dual frame of $\{h_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with invertible operator $A \in B(\mathcal{H})$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$. Assume that there exist constants $\lambda, \mu \geq 0$, such that*

$$\|\sum c_k(f_k - h_k)\| \leq \lambda\|\sum c_k h_k\| + \mu(\sum |c_k|^2)^{\frac{1}{2}}, \tag{6}$$

for all finite sequences $\{c_k\}$.
a)If $\lambda + \mu\sqrt{C_1}\|A\| < 1$, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$, where C_1 is an upper frame bound for $\{g_k\}_{k=1}^{\infty}$.
b)If $\lambda + \mu\sqrt{C_2}\|S^{-1}\| < 1$, then $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$, where S and C_2 are the frame operator and an upper frame bound of $\{h_k\}_{k=1}^{\infty}$, respectively.

Proof. It is easy to show that $\sum_{k=1}^{\infty} c_k(f_k - h_k)$ converges and the inequality (6) holds for all $\{c_k\}_{k=1}^{\infty} \in \ell_2(\mathbb{N})$. Hence the operator $W : \ell_2(\mathbb{N}) \longrightarrow \mathcal{H}$, defined by $W(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k(f_k - h_k)$ is well-defined and bounded. Therefore, by [6, Theorem 3.2.3], the sequence $\{f_k - h_k\}_{k=1}^{\infty}$ is Bessel and so is $\{f_k\}_{k=1}^{\infty}$.
Now let $\lambda + \mu\sqrt{C_1}\|A\| < 1$. For all $f \in \mathcal{H}$ we have

$$\begin{aligned}\|f - \sum_{k=1}^{\infty} < Af, g_k > f_k\| &= \|\sum_{k=1}^{\infty} < Af, g_k > (f_k - h_k)\| \\ &\leq \lambda\|\sum_{k=1}^{\infty} < Af, g_k > h_k\| + \mu(\sum_{k=1}^{\infty} | < Af, g_k > |^2)^{\frac{1}{2}} \\ &\leq (\lambda + \mu\sqrt{C_1}\|A\|)\|f\| < \|f\|.\end{aligned}$$

Therefore $\|I - TU^*A\| < 1$ and TU^* is invertible, where T and U are the pre-frame operators of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. Hence $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$. i.e., (a) holds.
If $\lambda + \mu\sqrt{C_2}\|S^{-1}\| < 1$, then a similar argument shows that

$$\|f - \sum_{k=1}^{\infty} < S^{-1}f, h_k > f_k\| < \|f\|, \quad \forall f \in \mathcal{H},$$

and hence $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$. i.e., (b) holds. □

Corollary 5.1. *Let $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ be g-dual frames for $\mathcal{H}$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$. If there exists a constant $\lambda \in [0, 1)$, such that*

$$\|\sum c_k(f_k - h_k)\| \leq \lambda\|\sum c_k h_k\|,$$

for all finite sequences $\{c_k\}$, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are g-dual frames for $\mathcal{H}$. Also are $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$.

Corollary 5.2. *Let $\{g_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S and upper frame bound C. Assume that $\{f_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{H}$ and there exist constants $\lambda, \mu \geq 0$, such that*

$$\|\sum c_k(f_k - g_k)\| \leq \lambda\|\sum c_k g_k\| + \mu(\sum |c_k|^2)^{\frac{1}{2}},$$

for all finite sequences $\{c_k\}$. *If* $\lambda + \mu\sqrt{C}\|S^{-1}\| < 1$, *then* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are g-dual frames for* $\mathcal{H}$.

In the case that $\{f_k - h_k\}_{k=1}^{\infty}$ is a Bessel sequence, the following theorem (as a consequence of Theorem 5.1) is useful.

Theorem 5.2. *Assume that* $\{g_k\}_{k=1}^{\infty}$ *is a g-dual frame of* $\{h_k\}_{k=1}^{\infty}$ *for* $\mathcal{H}$ *with invertible operator* $A \in B(\mathcal{H})$. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *and* B *be an invertible operator in* $B(\mathcal{H})$ *such that*

$$\sum_{k=1}^{\infty} | < Bf, f_k - h_k > |^2 \leq \mu \|f\|^2, \quad \forall f \in \mathcal{H}, \tag{7}$$

for some $\mu > 0$.
a) If $\mu < C_1^{-1}(\|A\|\|B^{-1}\|)^{-2}$, *then* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are g-dual frames for* $\mathcal{H}$, *where* C_1 *is an upper frame bound for* $\{g_k\}_{k=1}^{\infty}$.
b) If $\mu < C_2^{-1}(\|S^{-1}\|\|B^{-1}\|)^{-2}$, *then* $\{f_k\}_{k=1}^{\infty}$ *and* $\{h_k\}_{k=1}^{\infty}$ *are g-dual frames for* $\mathcal{H}$, *where* S *and* C_2 *are the frame operator and an upper frame bound of* $\{h_k\}_{k=1}^{\infty}$, *respectively.*

Proof. The inequality (7) shows that the sequence $\{f_k - h_k\}_{k=1}^{\infty}$ is a Bessel sequence with upper frame bound $\mu\|B^{-1}\|^2$ and hence

$$\|\sum c_k(f_k - h_k)\| \leq \sqrt{\mu}\|B^{-1}\|(\sum |c_k|^2)^{\frac{1}{2}},$$

for all $\{c_k\}_{k=1}^{\infty} \in \ell_2(\mathbb{N})$. Therefore the inequality (6) holds by constants 0 and $\sqrt{\mu}\|B^{-1}\|$. Now the results are obtained from Theorem 5.1. □

In the following we consider perturbation of g-dual Gabor frames. An important perturbation question is, if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ and $\phi \in L_2(\mathbb{R})$ is "close" to h, does it follows that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a g-dual frame of $\{E_{mb}T_{na}\phi\}_{m,n\in\mathbb{Z}}$. A sufficient condition for ϕ is given in the following theorem.

Theorem 5.3. *Let* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *be a g-dual frame of* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *for* $L_2(\mathbb{R})$ *with invertible operator* $A \in B(L_2(\mathbb{R}))$ *and* ϕ *be a function in* $L_2(\mathbb{R})$ *such that*

$$R := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} (\phi - h)(x - na)(\phi - h)(x - na - \frac{k}{b})| < \infty. \tag{8}$$

a) If $R < \frac{1}{C_1\|A\|^2}$, *then* $\{E_{mb}T_{na}\phi\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *are g-dual frames for* $L_2(\mathbb{R})$, *where* C_1 *is an upper frame bound for* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.
b) If $R < \frac{1}{C_2\|S^{-1}\|^2}$, *then* $\{E_{mb}T_{na}\phi\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *are g-dual frames for* $L_2(\mathbb{R})$, *where* S *and* C_2 *are the frame operator and an upper frame bound of* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$, *respectively.*

Proof. The inequality (8) shows that the sequence $\{E_{mb}T_{na}\phi - E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with upper frame bound R by [6, Theorem 8.4.4]. Now the results are obtained from Theorem 5.2. □

In the rest of section we consider perturbation of g-dual wavelet frames. A wavelet frame is a frame for $L_2(\mathbb{R})$ of the form $\{a^{j/2}\psi(a^j x - kb)\}_{j,k\in\mathbb{Z}}$, where $a > 1, b > 0$ and $\psi \in L_2(\mathbb{R})$. An important perturbation question is, if $\{a^{j/2}\tilde{\psi}(a^j x -$

$kb)\}_{j,k\in\mathbb{Z}}$ is a g-dual frame of $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ and $\phi\in L_2(\mathbb{R})$ is "close" to ψ, does it follows that $\{a^{j/2}\tilde{\psi}(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ is a g-dual frame of $\{a^{j/2}\phi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ for $L_2(\mathbb{R})$. A sufficient condition for ϕ is given in the following theorem.

Theorem 5.4. *Let $\{a^{j/2}\tilde{\psi}(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ be a g-dual frame of $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ with invertible operator $A\in B(L_2(\mathbb{R}))$ and ϕ be a function in $L_2(\mathbb{R})$ such that*

$$R:=\frac{1}{b}\sup_{|\gamma|\in[1,a]}\sum_{j,k\in\mathbb{Z}}|(\hat{\phi}-\hat{\psi})(a^j\gamma)(\hat{\phi}-\hat{\psi})(a^j\gamma+k/b)|<\infty. \tag{9}$$

a) If $R<\frac{1}{C_1\|A\|^2}$, then $\{a^{j/2}\phi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ and $\{a^{j/2}\tilde{\psi}(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ are g-dual frames for $L_2(\mathbb{R})$, where C_1 is an upper frame bound for $\{a^{j/2}\tilde{\psi}(a^jx-kb)\}_{j,k\in\mathbb{Z}}$.
b) If $R<\frac{1}{C_2\|S^{-1}\|^2}$, then $\{a^{j/2}\phi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ and $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ are g-dual frames for $L_2(\mathbb{R})$, where S and C_2 are the frame operator and an upper frame bound of $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$, respectively.

Proof. The inequality (9) shows that the sequence $\{a^{j/2}\phi(a^jx-kb)-a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ is a Bessel sequence with upper frame bound R by [6, Theorem 11.2.3]. Now the results are obtained from Theorem 5.2. □

REFERENCES

[1] *A. J. E. M. Janssen*,The duality condition for Weyl-Heisenberg frames. In Gabor analysis: theory and application (eds. H. G. Feichtinger and T. Strhmer). Birkhäuser, Boston, 1998.

[2] *C. Heil and D. Walnut*, Continuous and discrete wavelet transform, SIAM Rev., **31**(1969), 628-666.

[3] *I. Daubechies*,Ten Lectures on Wavelets, 1992.

[4] *I. Daubechies, A. Grossmann, and Y. meyer*, Painless nonorthogonal expansions, J. math. phys., **27**(1986), 1271-1283.

[5] *J. J. Benedetto*, Frame decomposition, sampling, and uncertainty priciple inequalities in wavelets, Mathmatics and applications (J. J. Benedetto and M. W. Frazier.), CRC Press, Boca Raton, FL, 1994,chapt. 7.

[6] *O. Christensen*, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, Basel, Berlin, 2002.

[7] *Ole Christensen and Richard S. Laugesen*, Approximately dual frames in Hilbert spaces and application to Gabor frames, Sampl. Theory Signal Image Process, **9**(2011), 77-90.

[8] *R. J. Duffin and A. C. Schaeffer*, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72**(1952), 341-366.

[9] *R. Young*, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.