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A NONLINEAR SECOND-ORDER HYPERBOLIC DIFFUSION SCHEME FOR IMAGE NOISE REDUCTION

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This article describes a novel nonlinear second-order PDE model for image filtering. The proposed denoising model is based on a second-order hyperbolic equation and provides effective detail-preserving image noise removal results. The well-posedness of this nonlinear PDE scheme is then investigated in this paper. A finite-difference based explicit numerical approximation scheme is constructed next for our continuous model. Our successfully performed denoising experiments are then described.

Keywords: image denoising, nonlinear PDE-based model, hyperbolic diffusion equation, well-posedness, numerical approximation scheme

1. Introduction

A high number of nonlinear partial differential equation (PDE) based models have been introduced in the last quarter of century to tackle some traditionally engineering problems, such as image denoising and restoration. They have important advantages over the conventional image filters and linear PDE-based approaches [1], such as overcoming the blurring effect, preserving edges and other image details, and having the localization property.

The vast majority of the nonlinear PDE schemes for image enhancement have a parabolic character [2]. We could mention here the anisotropic diffusion models inspired by the well-known Perona-Malik denoising scheme [2,3] and the variational PDE algorithms derived from the influential TV Denoising model [2, 4-6]. We also provided numerous nonlinear PDE-based image restoration approaches in the past, which use parabolic diffusion equations [7-9]. Unfortunately, the second-order nonlinear parabolic generate the undesired staircasing, or blocky, effect [10].

Numerous nonlinear diffusion-based denoising techniques that alleviate this blocky effect have been proposed in recent years. The nonlinear fourth-order PDE-based models, such as those inspired by You-Kaveh [11] or LLT [12] schemes, remove successfully the staircasing effect [13]. We also developed some effective fourth-order diffusion-based techniques that provide satisfactory

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denoising results [14]. The disadvantage of the most 4th-order PDE-based smoothing methods is that they are usually affected by other unintended effects, like blurring or speckle noise.

So, we consider hyperbolic diffusion-based denoising solutions that would overcome these undesired effects which destroy the image details. We developed linear second-order hyperbolic PDE models for image denoising, as modified Gaussian kernels, which were disseminated in some published papers [15] or papers being under consideration.

Our linear hyperbolic PDE-based techniques provide satisfactory noise reduction results, execute very fast, and also have the localization property [16], which means the solution is propagating with finite speed. Unfortunately they are still affected by blurring effect, therefore we consider their improvement in that direction. Thus, we can non-linearize the linear hyperbolic diffusion methods to obtain that improvement.

Such a nonlinear second-order hyperbolic PDE-based image noise removal technique is proposed in this article. Our novel PDE model is described in the next section, and a rigorous mathematical treatment on its well-posedness is performed in the third section. An explicit numerical approximation scheme based on the finite-difference method is proposed in the fourth section. Our successfully image denoising experiments and the performed method comparison are discussed in the fifth section. Our paper finalizes with a conclusions section and a list of references.

2. Second-order hyperbolic PDE-based denoising model

In this section we consider a PDE-based image noise reduction scheme that uses a second-order nonlinear hyperbolic diffusion model. The proposed PDE model is composed of a 2nd-order hyperbolic equation and several boundaries conditions:

$$\begin{cases} \gamma \dfrac{\partial^2 u}{\partial t^2} + \eta^2 \dfrac{\partial u}{\partial t} - \xi_u\left(\|\nabla u\|\right)\cdot \Delta u + \alpha\left(u - u_0\right) = 0 \\ u(0, x, y) = u_0(x, y) \\ u_t(0, x, y) = u_1(x, y) \\ u(t, x, y) = 0, \quad \forall t \geq 0, \ (x, y) \in \partial\Omega \end{cases}, (x, y) \in \Omega \quad (1)$$

where the parameters $\gamma, \eta \in (0,1]$, $\alpha \in (0, 0.4]$, $\Omega \subseteq R^2$ and u_0 represents the initial noisy image.

The edge-stopping function ξ_u of this PDE model must be properly modeled. We construct it in the following form that depends on the current state of the image *u* through a diffusivity conductance parameter modeled as a function, *k* (*u*):

$$\xi_u : [0,\infty) \to [0,\infty), \quad \xi_u(s) = \frac{\beta}{\left(\frac{k(u)}{s}\right)^2 + k(u)\left|\ln\left(\frac{k(u)}{s}\right)\right|} \tag{2}$$

where $\beta \in (1,17\ ]$, and the function *k* ( ) is modeled by using some statistics of the gradient of the evolving image, as follows:

$$k(u) = \varepsilon \cdot \mu(\|\nabla u\|) - \nu \cdot ord(u) \tag{3}$$

where $\varepsilon \in (2,3\ ]$, $\nu \in (0,1\ )$, $ord(u) \in \{1,...,N\}$ returns the order of the current state of u in the evolving image sequence, and $\mu(\ )$ represents the averaging (mean) operator.

The mathematical model given by (1) – (3) is constructed as an improved and nonlinear version of a past linear hyperbolic PDE model for image denoising proposed by us [15]. That linear PDE-based approach reduces successfully the Gaussian noise but cannot overcome completely the blurring effect. Given its nonlinear character, achieved by replacing a constant with a function of gradient magnitude $\xi_u(\|\nabla u\|)$, the PDE-based technique described here would provide a much better deblurring.

A mathematical treatment of this second-order hyperbolic PDE model is provided in the next section. The proper selection of function ξ_u and the well-posedness of this nonlinear PDE scheme will be rigorously investigated.

3. Mathematical investigation of the hyperbolic scheme

First, we analyze if the diffusivity function ξ_u provided by (2) is properly modeled for an effective restoration [2,3,8]. Thus, the considered function is positive, since $\xi_u(s) \geq 0, \forall s \geq 0$. Also, it is monotonically decreasing, because

$$\forall s_1 \geq s_2, \left(\frac{s_1}{k(u)}\right)^2 + k(u)\left|\ln\left(\frac{s_1}{k(u)}\right)\right| \geq \left(\frac{s_2}{k(u)}\right)^2 + k(u)\left|\ln\left(\frac{s_2}{k(u)}\right)\right|,$$

therefore we obtain

$$\xi_u(s_1) = \frac{\beta}{\left(\frac{s_1}{k(u)}\right)^2 + k(u)\ln\left(\frac{s_1}{k(u)}\right)} \leq \frac{\beta}{\left(\frac{s_2}{k(u)}\right)^2 + k(u)\ln\left(\frac{s_2}{k(u)}\right)} = \xi_u(s_2).$$

This function converges to 0, because $\lim_{s \to \infty} \xi_u(s) = 0$. So, ξ_u represents a good edge-stopping function for the PDE model (1), leading to satisfactory image denoising results.

The well-posedness of our hyperbolic model is another problem that has to be investigated. Thus, the PDE given by (1) is equivalent to the next equation:

$$\gamma \frac{\partial^2 u}{\partial t^2} + \eta^2 \frac{\partial u}{\partial t} - div\left(\tilde{\xi}_u \|\nabla u\|^2\right) + \alpha\left(u - u_0\right) = 0 \tag{4}$$

where $\tilde{\xi}_u{}'(s) = \xi_u\left(\sqrt{s}\right), \forall s \geq 0$.

The PDE model (4) accepts solutions if some certain conditions are met. Thus, we must have $\tilde{\xi}_u{}'(s) \geq 0$ that leads to $\xi_u(s) \geq 0$, a condition that is satisfied. Also, ξ_u must satisfy a bounding condition, that is

$$\exists K > 0 : \xi_u(s) \leq K\left(s^2 + s + 1\right), \forall s \geq 0 \tag{5}$$

If this condition is also satisfied, then there is a solution to (1) in some generalized sense, according to [17]. The relation given by (5) is equivalent to

$$\xi_u(s) = \frac{\beta}{\left(\frac{s}{k(u)}\right)^2 + k(u)\ln\left(\frac{s}{k(u)}\right)} \leq K\left(s^2 + s + 1\right)$$

which leads to $K \geq \frac{\beta}{\left(s^2 + s + 1\right)\left(\left(\frac{s}{k(u)}\right)^2 + k(u)\ln\left(\frac{s}{k(u)}\right)\right)}$. Of course, such a K value exists for any $s > 0$, therefore (5) holds. In fact, in [17,18] it is proved the existence and uniqueness of a solution $u = u(t, x)$, such that:

$$u \in L^2\left(0, T; H_0^1(\Omega)\right), \frac{\partial u}{\partial t} \in L^2\left((0, T) \times \Omega\right) \tag{6}$$

where H_0^1 represents the standard Sobolev space [18]. Our nonlinear hyperbolic diffusion-based model has also the localization property [16], its unique and weak solution propagating with finite speed. Besides the past linear model, the insights of our PDE denoising scheme draw also from this localization property. Because the PDE solution propagates with finite speed, the evolving image will remain quite close to the initial one. This solution is numerically approximated by the consistent PDE discretization scheme proposed in the following section.

4. Numerical approximation algorithm

We develop a consistent numerical approximation algorithm for our continuous model given by (1) - (3), which converges fast to the unique solution of its nonlinear second-order hyperbolic diffusion equation. The proposed numerical discretization scheme is based on the well-known finite-difference method [19]. Thus, we use a space grid size of h and a time step Δt. We quantize the space and time coordinates as follows:

$$x = ih, y = jh, t = n\Delta t, \forall i \in \{0,1,..., I\}, j \in \{0,1,..., J\}, n \in \{0,1,..., N\} \quad (7)$$

The main equation of this nonlinear hyperbolic PDE model, $\gamma \frac{\partial^2 u}{\partial t^2} + \eta^2 \frac{\partial u}{\partial t} - \xi_u(\|\nabla u\|) \cdot \nabla^2 u + \alpha(u - u_0) = 0$, can be written as:

$$\gamma \frac{\partial^2 u}{\partial t^2} + \eta^2 \frac{\partial u}{\partial t} - \xi_u\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\right) \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \alpha(u - u_0) = 0 \quad (8)$$

The equation (8) is then discretized, by using the finite differences [19], as following:

$$\gamma \frac{u^{n+\Delta t}(i,j) + u^{n-\Delta t}(i,j) - 2u^n(i,j)}{\Delta t^2} + \eta^2 \frac{u^{n+\Delta t}(i,j) - u^{n-\Delta t}(i,j)}{2\Delta t} -$$
$$- \xi_u\left(\sqrt{\left(\frac{u^n(i+h,j) - u^n(i-h,j)}{2h}\right)^2 + \left(\frac{u^n(i,j+h) - u^n(i,j-h)}{2h}\right)^2}\right) \cdot \quad (9)$$
$$\cdot \frac{u^n(i+h,j) + u^n(i-h,j) + u^n(i,j+h) + u^n(i,j-h) - 4u^n(i,j)}{h^2} +$$
$$+ \alpha\left(u^n(i,j) - u^0(i,j)\right) = 0$$

We may consider the parameter values $h = 1$ and $\Delta t = 1$, therefore (9) leads to the following explicit numerical approximation scheme:

$$u^{n+1}(i,j)=\frac{2\alpha-4\gamma}{2\gamma+\eta^2}u^n(i,j)+\frac{\eta^2-2\gamma}{2\gamma+\eta^2}u^{n-1}(i,j)+$$
$$+\frac{2}{\eta^2+2\gamma}\xi_u\left(\sqrt{\left(u^n(i+1,j)-u^n(i-1,j)\right)^2+\left(u^n(i,j+1)-u^n(i,j-1)\right)^2}\right)\cdot \tag{10}$$
$$\cdot\left(u^n(i+1,j)+u^n(i-1,j)+u^n(i,j+1)+u^n(i,j-1)-4u^n(i,j)\right)-\frac{2\alpha}{\eta^2+2\gamma}u^0(i,j)$$

where $\xi_u\left(\sqrt{\left(u^n(i+1,j)-u^n(i-1,j)\right)^2+\left(u^n(i,j+1)-u^n(i,j-1)\right)^2}\right)$ is computed by applying (2) and (3), $u^0(i,j)=u_0(i,j)$ and $n > 0$.

Our iterative filtering scheme receives an initial $[I\times J]$ degraded image as input and applies repeatedly the procedure given by (10), for each $n\in\{1,..., N\}$. This numerical approximation algorithm is consistent to the PDE model provided by (1), converging quite fast to an approximation of its solution, representing the optimal image denoising, u^{N+1}.

5. Experiments

The proposed nonlinear hyperbolic PDE-based noise removal approach was tested on numerous degraded images. The USC-SIPI database, containing 4 volumes, was the main collection used in our experiments. These tests were performed on Volume 2 (*Aerials*) of USC-SIPI, composed of 38 images of $[512\times512]$ and $[1024\times1024]$ sizes, Volume 3 (*Miscellaneous*), containing 44 images of $[256\times256]$, $[512\times512]$ and $[1024\times1024]$ sizes, and Volume 4 (*Sequences*), containing 69 images of $[256\times256]$ and $[512\times512]$ sizes. The images were corrupted with various levels of Gaussian noise, which were generated by considering various values for the μ (mean) and σ^2 (variance) parameters.

Our denoising model not only removes successfully the image noise, but also preserves the important details, such as the image boundaries. It also overcomes the undesired image effects, like image blurring effect, staircase (blocky) effect and speckle noise. We have identified on a trial and error basis the following set of PDE model's parameter values that provide optimal image denoising results:

$$\gamma=2.3,\eta=1.5,\beta=1.2,\alpha=0.12,\varepsilon=0.8,\nu=0.2,\Delta\mathrm{t}=1,\ h=1, N=19 \tag{11}$$

One can observe that the number of iterations *N* is quite low, which means the proposed filtering scheme runs fast enough. The execution time is around 0.5

seconds. Our filtering results are influenced by the power of noise. If the Gaussian noise parameters, μ and σ, are increased, the image will degrade much more and the denoising process will require a higher number of iterations, *N*, which means a greater time cost. The restoration result is also influenced by this number of iterations. An *N* value exceeding the number of steps related to the optimal denoising would produce a further degradation of the evolving image.

The performance of our image enhancement method was assessed by using measures like Structural Similarity Image Metric (SSIM), Peak Signal-to-Noise Ratio (PSNR) and Norm of Error (NE) Image. The performed method comparisons show that our nonlinear hyperbolic diffusion-based technique outperforms both the classic and PDE-based image enhancement approaches, producing higher SSIM values than those filtering solutions. This denoising method provides a considerably better image noise removal than well-known 2D conventional filters [1], such as Average, 2D Gaussian, and Wiener filters. Unlike these classic denoising schemes, the proposed hyperbolic model overcomes also the image blurring effect, preserving successfully the essential features, like image edges. Our nonlinear diffusion technique outperforms also the linear PDE-based denoising algorithms [15], providing an improved denoising and avoiding the undesired effects. We found it performs slighty better than the LLMMSE filter developed by Lee in 1980 [20].

Many state-of-the-art nonlinear PDE-based noise removal methods are also outperformed by our denoising scheme. This 2^{nd} - order hyperbolic diffusion-based model achieves much better denoising results than some popular second-order anisotropic diffusion-based schemes, such as Perona-Malik model [3], TV Denoising [4] and Weickert diffusion [2]. Also, it executes faster than these methods and, unlike them, do not generate staircasing effect [10]. Also, the described second-order PDE noise reduction technique outperforms some influential nonlinear fourth-order diffusion based techniques, like You-Kaveh scheme [11] or LLT [12]. The hyperbolic diffusion model removes successfully not only the blurring effect, but also the unintended speckle noise, that are often generated by the fourth-order PDE denoising models. Our iterative scheme runs much faster than those corresponding to the 4th-order PDE-based approaches. Method comparisons and restoration results are described in the next table and figure. As one can see in Table 1, the proposed model gets higher SSIM values than conventional filters, second-order and fourth-order PDE-based algorithms, and even the LLMMSE – Lee filter with a $[3 \times 3]$ noise estimation window.

Table 1

The SSIM values corresponding to several image filtering techniques

Filter	This model	Avg.	2D Gaussian	LLMMSE filter	Perona-Malik	TV Denoising	You-Kaveh
SSIM	0.6342	0.5625	0.5498	0.6239	0.6183	0.5857	0.6014

Fig. 1. Gaussian noise removal provided by various filtering techniques

The image enhancement results produced by these denoising techniques are displayed in Fig. 1. Original $[512 \times 512]$ *Lena* image is displayed in (a), while the image affected by Gaussian noise with parameters $\mu = 0.21$ and $\sigma^2 = 0.02$ is displayed in (b). The image denoising in (c), provided by our nonlinear PDE model looks better than the smoothing achieved by the $[3 \times 3]$ 2D filters from (d) - (f) (Average, Gaussian, LLMMSE), Perona-Malik scheme (g), TV denoising (h) and You-Kaveh algorithm (i). One can also see that the unintended image effects, which are still present in the figures (d) - (i), are completely removed in the (c) figure.

6. Conclusions

We have proposed a novel nonlinear hyperbolic PDE-based image noise removal technique in this article. The effective second-order hyperbolic diffusion model proposed here may be viewed as a nonlinear and improved variant of our past linear PDE denoising scheme [15].

Also, besides the linear PDE methods, our approach described here outperforms the most popular classic two-dimensional filters [1] and both second-order and fourth-order nonlinear diffusion-based methods [2-14]. It removes a greater amount of Gaussian noise and preserves better the boundaries and other image details. Our hyperbolic technique overcomes the blurring effect, generated by the conventional filters and fourth-order PDE-based models, and reduces considerably the staircasing effect that affects the most anisotropic diffusion models [10]. It also avoids successfully the speckle noise, often generated by the nonlinear 4^{th} - order diffusion schemes [11-13].

A mathematical treatment has been also provided for this proposed PDE denoising scheme. The proper modeling of its diffusivity function and its well-posedness is investigated in this paper. We have also constructed a robust and fast-converging finite-difference based numerical discretization scheme that is consistent to our nonlinear PDE model.

We also intend to further improve this nonlinear hyperbolic PDE image noise removal scheme. Thus, we are going to investigate other diffusivity functions for this model. We also consider transforming it into a possible more performant fourth-order PDE denoising scheme, as part of our future research in the image enhancement domain.

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