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FIXED POINT RESULTS IN M- METRIC SPACE WITH APPLICATION TO LCR CIRCUIT

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In this article, we introduce a fixed point theorem for M-metric space with relation theoretic principle and give an important application to solve the initial value problem (IVP) which illustrate the applicability and efficiency of our result.

Keywords: Binary relation, m-self-closedness, m-complete, electric circuit equation.

MSC2020: 47H10, 54H25.

1. Introduction

The Banach contraction principle (BCP) [1] is one of the most stalwart result in fixed point theory. BCP is basically depend on the domain and the nature of mapping. It assures the fixed points are unique and exist. There are numerous researchers who have worked on the generalization of BCP. We can extend the BCP via change the contraction mapping or alter the metric spaces such that Boyd and Wong-contraction [2, 3], Chatterjea-contraction [4], Wardowski-contraction [5], Kannan-contraction [6], partial metic space [7], M-metric space [8] etc.

Matthews [7] generalised the metric space as partial metric space and derived the specific results for it. After that, in 2014 Asadi et al. [8] extended partial metric space as M-metric space and obtained fixed point results for it. In 2015, Alam et al. [9] introduced the new and novel variant of BCP on a complete metric space with binary relation. After that many scientists have worked on relation theoretic principle and gave effective results, i.e. In 2015, Alam et al. [10] gave the results on the coincidence theorems in relation theoretic principle. After that, several researchers gave the relation theoretic contraction principle in different spaces.

In this paper, we are generalising the BCP and Boyd and Wong theorem for M-metric space via relation theoretic principle.

2. Preliminaries

In this section, we assemble some relevant definitions and essential results, which will be used in our subsequent discussion:

Notation: [8] In the sequel of M-metric space, the notations listed below are useful:

(i) $m_{uv} := m(u,u) \vee m(v,v) = \min\{m(u,u), m(v,v)\}$,

(ii) $M_{uv} := m(u,u) \wedge m(v,v) = \max\{m(u,u), m(v,v)\}$.

Definition 2.1. [8] *If $\varsigma \neq \emptyset$ and a map $m : \varsigma \times \varsigma \to \mathbb{R}^+$ is m-metric, when it satisfies the subsequent conditions:*

(1) $m(u,u) = m(v,v) = m(u,v) \iff u = v$,

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(2) $m_{uv} \leq m(u,v)$,
(3) $m(u,v) = m(v,u)$,
(4) $(m(u,v) - m_{uv}) \leq (m(u,z) - m_{uz}) + (m(z,v) - m_{zv})$.
Hence, the pair (ς, m) *is referred to as an* M*-metric space.*

Definition 2.2. [8] *Assume that a Sequence* $\{u_n\}$ *in* (ς, m)*. Then,*
(I) $\{u_n\}$ *converges to a point u iff*

$$\lim_{n\to\infty}(m(u_n,u) - m_{u_n,u}) = 0.$$

(II) $\{u_n\}$ *is m-Cauchy sequence iff*

$$\lim_{n,m\to\infty}(m(u_n,u_m) - m_{u_n,u_m}) \text{ and } \lim_{n,m\to\infty}(M(u_n,u_m) - m_{u_n,u_m})$$

exist and finite.
(III) *If each m-Cauchy sequence* $\{u_n\}$ *converges to a point u in a manner that*

$$\lim_{n\to\infty}(m(u_n,u) - m_{u_n,u}) = 0 \text{ and } \lim_{n\to\infty}(M(u_n,u) - m_{u_n,u}) = 0.$$

Then, M*-metric space is m-complete.*

Lemma 2.1. [8] *If* $u_n \to u$ *as* $n \to \infty$ *in* (ς, m)*, then* $\lim_{n\to\infty}(m(u_n,v) - m_{u_n,v}) = m(u,v) - m_{u,v}, \forall\, v \in \varsigma$.

Lemma 2.2. [8] *If* $u_n \to u$ *and* $v_n \to v$ *as* $n \to \infty$ *in* (ς, m)*, then* $\lim_{n\to\infty}(m(u_n,v_n) - m_{u_n,v_n}) = m(u,v) - m_{u,v}$.

Lemma 2.3. [8] *If* $u_n \to u$ *and* $v_n \to v$ *as* $n \to \infty$ *in* (ς, m)*, then* $m(u,v) = m_{u,v}$*. Further if* $m(u,u) = m(v,v)$*, then* $u = v$.

Definition 2.3. [9] *Let* $\varsigma \neq \emptyset$ *with a binary relation* $\mathfrak{R}$*. Then,*

(1) *u and v are* $\mathfrak{R}$*-comparative, if* $(u,v) \in \mathfrak{R}$ *or* $(v,u) \in \mathfrak{R}$ *and* $u, v \in \varsigma$*. It also represented by* $[u,v] \in \mathfrak{R}$.
(2) $\mathfrak{R}^{-1} = \{(u,v) \in \varsigma^2 : (v,u) \in \mathfrak{R}\}$*, where* $\mathfrak{R}^{-1}$ *is inverse or transpose or dual relation of* $\mathfrak{R}$.
(3) $\mathfrak{R}^s := \mathfrak{R} \cup \mathfrak{R}^{-1}$*, where* $\mathfrak{R}^s$ *is symmetric closure of* $\mathfrak{R}$.
(4) $(u,v) \in \mathfrak{R}^s \Longleftrightarrow [u,v] \in \mathfrak{R}$.
(5) *If* $(u_n, u_{n+1}) \in \mathfrak{R},\ \forall\, n \in \mathbb{N}_0$*. Then,* $\{u_n\}$ *is* $\mathfrak{R}$*-preserving sequence.*
(6) *Assume a map* $\mathcal{H} : \varsigma \to \varsigma$*. Then,* $\mathfrak{R}$ *is* $\mathcal{H}$*-closed,*

$$(u,v) \in \mathfrak{R} \Rightarrow (\mathcal{H}u, \mathcal{H}v) \in \mathfrak{R}, \text{ for any } u, v \in \varsigma.$$

(7) $\mathfrak{R}^s$ *is* $\mathcal{H}$*-closed, when* $\mathfrak{R}$ *is* $\mathcal{H}$*-closed.*

Proposition 2.1. [11] *Suppose* $\varsigma \neq \emptyset$ *with* $\mathfrak{R}$ *and* $\mathcal{H} : \varsigma \to \varsigma$*. If* $\mathfrak{R}$ *is* $\mathcal{H}$*-closed. Then,* $\mathfrak{R}$ *is also* $\mathcal{H}^n$*-closed,* $\forall\, n \in \mathbb{N}_0$.

Proposition 2.2. [9] *Assume that a metric space* (ς, d) *with* $\mathfrak{R}$ *and* $\mathcal{H} : \varsigma \to \varsigma$ *and* $\beta \in [0,1)$*, then the similar subsequent contractivity conditions are:*

(I) $d(\mathcal{H}u, \mathcal{H}v) \leq \beta d(u,v)\ \forall\, u, v \in \varsigma$ *with* $(u,v) \in \mathfrak{R}$,
(II) $d(\mathcal{H}u, \mathcal{H}v) \leq \beta d(u,v)\ \forall\, u, v \in \varsigma$ *with* $[u,v] \in \mathfrak{R}$.

Definition 2.4. [9, 10] *Let a metric space* (ς, d) *with* $\mathfrak{R}$ *on* ς *with* $u \in \varsigma$.

(i) *If there is any* $\mathfrak{R}$*-preserving sequence* $\{u_n\}$ *such that* $u_n \xrightarrow{d} u$*, we have* $\mathcal{H}(u_n) \xrightarrow{d} \mathcal{H}(u)$*. Then,* $\mathcal{H}$ *is* $\mathfrak{R}$*-continuous at u, Furthermore if* $\mathcal{H}$ *is* $\mathfrak{R}$*-continuous at every point of* ς*. Then, it is* $\mathfrak{R}$*-continuous.*
(ii) *If there exists a subsequence* $\{u_{n_k}\}$ *of any* $\mathfrak{R}$*-preserving sequence* $\{u_n\}$ *with* $[u_{n_k}, u]$ *in* $\mathfrak{R}$*,* $\forall\, k \in \mathbb{N}_0$*, then* $\mathfrak{R}$ *is d-self-closed.*

(iii) *If every $\mathfrak{R}$-preserving Cauchy sequence in ς converges. Then, (ς,d) is $\mathfrak{R}$-complete.*

Definition 2.5. [12] *Let $\varsigma \neq \emptyset$ with $\mathfrak{R}$. For $u,v \in \varsigma$, a path of length k in $\mathfrak{R}$ from u to v is a finite sequence $\{w_0,w_1,w_2,\cdots,w_k\} \subset \varsigma$ attaining these conditions:*

(i) $w_0 = u$ and $w_k = v$,

(ii) $(w_i,w_{i+1}) \in \mathfrak{R}$ *for each* i $(0 \leq i \leq k-1)$.

Definition 2.6. [11] *Let $\varsigma \neq \emptyset$ with $\mathfrak{R}$ and a self map $\mathcal{H}$. Then,*

(a) *$\mathfrak{R}$ is $\mathcal{H}$-transitive,*

$$(\mathcal{H}u,\mathcal{H}v),(\mathcal{H}v,\mathcal{H}z) \in \mathfrak{R} \Rightarrow (\mathcal{H}u,\mathcal{H}z) \in \mathfrak{R}, \text{ for any } u,v,w \in \varsigma.$$

(b) *If $\mathfrak{R}|_E$ is transitive for each $\mathfrak{R}$-preserving sequence $\{u_n\} \subset \varsigma$ (with range $E = \{u_n : n \in \mathbb{N}\}$), then $\mathfrak{R}$ is locally transitive.*

(c) *If $R|_E$ is transitive and for each $\mathfrak{R}$-preserving sequence $\{u_n\} \subset \mathcal{H}(\varsigma)$ (with range $E = \{u_n : n \in \mathbb{N}\}$) then $\mathfrak{R}$ is locally $\mathcal{H}$-transitive.*

These are collection of control functions is mentioned in Boyd and Wong, although it was subsequently utilised in Jotic [13].

$$\Psi = \{\phi : [0,\infty) \to [0,\infty) : \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{v \to t^+} \phi(v) < t \text{ for each } t > 0\}.$$

It is obvious that class Φ is enlarged by class Ψ, that is $\phi \subset \Phi$.

Proposition 2.3. [11] *If a metric space (ς,d) with $\mathfrak{R}$. Let $\mathcal{H} : \varsigma \to \varsigma$ a map and $\phi \in \Psi$, then these contractivity conditions are similar:*

(i) $d(\mathcal{H}u,\mathcal{H}v) \leq \phi(d(u,v))$ $\forall\, u,v \in \varsigma$ *with* $(u,v) \in \mathfrak{R}$,

(ii) $d(\mathcal{H}u,\mathcal{H}v) \leq \phi(d(u,v))$ $\forall\, u,v \in \varsigma$ *with* $[u,v] \in \mathfrak{R}$.

Lemma 2.4. [11] *Suppose that $\phi \in \Psi$. A sequence $\{s_n\} \subset (0,\infty)$ such as*

$$s_{n+1} \leq \phi(s_n) \quad \forall\, n \in \mathbb{N}_0,$$

therefore $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5. [11] *Assume that (ς,d) and $\{u_n\} \in \varsigma$. If $\{u_n\}$ is not a Cauchy, then $\exists\ \varepsilon > 0$ and two subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$, such that $\forall\, k \in \mathbb{N}$*

(i) $k \leq m_k < n_k$,

(ii) $d(u_{m_k},u_{n_k}) > \varepsilon$,

(iii) $d(u_{m_k},u_{n_{k-1}}) \leq \varepsilon$.

Furthermore, suppose that $\lim_{n\to\infty} d(u_n,u_{n+1}) = 0$, *then*

(iv) $\lim_{k\to\infty} d(u_{m_k},u_{n_k}) = \varepsilon$,

(v) $\lim_{k\to\infty} d(u_{m_{k+1}},u_{n_{k+1}}) = \varepsilon$.

There are some useful notations which are as follows:

(i) $F(\mathcal{H})$ is the set of all fixed points of $\mathcal{H}$.

(ii) $\varsigma(\mathcal{H},\mathfrak{R}) := \{u \in \varsigma : (u,\mathcal{H}u) \in \mathfrak{R}\}$.

3. Main result

The relation-theoretic version of BCP and Boyd and Wong contraction in M-metric space is shown in these results:

Theorem 3.1. *Assume that (ς,m) is an M-metric space with a binary relation $\mathfrak{R}$ and $m : \varsigma \times \varsigma \to \mathbb{R}^+$ is satisfying the assertions given below:*

1 . *(ς,m) is m-complete.*

2 . *$\varsigma(\mathcal{H},\mathfrak{R})$ is non-empty.*

3 . *$\mathfrak{R}$ is $\mathcal{H}$-closed.*
4 . *Either $\mathcal{H}$ is continuous or $\mathfrak{R}$ is m-self closed.*
5 . *$\exists\, \beta \in [0,1)$ such that $\forall\, u, v \in \varsigma$ with $(u,v) \in \mathfrak{R}$*

$$m(\mathcal{H}u, \mathcal{H}v) - m_{\mathcal{H}u,\mathcal{H}v} \leq \beta(m(u,v) - m_{u,v}).$$

Then, $\mathcal{H}$ has a fixed point.
6 . *$\gamma(u, v, \mathfrak{R}^s)$ is non-empty for each $u, v \in \varsigma$, then $\mathcal{H}$ has unique fixed point.*

Proof. Assume an arbitrary element $u_0 \in \varsigma(\mathcal{H}, \mathfrak{R})$. Construct the sequence $u_n = \mathcal{H}^n(u_0)\ \forall\, n \in \mathbb{N}_0$. As $(u_0, \mathcal{H}u_0) \in \mathfrak{R}$ using (2), we obtain

$$(\mathcal{H}u_0, \mathcal{H}^2 u_0), (\mathcal{H}^2 u_0, \mathcal{H}^3 u_0), \cdots (\mathcal{H}^n u_0, \mathcal{H}^{n+1} u_0) \in \mathfrak{R}.$$

So,

$$(u_n, u_{n+1}) \in \mathfrak{R},\ \forall\, n \in \mathbb{N}_0. \tag{3.1}$$

Thus, $\{u_n\}$ is $\mathfrak{R}$-preserving sequence. Next, applying the condition (4) to (3.1).
We conclude that

$$m(u_{n+1}, u_{n+2}) - m_{u_{n+1},u_{n+2}}) \leq \lambda(m(u_n, u_{n+1}) - m_{u_n,u_{n+1}}) \quad \forall\, n \in \mathbb{N}_0,$$

which by induction yields that

$$m(u_{n+1}, u_{n+2}) - m_{u_{n+1},u_{n+2}}) \leq \lambda^{n+1}(m(u_0, \mathcal{H}u_0) - m_{u_0,\mathcal{H}u_0}). \tag{3.2}$$

Using (3.2) the triangular inequality $\forall\, n \in \mathbb{N}_0,\ p \in \mathbb{N}$,

$$\begin{aligned} m(u_{n+1}, u_{n+p}) - m_{u_{n+1},u_{n+p}}) &\leq (m(u_{n+1}, u_{n+2}) - m_{u_{n+1},u_{n+2}}) + (m(u_{n+2}, u_{n+3}) - m_{u_{n+2},u_{n+3}}) \\ &\quad + \cdots + (m(u_{n+p-1}, u_{n+p}) - m_{u_{n+p-1},u_{n+p}}) \\ &\leq (\lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{n+p-1})(m(u_0, \mathcal{H}u_0) - m_{u_0,\mathcal{H}u_0}). \\ &\leq \lambda^n(m(u_0, \mathcal{H}u_0) - m_{u_0,\mathcal{H}u_0}) \sum_{j=1}^{p-1} \lambda^j. \end{aligned}$$

As $(m(u_{n+1}, u_{n+p}) - m_{u_{n+1},u_{n+p}}) \to 0$ and $(M(u_{n+1}, u_{n+p}) - m_{u_{n+1},u_{n+p}}) \to 0$ as $n \to \infty \Rightarrow$ sequence $\{u_n\}$ is m-Cauchy sequence in ς. As (ς, m) is m-complete. $\exists\, u \in \varsigma$ such that

$$u_n \xrightarrow{m} u.$$

Now, in lieu of (3) asserted $\mathcal{H}$ is continuous,
we have

$$u_{n+1} = \mathcal{H}(u_n) \xrightarrow{m} \mathcal{H}(u).$$

Now, $\mathcal{H}(u) = u$.
Alternatively, suppose $\mathfrak{R}$ is m-self closed and $\{u_n\}$ is $\mathfrak{R}$-preserving sequence

$$u_n \xrightarrow{m} u.$$

We have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u_n] \in \mathfrak{R}, \quad k \in \mathbb{N}_0.$$

Using (4), proposition 2.2 $[u_{n_k}, x] \in \mathfrak{R}$ and $u_{n_k} \xrightarrow{m} u$, we obtain

$$\begin{aligned} m(u_{n_k+1}, \mathcal{H}u) - m_{u_{n_k+1},\mathcal{H}u} &= m(\mathcal{H}u_{n_k}, \mathcal{H}u) - m_{\mathcal{H}u_{n_k},\mathcal{H}u} \\ &\leq \lambda(m(u_{n_k}, u) - m_{u_{n_k},u}) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

So that $u_{n_k+1} \xrightarrow{m} \mathcal{H}u$. Thus, $\mathcal{H}u = u$.
To show the uniqueness, we take $u, v \in F(\mathcal{H})$,

$$\mathcal{H}(u) = u \text{ and } \mathcal{H}(v) = v. \tag{3.3}$$

By assumption (5), $\exists$ a path of some function length k from u to v. Therefore,

$$w_0 = u \text{ and } w_k = v, \quad [w_i, w_{i+1}] \in \mathcal{R} \text{ for every } i \ (0 \leq i \leq k-1). \tag{3.4}$$

As $\mathfrak{R}$ is $\mathcal{H}$-closed by using proposition 2.2. We have

$$[\mathcal{H}^n w_i, \mathcal{H}^n w_{i+1}] \in \mathfrak{R} \text{ for every } (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \tag{3.5}$$

with the help of (3.3), (3.4), (3.5) and from hypothesis (4), we get

$$\begin{aligned}
m(u,v) - m_{u,v} &= m(\mathcal{H}^n w_0, \mathcal{H}^n w_k) - m_{\mathcal{H}^n w_0, \mathcal{H}^n w_k} \\
&\leq \sum_{i=0}^{k-1} m(\mathcal{H}^n w_i, \mathcal{H}^n w_{i+1}) - m_{\mathcal{H}^n w_0, \mathcal{H}^n w_{i+1}} \\
&\leq \lambda \sum_{i=0}^{k-1} m(\mathcal{H}^{n-1} w_i, \mathcal{H}^{n-1} w_k) - m_{\mathcal{H}^{n-1} w_0, \mathcal{H}^{n-1} w_{i+1}} \\
&\leq \lambda^2 \sum_{i=0}^{k-1} m(\mathcal{H}^{n-2} w_i, \mathcal{H}^{n-2} w_k) - m_{\mathcal{H}^{n-2} w_0, \mathcal{H}^{n-2} w_{i+1}} \\
&\leq \cdots \leq \lambda^n \sum_{i=0}^{k-1} m(w_i, w_{i+1}) - m_{w_i, w_{i+1}} \\
&\to 0 \ as \ n \to \infty
\end{aligned}$$

implies $u = v$. Hence, $\mathcal{H}$ has a fixed point, which is unique. □

Theorem 3.2. *Suppose M-metric space (ς, m) with $\mathfrak{R}$ and $\mathcal{H} : \varsigma \to \varsigma$. Assume that these conditions are true:*

1 . *(ς, m) is $\mathcal{H}$-closed.*
2 . *$\mathcal{R}$ is T-closed and locally $\mathcal{H}$-transitive.*
3 . *Either $\mathcal{H}$ is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is m-self closed.*
4 . *$\varsigma(\mathcal{H}, \mathfrak{R})$ is non-empty.*
5 . *$\exists\ \phi \in \Phi$ such that*

$$m(\mathcal{H}u, \mathcal{H}v) \leq \phi(m(u,v)), \ \forall\ u, v \in \varsigma \text{ with } (u,v) \in \mathfrak{R}.$$

Then, $\mathcal{H}$ has unique fixed point.

Proof. In view of assumption (4), consider an arbitrary point $u_0 \in \varsigma(\mathcal{H}, \mathfrak{R})$.
Construct $\{u_n\}$ with the initial point u_0,

$$u_n = \mathcal{H}^n(u_0), \ \forall\ n \in \mathbb{N}_0.$$

As $(u_0, \mathcal{H}u_0) \in \mathfrak{R}$, As $\mathfrak{R}$ is $\mathcal{H}$-closed and using proposition 2.1, $(\mathcal{H}^n u_0, \mathcal{H}^{n+1} u_0) \in \mathfrak{R}$. Therefore,

$$(u_n, u_{n+1}) \in \mathfrak{R}, \quad \forall\ n \in \mathbb{N}_0. \tag{3.6}$$

Thus, $\{u_n\}$ is $\mathfrak{R}$-preserving sequence. Using the condition (5) to (3.6), we conclude $\forall\ n \in \mathbb{N}_0$, i.e.

$$\begin{aligned}
m(u_{n+1}, u_{n+2}) - m_{u_{n+1}, u_{n+2}} &\leq \phi(m(u_n, u_{n+1}) - m_{u_n, u_{n+1}}) \\
\lim_{n \to \infty} m(u_{n+1}, u_{n+1}) - m_{u_{n+1}, u_{n+1}} &= 0.
\end{aligned}$$

Next, we assume that $\{u_n\}$ is not a Cauchy sequence. By Lemma 2.5, $\exists\ \varepsilon > 0$ and the subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$, i.e.

$$k \leq m_k < n_k, \ m(u_{m_k}, u_{n_k}) - m_{u_{m_k}, u_{n_k}} > \varepsilon \geq m(u_{m_k+1}, u_{n_k-1}) - m_{u_{m_k+1}, u_{n_k-1}}, \ \forall\ k \in \mathbb{N}. \tag{3.7}$$

Denote $r_k = m(u_{m_k}, u_{n_k}) - m_{u_{m_k}, u_{n_k}}$. As $\{u_n\}$ is $\mathfrak{R}$-preserving and $\{u_n\} \subset \mathcal{H}(u)$, by locally $\mathcal{H}$-transivity of $\mathfrak{R}$, we have $(u_{m_k}, u_{n_k}) \in \mathfrak{R}$.
Applying the contractivity condition (5), we have

$$\begin{aligned} m(u_{m_k+1}, u_{n_k+1}) - m_{u_{m_k+1}, u_{n_k+1}} &= m(\mathcal{H}u_{m_k}, \mathcal{H}u_{n_k}) - m_{\mathcal{H}u_{m_k}, \mathcal{H}u_{n_k}} \\ &\leq \phi(m(u_{m_k}, u_{n_k}) - m_{u_{m_k}, u_{n_k}}) = \phi(\nu_k). \end{aligned}$$

So that,

$$m(u_{m_k+1}, u_{n_k+1}) - m_{u_{m_k+1}, u_{n_k+1}} \leq \phi(\nu_k). \tag{3.8}$$

Using the notion $\nu_k \Rightarrow \varepsilon$ in the real line as $k \to \infty$ (owing to (3.7)) and $\nu_k > \varepsilon,\ k \in \mathbb{N}$. We get

$$\lim_{n\to\infty} \sup \phi(\nu_k) = \lim_{\nu\to\varepsilon^+} \sup \phi(\nu) < \varepsilon.$$

As we take $k \to \infty$ in (3.8), then $\varepsilon = \lim_{k\to\infty} \sup m(u_{m_k+1}, u_{n_k+1}) - m_{u_{m_k+1}, u_{n_k+1}} \leq \lim_{k\to\infty} \sup \phi(\nu_k) < \varepsilon$, a contradiction. Then, $\{u_n\}$ is a m-Cauchy sequence. In consequence $\{u_n\}$ is $\mathfrak{R}$-preserving m-Cauchy sequence. Under $\mathfrak{R}$-completeness, $\exists\, u \in \varsigma$ i.e. $u_n \xrightarrow{m} u$.
Lastly, to prove that u is a fixed point of $\mathcal{H}$, using assumption (3). Suppose $\mathcal{H}$ is $\mathcal{R}$-continuous, while $\{u_n\}$ is $\mathfrak{R}$-preserving with $u_n \xrightarrow{m} u$, implies that

$$u_{n+1} = \mathcal{H}u_n = \mathcal{H}(u_n) \xrightarrow{m} \mathcal{H}(u).$$

Using the limit's of uniqueness, we get $\mathcal{H}(u) = u$.

Alternatively, assume that $\mathfrak{R}$ is m-self closed. As $\{u_n\}$ is $\mathfrak{R}$-preserving, i.e. $\{u_n\} \xrightarrow{m} u, m$ self closedness of $\mathfrak{R}$ ensures the existence of $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathfrak{R}$, assumption (5) and proposition 2.3, we have obtain $\forall\ k\ \in \mathbb{N}_0$.

$$m(u_{n_k+1}, \mathcal{H}u) - m_{u_{n_k+1}, \mathcal{H}u} = m(\mathcal{H}u_{n_k}, \mathcal{H}u) - m_{\mathcal{H}u_{n_k}, \mathcal{H}u} \leq \phi(m(u_{n_k}, u) - m_{u_{n_k}, u}).$$

We claim that

$$m(u_{n_k+1}, \mathcal{H}u) - m_{u_{n_k+1}, \mathcal{H}u} \leq m(u_{n_k}, u) - m_{u_{n_k}, u}, \quad \forall\ k\ \in \mathbb{N}. \tag{3.9}$$

Consider a partition $\{\mathbb{N}_0, \mathbb{N}^+\}$ of $\mathbb{N}$ ($\mathbb{N}_0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}_0 \cap \mathbb{N}^+ = \emptyset$). Confirming that

(1) $m(u_{n_k}, u) - m_{u_{n_k}, u} = 0, \quad \forall\, k \in \mathbb{N}_0,$

(2) $m(u_{n_k}, u) - m_{u_{n_k}, u} > 0, \quad \forall\, k \in \mathbb{N}^+.$

In case (1) we have $m(\mathcal{H}u_{n_k}, \mathcal{H}u) - m_{\mathcal{H}u_{n_k}, \mathcal{H}u} = 0,\ \forall\ k \in \mathbb{N}_0$ which implies that $m(u_{n_k+1}, \mathcal{H}u) - m_{u_{n_k+1}, \mathcal{H}u} = 0,\ \forall\, k \in \mathbb{N}_0$, so (3.9) holds $\forall\, k \in \mathbb{N}_0$.
In case (2), $m(u_{n_k+1}, \mathcal{H}u) - m_{u_{n_k+1}, \mathcal{H}u} \leq \phi(m(u_{n_k}, u) - m_{u_{n_k}, u}) \quad \forall\ k \in \mathbb{N}^+$, therefore (3.9) holds, $\forall\, k \in \mathbb{N}^+$. Then, (3.9) true $\forall\, k \in \mathbb{N}$.
As $k \to \infty$ and using $u_n \xrightarrow{m} x$, we get $u_{n_k+1} \xrightarrow{m} \mathcal{H}u$. Hence, $\mathcal{H}(u) = u$. □

Example 1: Suppose $\varsigma = [0, \infty)$ and $m(u, v) = \frac{u+v}{2}$. Define $\mathfrak{R} = \{(u, v) \in [0, \infty) \times [0, \infty) : u - v \geq 0,\ x \in \mathbb{Q}\}$ on ς.

Assume $\mathcal{H} : \varsigma \to \varsigma, \mathcal{H}(u) = \frac{u}{5}$. Clearly, $\mathfrak{R}$ is $\mathcal{H}$-closed and $\mathcal{H}$ is continuous.

Now for $u, v \in \varsigma$ with $u, v \in \mathfrak{R}$. We have

$$\begin{aligned} m(\mathcal{H}u, \mathcal{H}v) &= \frac{\mathcal{H}u + \mathcal{H}v}{2} = \frac{\frac{u}{5} + \frac{v}{5}}{2} \\ &= \frac{1}{5}\left(\frac{u+v}{2}\right) = \frac{1}{5}m(u, v) < \frac{2}{5}m(u, v). \end{aligned}$$

$\mathcal{H}$ satisfies assumption (4) of Theorem 3.1 for $\beta = \frac{2}{5}$. Thus, all conditions (1-4) of Theorem 3.1 are satisfied. Moreover here assumption (5) of Theorem 3.1 also holds and $\mathcal{H}$ has a unique fixed point.

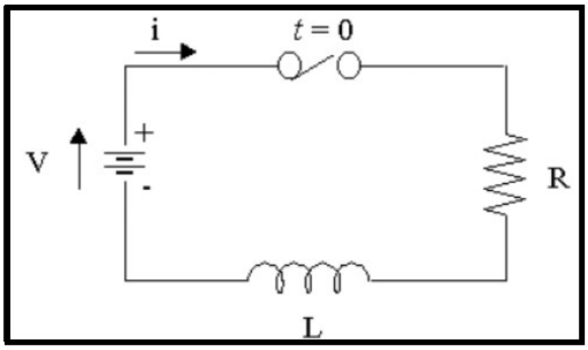

Figure 1: Kirhchoff's voltage law.

4. **Applications**

As an application of Theorem 3.1, using a series-connected resistor R, Emf E, inductance L, and voltage V, we may solve an electric circuit equation..
The relation shown in Fig. (1) are as follows:
(i) $V = iR$.
(ii) $V = L\frac{di}{dt}$.
Here, $i = \frac{du}{dt}$. According to Kirchhoff's law, the total voltage around a closed loop in a circuit must be equal to zero. So,

$$L\frac{di}{dt} + iR = V(t)$$

or

$$L\frac{d^2u}{dt^2} + RL\frac{du}{dt} = V(t), \quad u(0) = 0 \quad u'(0) = a \tag{4.1}$$

Thus, the Green function related to the IVP (4.1) is

$$\mathcal{S}(t,\eta) = \begin{cases} (t-\eta)e^{\tau(t-\eta)}, & 0 \le \eta \le t \le 1 \\ 0, & 0 \le t \le \eta \le 1. \end{cases}$$

where $\tau = \frac{R}{2L}$. IVP (4.1) corresponds to $u(t) = \int_0^t \mathcal{S}(t,\eta)\mathcal{F}(\eta, u(\eta))d\eta, \quad \eta \in [0,1]$. Define $m : \varsigma \times \varsigma \longrightarrow \mathbb{R}^+, m(u,v) = \sup_{t\in[0,1]} \frac{|u+v|}{2}, \quad u,v \in \varsigma$.

Theorem 4.1. *Assume that relational M-metric space $(\varsigma, m, \mathfrak{R})$ with a increasing non-linear mapping $\mathcal{H} : \varsigma \to \varsigma$ fulfils the subsequent axioms:*
(i) $\mathcal{S}(t,\eta) : [0,1] \times [0,1] \to [0,\infty)$ is a continuous map.
(ii) A mapping $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathbb{R}$ fulfils $|\mathcal{F}(t,u(t)) + \mathcal{F}(t,v(t))| \le \frac{1}{4}\tau^2(u+v)$, for any $\tau \in \mathbb{R}^+, u \ge v$.
After that, the IVP (4.1) that occurs in an electric circuit has a solution.

Proof. Denote $\mathcal{H} : \varsigma \to \varsigma, \mathcal{H}(u(t)) = \int_0^t \mathcal{S}(t,\eta)\mathcal{F}(\eta, u(\eta))d\eta$, and $\mathfrak{R} = \{(u,v) \in \varsigma \times \varsigma,\ u(t) \ge v(t), \quad t \in [0,1]\}$. It is obvious that the fixed point of $\mathcal{H}$ is a solution of (4.1).

(1) Consider $v = \mathbb{C}[0,1]$ and satisfies $\mathcal{H}\varsigma \subseteq v \subseteq \varsigma$.
(2) (v,m) is $\mathfrak{R}$-complete, when (ς,m) is $\mathfrak{R}$-complete.
(3) If $\tau = 1$, thus $(u, \mathcal{H}u) \in \mathfrak{R}$.
(4) Next, $\{u_n\} \subseteq \varsigma$ is $\mathfrak{R}$-preserving Cauchy sequence if $u_n \to u$ as $n \to \infty$. Then, we get $u_n(t) \ge u_{n+1}(t) \quad t \in [0,1]$ and $n \in \mathbb{N}_0$.
Since $u_{n+1} = \mathcal{H}u_n$ and $\{u_n\}$ is monotonically decreasing, $u_n(t) \ge u(t),\ t \in [0,1]$. Now we

can select a subsequence in this manner i.e. $(u_{n_k}, u) \in \mathfrak{R}, \quad \mathrm{t} \in [0,1],\ n \in \mathbb{N}_0$. Hence, $\mathfrak{R}|_v$ is m-closed.

(5) For $u(\mathrm{t}) \geq v(\mathrm{t})$,

$$\begin{aligned} m(\mathcal{H}u, \mathcal{H}v) &= \frac{|\mathcal{H}u + \mathcal{H}v|}{2} \\ &= \frac{1}{2}\left|\int_0^{\mathrm{t}} (\mathcal{S}(\mathrm{t},\eta)\mathcal{F}(\eta, u(\eta))d\eta) + \int_0^{\mathrm{t}} \mathcal{S}(\mathrm{t},\eta)\mathcal{F}(\eta, v(\eta))d\eta\right| \\ &\leq \frac{1}{2}.\frac{1}{4}\tau^2 \sup_{\mathrm{t}\in[0,1]} |u+v| \int_0^{\mathrm{t}} \mathcal{S}(\mathrm{t},\eta)d\eta \\ &= \frac{1}{8}\tau^2 \sup_{\mathrm{t}\in[0,1]} |u+v| \left(\int_0^1 (\mathrm{t}-\eta)e^{\tau(\mathrm{t}-\eta)}d\eta - \frac{\eta^2}{2}e^{\tau(\mathrm{t}-\eta)}.\tau.(\frac{-\eta^2}{2})\right) \\ &= \frac{1}{8}\tau^2 \sup_{\mathrm{t}\in[0,1]} |u+v|(1-\tau e^{-\tau \mathrm{t}} - e^{-\tau \mathrm{t}}) \leq \frac{1}{4}m(u,v), \end{aligned}$$

where $(1-\tau e^{-\tau \mathrm{t}} - e^{-\tau \mathrm{t}}) \leq 1$. Therefore, Theorem 3.1 is satisfied and IVP has a solution.

□

5. Conclusion

In this research paper, we generalized fixed point results for BCP and Boyd and Wong contraction for M-metric space with the respect to relation theoretic principle. We also gave an application to show the existence of the solution of IVP which supported our fixed point theorem.

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