CERTAIN TYPES OF VAGUE GRAPHS

Muhammad AKRAM[1], Feng FENG[2], Shahzad SARWAR[3] and Youne Bae JUN[4]

In this article, we propose certain types of vague graphs such as neighbourly irregular vague graphs, neighbourly total irregular vague graphs, highly irregular vague graphs and highly total irregular vague graphs. Some basic properties associated with these new vague graphs are investigated, and a necessary and sufficient condition under which neighbourly irregular and highly irregular vague graphs are equivalent is obtained.

Keywords: Irregular vague graph, neighbourly irregular vague graphs.

2000 Mathematics Subject Classification: 05C99.

1. Introduction

In the classical set theory introduced by Cantor, values of elements in a set are either 0 or 1. That is, for any element, there are only two possibilities: the element is either in the set or it is not. Therefore, Cantor's set theory cannot handle data with ambiguity and uncertainty. In 1965, Zadeh [25] proposed fuzzy theory and introduced fuzzy set theory. The most important feature of a fuzzy set is that it consists of a class of objects that satisfy a certain (or several) property. For example, for a fuzzy set A, each object x has a membership degree of A, denoted as $\mu_A(x)$. This membership function has the characteristics: The single degree contains the evidences for both supporting and opposing x. It can not only represent one of the two evidences, but it can represent both at the same time too. In order to deal with this problem, Gau and Buehrer [14] proposed the concept of vague set in 1993, by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, a true-membership function $t_v(x)$ and a false-membership function $f_v(x)$ are used to describe the boundaries of the membership degree. These two boundaries form a subinterval $[t_v(x), 1 - f_v(x)]$ of $[0, 1]$. The vague set theory improves the description of the objective real world, becoming a promising tool to deal with inexact, uncertain or vague knowledge.

[1] Punjab University College of Information Technology, University of the Punjab, Old Campus, Lahore-54000, Pakistan, E-mail: m.akram@pucit.edu.pk

[2] Department of Applied Mathematics, Xi'an University of Posts and Telecommunications, Xi'an 710121, China, E-mail: fengnix@hotmail.com

[3] Punjab University College of Information Technology, University of the Punjab, Old Campus, Lahore-54000, Pakistan, E-mail: s.sarwar@pucit.edu.pk

[4] Department of Mathematics Education (and RINS), Gyeongsang National University Chinju 660-701, Republic of Korea, E-mail: skywine@gmail.com

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Kaufmann's initial definition of a fuzzy graph [16] was based on Zadeh's fuzzy relations [26]. Rosenfeld [20] introduced the fuzzy analogue of several basic graph-theoretic concepts and Bhattacharya [8] gave some remarks on fuzzy graphs. Mordeson and Peng [18] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. In [22], the definition of complement of a fuzzy graph was modified so that the complement of the complement is the original fuzzy graph, which agrees with the crisp graph case. Atanassov [5] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs. Ramakrishna [19] introduced the concept of vague graphs and studied some of their properties. Akram et al. [1-4] introduced many new concepts, including bipolar fuzzy graphs, interval valued line fuzzy graphs and strong intuitionistic fuzzy graphs. In this paper, we introduce certain types of irregular vague graphs. Specifically, we introduce the concepts of neighbourly irregular vague graphs, neighbourly total irregular vague graphs, highly irregular vague graphs and highly total irregular vague graphs. We prove a necessary and sufficient condition under which neighbourly irregular and highly irregular vague graphs are equivalent.

2. Preliminaries

By a graph $G^* = (V, E)$, we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. Formally, given a graph $G^* = (V, E)$, two vertices $x, y \in V$ are said to be *neighbors, or adjacent nodes*, if $xy \in E$. A *path* in a graph G^* is an alternating sequence of vertices and edges v_0, e_1, v_1, e_2, $\cdots$,v_{n-1}, e_n, v_n. The path with $n+1$ vertices is denoted by P_n. A path is sometime denoted by $P_n : v_0v_1 \cdots v_n$ $(n > 0)$. A path $P_n : v_0v_1 \cdots v_n$ in G^* is called a *cycle* if $v_0 = v_n$ and $n \geq 3$. Note that path graph, P_n, has n edges and can be obtained from cycle graph, C_n, by removing any edge. The *neighbourhood* of a vertex v in a graph G^* is the induced subgraph of G^* consisting of all vertices adjacent to v and all edges connecting two such vertices. The neighbourhood is often denoted $N(v)$. The degree $\deg(v)$ of vertex v is the number of edges incident on v. The set of neighbors, called a *(open) neighborhood* $N(v)$ for a vertex v in a graph G^*, consists of all vertices adjacent to v but not including v, that is, $N(v) = \{u \in V \mid vu \in E\}$. When v is also included, it is called a *closed neighborhood* $N[v]$, that is, $N[v] = N(v) \cup \{v\}$. A *regular graph* is a graph where each vertex has the same number of neighbors, i.e., all the vertices have the same closed neighbourhood degree. An undirected graph G^* is *connected* if there is a path between each pair of distinct vertices. A connected graph is *highly irregular* if each of its vertices is adjacent only to vertices with distinct degrees. Equivalently, a graph G^* is highly irregular if every two vertices of

G^* connected by a path of length 2 have distinct degrees. A connected graph is said to be *neighbourly irregular* if no two adjacent vertices of G^* have the same degree. Equivalently, a connected graph G^* is called neighbourly irregular if every two adjacent vertices of G have distinct degree. For further information, the readers are referred to [7, 9].

Definition 2.1. [25, 26] *A fuzzy subset μ on a set X is a map $\mu : X \to [0,1]$. A fuzzy binary relation on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : X \times X \to [0,1]$.*

Definition 2.2. [5] *An intuitionistic fuzzy set (IFS, for short) on a universe X is an object of the form*

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\},$$

where $\mu_A(x)(\in [0,1])$ is called degree of membership of x in A and $\nu_A(x)(\in [0,1])$ is called degree of nonmembership of x in A, and μ_A, ν_A satisfy the following condition for all $x \in X$, $\mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.3. [14] *A vague set A in an ordinary finite nonempty set X is a pair (t_A, f_A), where $t_A : X \to [0,1]$, $f_A : X \to [0,1]$ are true and false membership functions, respectively such that $0 \leq t_A(x) + f_A(x) \leq 1$ for all $x \in X$.*

In the above definition, $t_A(x)$ is considered as the lower bound for degree of membership of x in A (based on evidence), and $f_A(x)$ is the lower bound for negation of membership of x in A (based on evidence against). Therefore, the degree of membership of x in the vague set A is characterized by the interval $[t_A(x), 1 - f_A(x)]$. So, a vague set is a special case of interval valued sets studied by many mathematicians and applied in many branches of mathematics (see for example [3, 23, 24]). Also vague sets have many applications (cf. [6, 11, 21]). The interval $[t_A(x), 1 - f_A(x)]$ is called the *vague value* of x in A, and is denoted by $V_A(x)$. We denote zero vague and unit vague value by $\mathbf{0} = [0,0]$ and $\mathbf{1} = [1,1]$, respectively. It is worth to mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval valued membership value is assigned to each element of the universe considering the "evidence for x" only, without considering "evidence against x". In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership.
A vague relation is a generalization of a fuzzy relation.

Definition 2.4. *Let X and Y be ordinary finite nonempty sets. We call a vague relation to be a vague subset of $X \times Y$, that is, an expression R defined by:*

$$R = \{< (x, y), t_R(x, y), f_R(x, y) > | x \in X, y \in Y\}$$

where $t_R : X \times Y \to [0,1]$, $f_R : X \times Y \to [0,1]$, which satisfies the condition $0 \leq t_R(x, y) + f_R(x, y) \leq 1$, for all $(x, y) \in X \times Y$. A vague relation R on X is called reflexive if $t_R(x, x) = 1$ and $f_R(x, x) = 0$ for all $x \in X$. A vague relation R on X is symmetric if $t_R(x, y) = t_R(y, x)$ and $f_R(x, y) = f_R(y, x)$ for all $x, y \in X$.

A vague set, as well as an intuitionistic fuzzy set [5], is a further generalization of a fuzzy set. In the literature, the notions of intuitionistic fuzzy sets and vague sets are regarded as equivalent, in the sense that an intuitionistic fuzzy set is isomorphic to a vague set [12].

3. Certain types of vague graphs

Throughout this paper, G^* will be a crisp graph (V, E), and G a vague graph (A, B). Since an edge $xy \in E$ is identified with an ordered pair $(x, y) \in V \times V$, a vague relation on E can be identified with a vague set on E. This gives a possibility to define a vague graph as a pair of vague sets.

Definition 3.1. [19] *Let $G^* = (V, E)$ be a crisp graph. A pair $G = (A, B)$ is called a vague graph on a crisp graph $G^* = (V, E)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that*

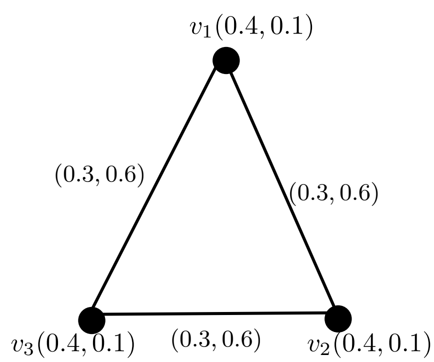
$$t_B(xy) \leq \min(t_A(x), t_A(y)) \text{ and } f_B(xy) \geq \max(f_A(x), f_A(y))$$

for each edge $xy \in E$.

Definition 3.2. *Let G be a vague graph on G^*. If all the vertices have the same neighbourhood degree m, then G is called a regular vague graph. The neighbourhood degree of a vertex x in G is defined by $\deg(x) = (\deg_t(x), \deg_f(x))$, where $\deg_t(x) = \sum_{y \in N(x)} t_A(y)$ and $\deg_f(x) = \sum_{y \in N(x)} f_A(y)$.*

Example 3.1. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\}.$$

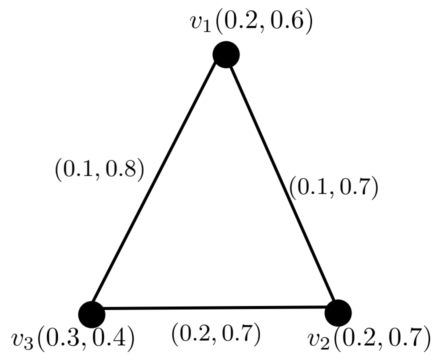


By routine computations, we have $\deg(v_1) = \deg(v_2) = \deg(v_3) = (0.8, 0.2)$. It is clear that G is a regular vague graph.

Definition 3.3. *Let G be a vague graph on G^*. If there is a vertex which is adjacent to vertices with distinct neighbourhood degrees, then G is called an irregular vague graph. That is, $\deg(x) \neq m$ for all $x \in V$.*

Example 3.2. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\}.$$



By routine computations, we have $\deg(v_1) = (0.5, 1.1)$, $\deg(v_2) = (0.5, 1.0)$ *and* $\deg(v_3) = (0.4, 1.3)$. *It is clear that G is an irregular vague graph.*

Definition 3.4. *Let G be a vague graph. The closed neighbourhood degree of a vertex x is defined by* $\deg[x] = (\deg_t[x], \deg_f[x])$, *where*

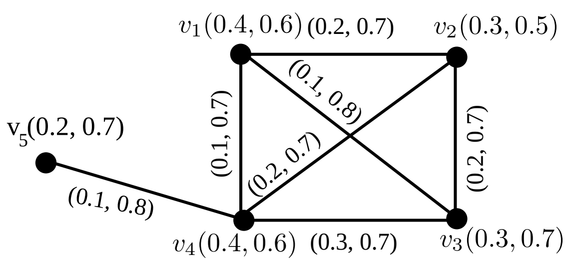
$$\deg_t[x] = \deg_t(x) + t_A(x),$$

$$\deg_f[x] = \deg_f(x) + f_A(x).$$

If there is a vertex which is adjacent to vertices with distinct closed neighbourhood degrees, then G is called a totally irregular vague graph.

Example 3.3. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3, v_4, v_5\}, E = \{v_1v_2, v_2v_3, v_2v_4, v_3v_1, v_3v_4, v_4v_1, v_4v_5\}.$$

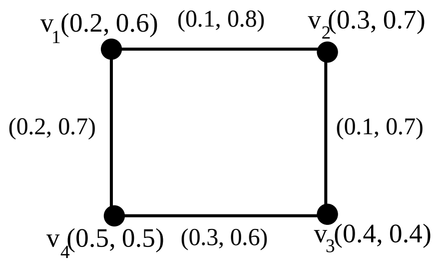


By routine computations, we have $\deg[v_1] = (1.4, 2.4)$, $\deg[v_2] = (1.4, 2.4)$, $\deg[v_3] = (1.4, 2.4)$, $\deg[v_4] = (1.6, 3.1)$ *and* $\deg[v_5] = (0.6, 1.3)$. *It is clear from calculations that G is a totally irregular vague graph.*

Definition 3.5. *A connected vague graph G is said to be a neighbourly irregular vague graph if every two adjacent vertices of G have distinct neighbourhood degree.*

Example 3.4. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3, v_4\}, E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}.$$

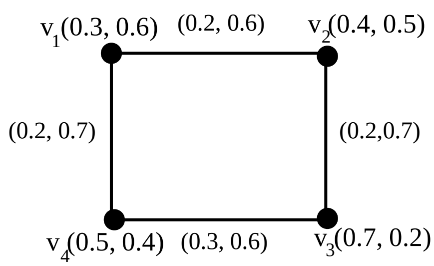


By routine computations, we have $\deg(v_1) = (0.8, 1.2)$, $\deg(v_2) = (0.6, 1.0)$, $\deg(v_3) = (0.8, 1.2)$ *and* $\deg(v_4) = (0.6, 1.0)$. *Hence G is neighbourly irregular vague graph.*

Definition 3.6. *A connected vague graph G is said to be a neighbourly total irregular vague graph if if every two adjacent vertices of G have distinct closed neighbourhood degree.*

Example 3.5. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\}.$$



By routine computations, we have $\deg[v_1] = (1.2, 1.5)$, $\deg[v_2] = (1.4, 1.3)$, $\deg[v_3] = (1.6, 1.1)$ *and* $\deg[v_4] = (1.5, 1.2)$. *Hence G is neighbourly total irregular vague graph.*

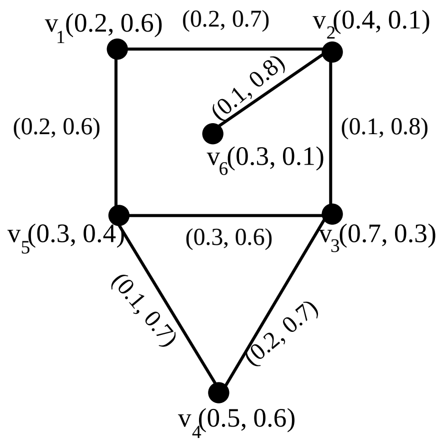
Definition 3.7. *Let G be a connected vague graph. G is called a highly irregular vague graph if every vertex of G is adjacent to vertices with distinct neighbourhood degrees.*

Lemma 3.1. *A highly irregular vague graph may not be a neighbourly irregular vague graph.*

Lemma 3.13 follows from the following example.

Example 3.6. *Consider a vague graph G such that*

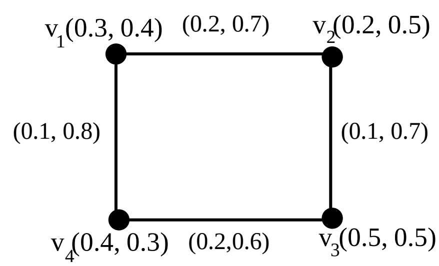
$$V = \{v_1, v_2, v_3, v_4, v_5\}, E = \{v_1v_2, v_2v_3, v_2v_6, v_3v_4, v_3v_5, v_4v_5, v_5v_1\}.$$



By routine computations, we have $\deg(v_1) = (0.7, 0.5)$, $\deg(v_2) = (1.2, 1.0)$, $\deg(v_3) = (1.2, 1.0)$, $\deg(v_4) = (1.0, 0.7)$, $\deg(v_5) = (1.4, 1.4)$ *and* $\deg(v_6) = (0.4, 0.1)$. *Consider a vertex* $v_2 \in V$ *which is adjacent to the vertices* v_1, v_3 *and* v_6 *with distinct neighbourhood degrees. But* $\deg(v_2) = \deg(v_3)$. *Hence* G *is highly irregular vague graph but it is not a neighborly irregular vague graph.*

Example 3.7. *Consider a vague graph* G *such that*

$$V = \{v_1, v_2, v_3, v_4\}, E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}.$$



By routine computations, we have $\deg(v_1) = (0.6, 0.8)$, $\deg(v_2) = (0.8, 0.9)$, $\deg(v_3) = (0.6, 0.8)$, $\deg(v_4) = (0.8, 0.9)$. *We see that every two adjacent vertices have distinct neighbourhood degree. But consider a vertex* v_2 *which is adjacent to the vertices* v_1 *and* v_3 *has same degree, that is,* $\deg(v_1) = \deg(v_3)$. *Hence* G *is neighbourly irregular vague graph but not a highly irregular vague graph.*

Lemma 3.2. *A neighbourly irregular vague graph may not be a highly irregular vague graph.*

Theorem 3.1. *Let* G *be a vague graph. Then* G *is highly irregular vague graph and neighbourly vague graph if and only if the neighbourhood degrees of all the vertices of* G *are distinct.*

Proof. Let G be a vague graph with n-vertices $v_1, v_2, \ldots, v_n$. Assume that G is highly irregular vague graph and neighbourly vague graph.
Claim: The neighbourhood degrees of all vertices of G are distinct. Let $\deg(v_i) =$

$(k_i, l_i), i = 1, 2, ..., n$. Let the adjacent vertices of v_1 be $v_2, v_3, \ldots, v_n$ with neighbourhood degrees

$$(k_2, l_2), (k_3, l_3), \ldots, (k_n, l_n),$$

respectively. Then we've $k_2 \neq k_3 \neq \cdots \neq k_n$ and $l_2 \neq l_3 \neq ... \neq l_n$, since G is highly irregular. Also $k_1 \neq k_2 \neq k_3 \neq \ldots \neq k_n$ and $l_1 \neq l_2 \neq l_3 \neq \ldots \neq l_n$, since G is neighbourly irregular. Hence, the neighbourhood degree of all the vertices of G are distinct.

Conversely, assume that the neighbourhood degrees of all the vertices of G are distinct.

Claim: G is highly irregular and neighbourly irregular vague graph.

Let $\deg(v_i) = (k_i, l_i), i = 1, 2, \ldots, n$. Given that $k_1 \neq k_2 \neq k_3 \neq ... \neq k_n$ and $l_1 \neq l_2 \neq l_3 \neq ... \neq l_n$, which implies that every two adjacent vertices have distinct neighbourhood degrees and to every vertex, the adjacent vertices have distinct neighbourhood degrees. □

Theorem 3.2. *A vague graph G of G^*, where G^* is a cycle with vertices 3 is neighbourly irregular and highly irregular vague graph if and only if the true membership and false membership value of the vertices between every pair of vertices are all distinct.*

Proof. Assume that true membership and false membership value of the vertices are all distinct. Claim: G is neighbourly irregular and highly irregular vague graph.

Let $v_i, v_j, v_k \in V$. Given that, $t_A(v_i) \neq t_A(v_j) \neq t_A(v_k)$ and $f_A(v_i) \neq f_A(v_j) \neq f_A(v_k)$, which implies that $\sum_{x \in N(x)} t_A(v_i) \neq \sum_{x \in N(x)} t_A(v_j) \neq \sum_{x \in N(x)} t_A(v_k)$ and $\sum_{x \in N(x)} f_A(v_i) \neq \sum_{x \in N(x)} f_A(v_j) \neq \sum_{x \in N(x)} f_A(v_k)$. That is, $\deg(v_i) \neq \deg(v_j) \neq \deg(v_k)$. Hence G is neighbourly irregular and highly irregular vague graph.

Conversely, assume that G is neighbourly irregular and highly irregular.

Claim: true membership and false membership value of the vertices are all distinct.

Let $\deg(v_i) = (k_i, l_i), i = 1, 2, \ldots, n$. Suppose that true membership and false membership value of any two vertices are same. Let $v_1, v_2 \in V$. Let $t_A(v_1) = t_A(v_2)$ and $f_A(v_1) = f_A(v_2)$. Then $\deg(v_1) = \deg(v_2)$, since G^* is cycle, which is a contradiction to the fact that G is neighbourly irregular and highly irregular vague graph. Hence true membership and false membership value of the vertices are all distinct. □

Definition 3.8. *A vague graph G is called complete if*

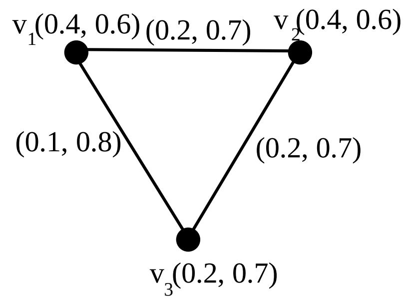
$$t_B(xy) = \min(t_A(x), t_A(y)) \text{ and } f_B(xy) = \max(f_A(x), f_A(y))$$

for each edge $xy \in E$.

Proposition 3.1. *A complete vague graph needed not be neighbourly irregular vague graph.*

Example 3.8. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\}.$$

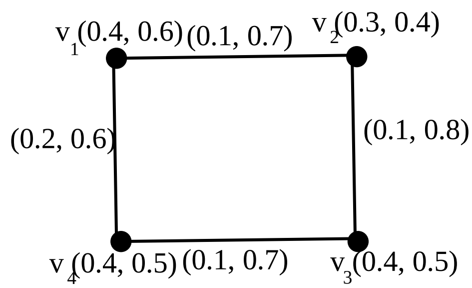


By routine computations, we have $\deg(v_1) = (0.6, 1.3)$, $\deg(v_2) = (0.6, 1.3)$ *and* $\deg(v_3) = (0.8, 1.2)$. *We see that neighbourhood degree of* v_1 *and* v_2 *are not distinct. Hence* G *is not neighbourly irregular vague graph but complete vague graph.*

Proposition 3.2. *A neighbourly irregular vague graph needed not be a neighbourly total irregular vague graph*

Example 3.9. *Consider a vague graph* G *such that*

$$V = \{v_1, v_2, v_3, v_4\}, E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}.$$

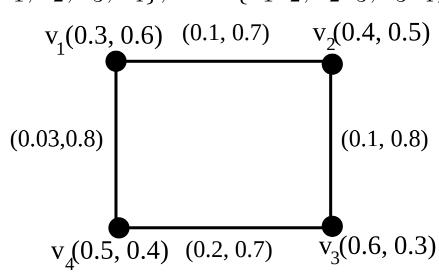


By routine computations, we have $\deg(v_1) = (0.7, 0.9)$, $\deg(v_2) = (0.8, 1.1)$, $\deg(v_3) = (0.7, 0.9)$, $\deg(v_4) = (0.8, 1.1)$ *and* $\deg[v_1] = (1.1, 1.5)$, $\deg[v_2] = (1.1, 1.5)$, $\deg[v_3] = (1.1, 1.4)$, $\deg[v_4] = (1.2, 1.6)$. *We see that* $\deg[v_1] = \deg[v_2]$. *Hence* G *is neighbourly irregular vague graph but not a neighbourly total vague graph.*

Proposition 3.3. *A neighbourly total irregular vague graph need not be a neighbourly irregular vague graph.*

Example 3.10. *Consider a vague graph* G *such that*

$$V = \{v_1, v_2, v_3, v_4\}, E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}.$$



By routine computations, we have $\deg[v_1] = (1.2, 1.5)$, $\deg[v_2] = (1.3, 1.4)$, $\deg[v_3] = (1.5, 1.2)$, $\deg[v_4] = (1.4, 1.3)$. *But* $\deg(v_1) = \deg(v_2) = \deg(v_2) = (0.9, 0.9)$. *Hence*

G is neighbourly total irregular vague graph but not a neighbourly irregular vague graph.

Proposition 3.4. *Let G be a vague graph. If G is neighbourly irregular vague graph and (t_A, f_A) is a constant function, then G is a neighbourly total irregular vague graph.*

Proof. Assume that G is a neighbourly irregular vague graph. That is the neighbourhood degrees of every two adjacent vertices are distinct. Let $v_i, v_j \in V$, where v_i and v_j are adjacent vertices with distinct neighbourhood degrees (k_1, l_1) and (k_2, l_2) respectively. That is $\deg(v_i) = (k_1, l_1)$ and $\deg(v_j) = (k_2, l_2)$, where $k_1 \neq k_2$, $l_1 \neq 1_2$. Let us assume that $(t_1(v_i), f_1(v_i)) = (t_1(v_j), f_1(v_j)) = (c_1, c_2)$, where c_1, c_2 are constant and $c_1, c_2 \in [0, 1]$. Therefore, $\deg_t[v_i] = \deg_t(v_i) + t_1(v_i) = k_1 + c_1$ and $\deg_f[v_i] = \deg_f(v_i) + f_1(v_i) = l_1 + c_2$ $\deg_t[v_j] = \deg_t(v_j) + t_1(v_j) = k_2 + c_1$ and $\deg_f[v_j] = \deg_f(v_j) + f_1(v_j) = l_2 + c_2$
Claim: $\deg_t[v_i] \neq \deg_t[v_j]$ and $\deg_f[v_i] \neq \deg_f[v_j]$. Suppose that, $\deg_t[v_i] = \deg_t[v_j]$ and $\deg_f[v_i] = \deg_f[v_j]$. Consider

$$\begin{aligned}
\deg_t[v_i] &= \deg_t[v_j] \\
k_1 + c_1 &= k_2 + c_1 \\
k_1 - k_2 &= c_1 - c_1 = 0 \\
k_1 &= k_2, \text{ which is a contradiction to } k_1 \neq k_2.
\end{aligned}$$

Therefore, $\deg_t[v_i] \neq \deg_t[v_j]$. Similarly, we consider

$$\begin{aligned}
\deg_f[v_i] &= \deg_f[v_j] \\
l_1 + c_2 &= l_2 + c_2 \\
l_1 - l_2 &= c_2 - c_2 = 0 \\
l_1 &= l_2, \text{ which is a contradiction to } l_1 \neq l_2.
\end{aligned}$$

Therefore, $\deg_f[v_i] \neq \deg_f[v_j]$. Hence G is a neighbourly total irregular vague graph. □

Proposition 3.5. *Let G be a vague graph. If G is a neighbourly total irregular and (t_A, f_A) is a constant function, then G is a neighbourly irregular vague graph.*

Proof. Assume that G is a neighbourly total irregular vague graph. That is the closed neighbourhood degree of every two adjacent vertices are distinct. Let $v_i, v_j \in V$ and $\deg[v_i] = (k_1, l_1)$, $\deg[v_j] = (k_2, l_2)$, where $k_1 \neq k_2$ and $l_1 \neq l_2$. Assume that, $(t_1(v_i), f_1(v_i)) = (c_1, c_2)$ and $(t_1(v_j), f_1(v_j)) = (c_1, c_2)$, where $c_1, c_2 \in [0, 1]$ are constant and $\deg[v_i] \neq \deg[v_j]$.
Claim: $\deg(v_i) \neq \deg(v_j)$
Given that $\deg[v_i] \neq \deg[v_j]$ which implies $\deg_t[v_i] \neq \deg_t[v_j]$ and $\deg_f[v_i] \neq$

$\deg_f[v_j]$. Now, we consider

$$\begin{aligned}\deg_t[v_i] &\neq \deg_t[v_j]\\ k_1 + c_1 &\neq k_2 + c_1\\ k_1 &\neq k_2.\end{aligned}$$

We now consider

$$\begin{aligned}\deg_f[v_i] &\neq \deg_f[v_j]\\ l_1 + c_2 &\neq l_2 + c_2\\ l_1 &\neq l_2.\end{aligned}$$

that is, the neighbourhood degrees of adjacent vertices of G are distinct. Hence neighbourhood degree of every pair of adjacent vertices is distinct in G. □

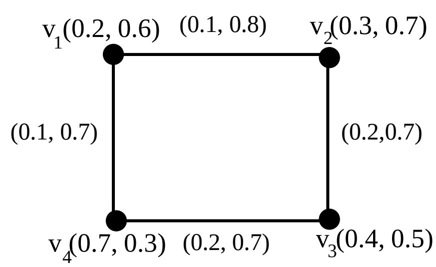
From propositions 3.26 and 3.27, we conclude that:

Theorem 3.3. *Let G be a vague graph such that (t_A, f_A) is a constant vague set. Then G is a neighbourly total irregular vague graph if and only if G is a neighbourly irregular vague graph.*

Remark 3.1. *Let G be a vague graph. If G is both neighbourly irregular and neighbourly total irregular vague graph, then (t_A, f_A) may not be a constant function.*

Example 3.11. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\}.$$



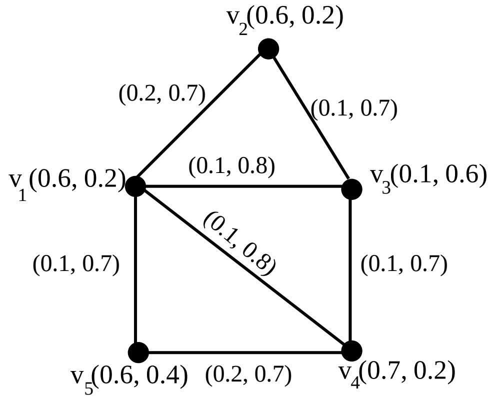
Clearly, G is neighbourly irregular and neighbourly total irregular vague graph, but membership and non-membership value of the vertices are not a constant function.

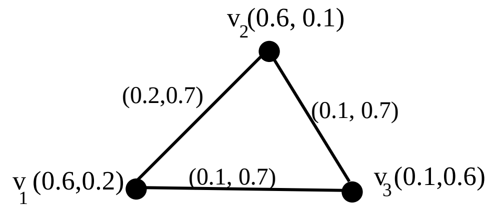
Remark 3.2. *If G is neighbourly irregular vague graph, then vague subgraph $H = (A', B')$ of G may not be neighbourly irregular vague graph.*

Example 3.12. *Consider a vague graph G such that*

$$V = \{v_1, v_2, v_3, v_4, v_5\}, E = \{v_1v_2, v_2v_3, v_3v_1, v_3v_4, v_4v_1, v_4v_5, v_5v_1\}.$$

Consider $H = (A', B')$such that $V' = \{v_1, v_2, v_3\}$, $E' = \{v_1v_2, v_2v_3, v_3v_1\}$.

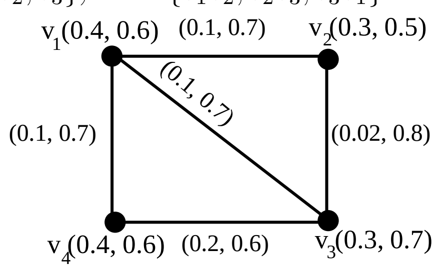


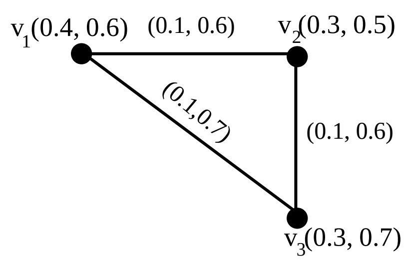


(1)For G: by routine computations, we have $\deg(v_1) = (2.0, 1.4)$, $\deg(v_2) = (0.7, 0.8)$, $\deg(v_3) = (1.9, 0.6)$, $\deg(v_4) = (1.3, 1.2)$ *and* $\deg(v_5) = (1.3, 0.4)$.
(2) For H: by routine computations, we have $\deg(v_1) = (0.7, 0.8)$, $\deg(v_2) = (0.7, 0.8)$ *and* $\deg(v_3) = (1.2, 0.4)$. *It is easy to see that* v_1 *and* v_2 *are adjacent vertices with same neighbourhood degree in* H. *Hence* H *is not a neighbourly irregular vague graph but* G *is neighbourly irregular vague graph.*

Proposition 3.6. *If* G *is total irregular vague graph, then vague subgraph* $H = (A', B')$ *of* G *may not be total irregular vague graph.*

Example 3.13. *Consider a vague graph* G *such that* $V = \{v_1, v_2, v_3, v_4\}$, $E = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_3v_1, v_4v_1\}$. *Consider* $H = (A', B')$, *such that* $V' = \{v_1, v_2, v_3\}$, $E' = \{v_1v_2, v_2v_3, v_3v_1\}$.





(1) G: By routine computations, we have $\deg[v_1] = (1.4, 2.4)$, $\deg[v_2] = (1.0, 1.8)$, $\deg[v_3] = (1.4, 2.4)$, $\deg[v_4] = (1.1, 1.9)$. *Here there is a vertex* v_3 *which is adjacent to* v_1, v_2 *and* v_4, *where* $\deg[v_1] \neq \deg[v_2] \neq \deg[v_4]$.
(2) H: By routine computations, we have $\deg[v_1] = (1.0, 1.8)$, $\deg[v_2] = (1.0, 1.8)$ *and* $\deg[v_3] = (1.0, 1.8)$. *Here, there is a vertex* v_1 *which is adjacent to the vertices* v_2 *and* v_3 *with same closed neighbourhood degree. Also,* v_2 *which is adjacent to the vertices* v_1 *and* v_3 *with same closed neighbourhood degree and* v_3 *which is adjacent to the vertices* v_1 *and* v_2 *with same closed neighbourhood degree. Hence* H *is not a total irregular vague graph but* G *is total irregular vague graph.*

4. **Conclusions**

In the real world there are vaguely specified data values in many applications. Fuzzy set theory has been proposed to handle such vagueness by generalizing the notion of membership in a set. Essentially, in a fuzzy set each element is associated with a point-value selected from the unit interval $[0, 1]$, which is termed the grade of membership in the set. Instead of using point-based membership as in fuzzy sets, interval-based membership is used in a vague set. The interval-based membership in vague sets is more expressive in capturing vagueness of data. There are some interesting features for handling vague data that are unique to vague sets, such as vague sets allow for a more intuitive graphical representation of vague data, which facilitates significantly better analysis in data relationships, incompleteness, and similarity measures. The notion of vague sets was initially incorporated into relations. Based on vague relations, we have introduced the certain types of vague graphs in this paper. The natural extension of this research work is application of vague graphs in the area of computing including neural networks, expert systems, database theory, and geographical information systems.

Acknowledgements. This work was partially supported by National Natural Science Foundation of China (Program No. 11301415), Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2013JQ1020) and Scientific Research Program Funded by Shaanxi Provincial Education Department of China (Program No. 2013JK1098). The authors are highly grateful to the referees for their valuable comments and suggestions.

REFERENCES

[1] *M. Akram*, Bipolar fuzzy graphs, Information Sci., **181** (2011), 5548-5564.

[2] *M. Akram*, Interval-valued fuzzy line graphs, Neural Computing & Applications, **21**(2012), 145-150.

[3] *M. Akram and W.A. Dudek*, Interval-valued fuzzy graphs, Computers Math. Appl., **61** (2011) 289-299.

[4] *M. Akram and B. Davvaz*, Strong intuitionistic fuzzy graphs, Filomat, **26** (2012), 177-196.

[5] *K.T. Atanassov*, Intuitionistic fuzzy sets: Theory and applications, Studies in fuzziness and soft computing, Heidelberg, New York, Physica-Verl., 1999.

[6] *R. Biswas*, Vague groups, Int. J. Comput. Cognition, **4**(2) (2006), 20-23.

[7] *M. Behzad and G. Chartrand*, No graph is perfect, Amer. Math. Month., **74** (1967), 962-963.

[8] *P. Bhattacharya*, Some remarks on fuzzy graphs, Pattern Recog. Lett., **6**(1987), 297-302.

[9] *S.G. Bhragsam and S.K. Ayyaswamy*, Neighbourly irregular graphs, Indian J. pure appl. Math., **35**(3)(2004), 389-399.

[10] *I. Bloch*, Dilation and erosion of spatial vague sets, Lecture Notes in Artificial Intelligence, 2007, 385-393.

[11] *I. Bloch*, Geometry of spatial vague sets based on vague numbers and mathematical morphology, Fuzzy Logic and Applications, Lecture Notes in Computer Science, **5571** (2009), 237-245.

[12] *H. Bustince and P. Burillo*, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems **79**(1996)403-405.

[13] *A.N. Gani and S. R. Latha*, On irregular fuzzy graphs, Appl. Math. Sci., **6**(2012), 517-523.

[14] *W.L. Gau and D.J. Buehrer*, Vague sets, IEEE Transactions on Systems, Man and Cybernetics, **23** (1993), 610-614.

[15] *H. Khan, M. Ahmad and R. Biswas*, Vague relations, Inter. Jour. Compu. Cog., **5**(1) (2007), 31-35.

[16] *A. Kauffman*, Introduction a la Theorie des Sous-emsembles Flous, Masson et Cie, Vol.1, 1973.

[17] *J. N. Mordeson and C. S. Peng*, Operations on fuzzy graphs, Information Sci., **79** (1994), 159-170.

[18] *J.N. Mordeson and P.S. Nair*, Fuzzy graphs and fuzzy hypergraphs, Physica Verlag, Heidelberg 1998; Second Edition 2001.

[19] *N. Ramakrishna*, Vague graphs, Internat. J. Computational Cognition, **7** (2009), 51-58.

[20] *A. Rosenfeld*, Fuzzy graphs, Fuzzy Sets and their Applications (L.A.Zadeh, K.S.Fu, M.Shimura, Eds.), Academic Press, New York, 1975, 77-95.

[21] *S. Sezer*, Vague rings and vague ideals, Iranian J. Fuzzy Syst., **18** (2011), 145-157.

[22] *M.S. Sunitha and A. Vijayakumar*, Complement of a fuzzy graph, Indian J. Pure Appl. Math., **33**(2002), 1451-1464.

[23] *H.C. Wu*, Interval-valued optimization problems based on different solution concepts, Pac. J. Optim., **7** (2011), 173-193.

[24] *X. Yang, T.Y. Lin, J. Yang, Y. Li and D. Yu*, Combination of interval-valued fuzzy set and soft set, Comput. Math. Appl., **58** (2009), 521-527.

[25] *L.A. Zadeh*, Fuzzy sets, Information and Control, **8** (1965), 338-353.

[26] *L.A. Zadeh*, Similarity relations and fuzzy orderings, Inform. Sci., **3** (1971), 177-200.