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COVARIANT VERSION OF THE STINESPRING TYPE THEOREM FOR n-TUPLES OF COMPLETELY POSITIVE MAPS ON HILBERT C^*-MODULES

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In this paper, we give a covariant version of the Sinespring construction for n-tuples of covariant completely positive maps on full Hilbert C^*-modules.

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1. Introduction and Preliminaries

Completely positive linear maps are an important tool in quantum information theory. The structure of completely positive maps, the description of the order relation on their set, the characterization of pure maps and extremal maps according to their structure play an important role in understanding a lot of problems. Stinespring's representation theorem is a structure theorem for operator valued completely positive maps on C^*-algebras. A completely positive map $\varphi : A \to L(\mathcal{H})$ is of the form $\varphi(\cdot) = V^*\pi(\cdot)V$ where π is a $*$-representation of A on a Hilbert space $\mathcal{K}$ and V is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$ [8]. A Stinespring type theorem for a class of unital maps on Hilbert C^*-modules was proved by Asadi in [1] and Bhat, Ramesh and Sumesh in [3]. In [5], Joiţa introduced the notion of operator valued covariant completely positive map on Hilbert C^*-modules and she provided a covariant Stinespring construction associated to such map. Pliev and Tsopanov [7] introduced the notion of n-tuple of completely positive maps on Hilbert C^*-modules and they provided a Stinespring construction for n-tuples of completely positive maps on Hilbert C^*-modules. In this note, we introduce the notion of n-tuple of covariant completely positive maps on Hilbert C^*-modules and we prove a covariant Stinespring type theorem for such maps.

A *Hilbert C^*-module* over a C^*-algebra A is linear space X which is also a right A-module equipped with an inner product $\langle \cdot, \cdot \rangle$ which is A-linear in the second variable, $\langle x, x \rangle \geq 0$ for all $x \in X$, $\langle x, x \rangle = 0$ if and only if $x = 0$ and $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in X$ and such that X is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. It will be denoted by (X, A). If the closed bilateral $*$-ideal $\langle X, X \rangle$ of A generated by $\{\langle x, y \rangle ; x, y \in X\}$ coincides with A, we say that X is *full.* For more details we refer the reader [6].

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then $L(\mathcal{H}, \mathcal{K})$ has a natural structure of Hilbert C^*-module over $L(\mathcal{H})$ with the module structure given by

$$T \cdot S = T \circ S \text{ for all } T \in L(\mathcal{H}, \mathcal{K}) \text{ and } S \in L(\mathcal{H})$$

and the inner product given by

$$\langle T_1, T_2 \rangle = (T_1)^* \circ T_2 \text{ for all } T_1, T_2 \in L(\mathcal{H}, \mathcal{K}).$$

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A *morphism of Hilbert C^*-modules* from a Hilbert C^*-module (X, A) to a Hilbert C^*-module (Y, B) is a map $\Psi : X \to Y$ with the property that there is a C^*-morphism $\psi : A \to B$ such that $\langle \Psi(x), \Psi(y) \rangle = \psi(\langle x, y \rangle)$ for all $x, y \in X$ [2, p. 184]. If (X, A) is full, then the underlying C^*-morphism ψ of Ψ is unique. An *isomorphism of Hilbert C^*-modules* is a bijective morphism of Hilbert C^*-modules Ψ such that Ψ^{-1} is also a morphism of Hilbert C^*-modules.

An *action* of a locally compact group G on a full Hilbert C^*-module (X, A) is a pair (γ, α), where α is an action of G on A, γ is a group morphism from G to the group $\text{Aut}(X, A)$ of all isomorphisms of Hilbert C^*-modules on (X, A) such that, for each $x \in X$, the map $g \to \gamma_g(x)$ from G to X is continuous, and for each $g \in G$, (γ_g, α_g) is an isomorphism of Hilbert C^*-modules.

A *representation* of a full Hilbert C^*-module (X, A) on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is morphism of Hilbert C^*-modules from (X, A) to $(L(\mathcal{H}, \mathcal{K}), L(\mathcal{H}))$.

A 5-tuple $((\pi_X, \pi_A), u, v, \mathcal{H}, \mathcal{K})$ consisting of a representation (π_X, π_A) of (X, A) on $\mathcal{H}$ and $\mathcal{K}$ and two unitary representations $(u, \mathcal{H})$ and $(v, \mathcal{K})$ of G such that

$$\pi_X(\gamma_g(x)) = v_g \pi_X(x) u_g^*$$

for all $g \in G$ and for all $x \in X$ is called a *representation* of (X, A), *covariant* with respect to the action (γ, α) of G on (X, A) [5]. If $((\pi_X, \pi_A), u, v, \mathcal{H}, \mathcal{K})$ is a representation of (X, A), covariant with respect to (γ, α) and X is full, then $(\pi_A, u, \mathcal{H})$ is a representation of A, covariant with respect to the action α of G on A.

Asadi [1] introduced the notion of operator valued complete positive map on Hilbert C^*-modules. A map $\Phi : (X, A) \to (L(\mathcal{H}, \mathcal{K}), L(\mathcal{H}))$ is *completely positive* if there is a completely positive map $\varphi : A \to L(\mathcal{H})$ such that

$$\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$$

for all $x, y \in X$. Bhat, Ramesh and Sumesh [3] provided a Stinespring construction associated to a unital completely positive map $\Phi : (X, A) \to (L(\mathcal{H}, \mathcal{K}), L(\mathcal{H}))$ in terms of the Stinespring construction associated to the underlying completely positive map $\varphi : A \to L(\mathcal{H})$.

Let $(u, \mathcal{H})$ and $(v, \mathcal{K})$ be two unitary representations of G. A completely positive map $\Phi : (X, A) \to (L(\mathcal{H}, \mathcal{K}), L(\mathcal{H}))$ is $(u, v)-$*covariant with respect to* (γ, α), if

$$\Phi(\gamma_g(x)) = v_g \Phi(x) u_g^*$$

for all $g \in G$ and for all $x \in X$ [5].

2. n-tuples of completely positive maps

Let (X, A) be a Hilbert C^*-module.

We recall that a *completely n-positive linear map* $[\varphi]$ from a C^*-algebra A to another C^*-algebra B is a matrix $[\varphi_{ij}]_{i,j=1}^n$ of linear maps from A to B such that $[\varphi]\left([a_{ij}]_{i,j=1}^n\right) = [\varphi_{ij}(a_{ij})]_{i,j=1}^n$ and $[\varphi] : M_n(A) \to M_n(B)$ is completely positive.

Definition 2.1. [7] *An n-tuple $\Phi = (\Phi_1, \Phi_2, ..., \Phi_n)$ of linear maps from X to $L(\mathcal{H}, \mathcal{K})$ is completely positive if there is a completely n-positive linear map $[\varphi] = [\varphi_{ij}]_{i,j=1}^n$ from A to $L(\mathcal{H})$ such that*

$$[\langle \Phi_i(x), \Phi_j(y) \rangle]_{i,j=1}^n = [\varphi_{ij}(\langle x, y \rangle)]_{i,j=1}^n$$

for all $x, y \in X$.

Example 2.1. *Let $X = M_2(\mathbb{C})$ and $A = M_2(\mathbb{C})$. Consider the maps $\Phi_1, \Phi_2 : M_2(\mathbb{C}) \to L\left(\mathbb{C}^2, \mathbb{C}^4\right)$ given by*

$$\Phi_1(a) = \begin{bmatrix} a \\ -a \end{bmatrix}, \text{ respectively } \Phi_2(a) = \begin{bmatrix} ia \\ -ia \end{bmatrix}$$

and the linear maps $\varphi_{kj} : M_2(\mathbb{C}) \to L\left(\mathbb{C}^2\right), k, j \in \{1, 2\}$, *given by*

$$\varphi_{11}(a) = \varphi_{22}(a) = 2a; \varphi_{12}(a) = 2ia \text{ and } \varphi_{21}(a) = -2ia.$$

Clearly,

$$\Phi_k(a)^* \Phi_j(b) = \varphi_{kj}(a^*b)$$

for all $a, b \in M_2(\mathbb{C})$ *and for all* $k, j \in \{1, 2\}$. *Since*

$$\begin{aligned} \begin{bmatrix} \varphi_{11}(a) & \varphi_{12}(b) \\ \varphi_{21}(c) & \varphi_{22}(d) \end{bmatrix} &= \begin{bmatrix} 2a & 2ib \\ -2ic & 2d \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2}I_2 & 0 \\ 0 & -\sqrt{2}iI_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \sqrt{2}I_2 & 0 \\ 0 & \sqrt{2}iI_2 \end{bmatrix} \end{aligned}$$

for all $a, b, c, d \in M_2(\mathbb{C})$, *where* I_2 *denotes the unity of* $M_2(\mathbb{C})$ *and* $i \in \mathbb{C}$ *with* $i^2 = -1$, *the map* $[\varphi] = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ *is a completely 2-positive map from* $M_2(\mathbb{C})$ *to* $L\left(\mathbb{C}^2\right)$. *Therefore, the pair* (Φ_1, Φ_2) *is completely positive.*

Example 2.2. *Let* $X = M_2(\mathbb{C}), A = M_2(\mathbb{C})$. *The map* $\pi_X : M_2(\mathbb{C}) \to L(\mathbb{C}^4)$ *given by*

$$\pi_X(a) = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

is a representation of X, *since*

$$\pi_X(a)^* \pi_X(b) = \begin{bmatrix} a^*b & 0 \\ 0 & a^*b \end{bmatrix}$$

for all $a, b \in M_2(\mathbb{C})$ *and the map* $\pi_A : M_2(\mathbb{C}) \to L(\mathbb{C}^4)$ *given by*

$$\pi_A(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

is a representation of $M_2(\mathbb{C})$ *on* $\mathbb{C}^4$.

Let $V_1 = \begin{bmatrix} I_2 & I_2 \\ 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 & I_2 \\ I_2 & I_2 \end{bmatrix}, W_1 = \begin{bmatrix} 0 & 0 \\ I_2 & 0 \end{bmatrix}$ *and* $W_2 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}$.

For each $i, j \in \{1, 2\}$, *consider the map* $\varphi_{ij} : M_2(\mathbb{C}) \to L(\mathbb{C}^4)$ *given by*

$$\varphi_{ij}(a) = V_i^* \pi_A(a) V_j$$

and for each $i \in \{1, 2\}$, *consider the map* $\Phi_i : M_2(\mathbb{C}) \to L(\mathbb{C}^4)$ *given by*

$$\Phi_i(a) = W_i^* \pi_X(a) V_i.$$

Then $[\varphi] = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ *is a completely 2-positive linear map from* $M_2(\mathbb{C})$ *to* $L(\mathbb{C}^4)$ [4, Proposition 4.1.7] *and it is easy to check that*

$$V_i^* \pi_X(a)^* W_i W_j^* \pi_X(a) V_j = \varphi_{ij}(a^*b)$$

for all $a, b \in M_2(\mathbb{C})$ *and for all* $i, j \in \{1, 2\}$. *Therefore,* $\Phi_i(a)^* \Phi_j(b) = \varphi_{ij}(a^*b)$ *for all* $a, b \in M_2(\mathbb{C})$ *and for all* $i, j \in \{1, 2\}$, *and so the pair* (Φ_1, Φ_2) *is completely positive.*

In the general case, we have the following result.

Proposition 2.1. *Let* (X, A) *be a Hilbert* C^**-module, let* (π_X, π_A) *be a representation of* (X, A) *on* $\mathcal{H}$ *and* $\mathcal{K}$, $V_1, ..., V_n \in L(\mathcal{H})$ *and* $W_1, ..., W_n \in L(\mathcal{K})$ *such that*

$$V_i^* \pi_X(x)^* W_i W_j^* \pi_X(y) V_j = V_i^* \pi_A(\langle x, y\rangle) V_j$$

for all $x, y \in X$ and for all $i, j \in \{1, 2, ..., n\}$. Then, the n-tuple $(\Phi_1, \Phi_2, ..., \Phi_n)$ of linear maps from (X, A) to $L(\mathcal{H}, \mathcal{K})$, where

$$\Phi_i (x) = W_i^* \pi_X (x) V_i, x \in X, i \in \{1, ..., n\},$$

is completely positive.

Proof. For each $i, j \in \{1, 2, ..., n\}$, consider the linear map $\varphi_{ij} : A \to L(\mathcal{H})$ defined by

$$\varphi_{ij} (a) = V_i^* \pi_A (a) V_j.$$

Then $[\varphi] = [\varphi_{ij}]_{i,j=1}^n$ is a completely n-positive map from A to $L(\mathcal{H})$. Since

$$\begin{aligned} \Phi_i (x)^* \Phi_j (y) &= V_i^* \pi_X (x)^* W_i W_j^* \pi_X (y) V_j \\ &= V_i^* \pi_A (\langle x, y\rangle) V_j = \varphi_{ij} (\langle x, y\rangle) \end{aligned}$$

for all $x, y \in X$ and for all $i, j \in \{1, 2, ..., n\}$, the n-tuple $(\Phi_1, \Phi_2, ..., \Phi_n)$ is completely positive. □

In [7, Theorem 2.1], Pliev and Tsopanov shown that for an n-tuple $\Phi = (\Phi_1, \Phi_2, ..., \Phi_n)$ of completely positive maps from (X, A) to $L(\mathcal{H}, \mathcal{K})$, there are a representation (π_X, π_A) of (X, A) on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{K}_1$, $V_i \in L(\mathcal{H}, \mathcal{H}_1)$ and $W_i \in L(\mathcal{K}, \mathcal{K}_1)$, $i \in \{1, 2, ..., n\}$ such that

$$\Phi_i (x) = W_i^* \pi_X (x) V_i, i \in \{1, 2, ..., n\}.$$

Moreover, if $\mathcal{H}_1 = \overline{\text{Span}}\{\pi_A (a) V_i\xi, a \in A, \xi \in \mathcal{H}, i = 1, .., n\}$ and $\mathcal{K}_1 = \overline{\text{Span}}\{\pi_X (x) V_i\xi, x \in X, \xi \in \mathcal{H}, i = 1, .., n\}$,then the above writing is unique up to unitary equivalence [7, Theorem 2.2]. The $2n + 3$-tuple

$$((\pi_X, \pi_A), \mathcal{H}_1, \mathcal{K}_1, V_1, ..., V_n, W_1, ..., W_n)$$

is called the minimal Stinespring construction associated to the n-tuple $\Phi = (\Phi_1, \Phi_2, ..., \Phi_n)$ of completely positive maps from (X, A) to $L(\mathcal{H}, \mathcal{K})$.

3. n-tuples of covariant completely positive maps

Let (X, A) be a full Hilbert C^*-module, let G be a locally compact group G, let (γ, α) be an action of G on X, and let $(u, \mathcal{H})$ and $(v, \mathcal{K})$ be two unitary representations of G.

Definition 3.1. *An n-tuple $(\Phi_1, \Phi_2, ..., \Phi_n)$ of linear maps from (X, A) to $L(\mathcal{H}, \mathcal{K})$ is completely positive, (u, v)-covariant with respect to the action (γ, α) if there is a completely n-positive linear map $[\varphi] = [\varphi_{ij}]_{i,j=1}^n$ from A to $L(\mathcal{H})$ such that*

$$[\langle \Phi_i (x), \Phi_j (y)\rangle]_{i,j=1}^n = [\varphi_{ij} (\langle x, y\rangle)]_{i,j=1}^n$$

for all $x, y \in X$ and

$$\Phi_i (\gamma_g (x)) = v_g \Phi_i (x) u_g^*$$

for all $g \in G$, for all $x \in X$ and for all $i \in \{1, 2, ..., n\}$.

Remark 3.1. *From*

$$\begin{aligned} \varphi_{ij} (\alpha_g \langle x, y\rangle) &= \varphi_{ij} (\langle \gamma_g (x), \gamma_g (y)\rangle) = \langle \Phi_i (\gamma_g (x)), \Phi_j (\gamma_g (y))\rangle \\ &= \left\langle v_g \Phi_i (x) u_g^*, v_g \Phi_j (y) u_g^* \right\rangle = u_g \varphi_{ij} (\langle x, y\rangle) u_g^* \end{aligned}$$

for all $g \in G$, for all $x, y \in X$ and for all $i, j \in \{1, 2, ..., n\}$, we conclude that $[\varphi]$ is u -covariant with respect to α.

Proposition 3.1. *Let (X, A) be a full Hilbert C^*-module, let $((\pi_X, \pi_A), u', v', \mathcal{H}, \mathcal{K})$ be a representation of (X, A), covariant with respect to (γ, α), $V_1, ..., V_n \in L(\mathcal{H})$ and $W_1, ..., W_n \in L(\mathcal{K})$ such that*

(1) $u'_g V_i = V_i u_g$ *for all $g \in G$ and for all $i \in \{1, 2, ..., n\}$;*

(2) $v'_g W_i = W_i v_g$ *for all* $g \in G$ *and for all* $i \in \{1, 2, ..., n\}$;
(3) $V_i^* \pi_A (\langle x, y\rangle) V_j = V_i^* \pi_X (x)^* W_i W_j^* \pi_X (y) V_j$ *for all* $x, y \in X$, *and for all* $i, j \in \{1, 2, ..., n\}$.

Then, the n-tuple $(\Phi_1, \Phi_2, ..., \Phi_n)$ *of linear maps from* (X, A) *to* $L(\mathcal{H}, \mathcal{K})$, *where*

$$\Phi_i (x) = W_i^* \pi_X (x) V_i, x \in X, i \in \{1, ..., n\},$$

is completely positive, (u, v)*-covariant with respect the action* (γ, α).

Proof. By Proposition 2.1, $(\Phi_1, \Phi_2, ..., \Phi_n)$ is completely positive. For each $i \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} \Phi_i (\gamma_g (x)) &= W_i^* v'_g \pi_X (x) \left(u'_g\right)^* V_i \\ &= v_g W_i^* \pi_X (x) V_i u_g^* = v_g \Phi_i ((x)) u_g^* \end{aligned}$$

for all $g \in G$ and for all $x \in X$, and so $(\Phi_1, \Phi_2, ..., \Phi_n)$ is (u, v)-covariant with respect the action (γ, α). □

Theorem 3.1. *Let* $\Phi = (\Phi_1, \Phi_2, ..., \Phi_n)$ *be an n-tuple of completely positive maps from* (X, A) *to* $L(\mathcal{H}, \mathcal{K})$, (u, v)*-covariant with respect to the action* (γ, α). *Then there is a covariant representation* $\left(\left(\pi_\Phi, \pi_{[\varphi]}\right), v^\Phi, w^\Phi, \mathcal{H}_\Phi, \mathcal{K}_\Phi\right)$ *of* (X, A) *on the Hilbert spaces* $\mathcal{H}_\Phi$ *and* $\mathcal{K}_\Phi$ *and there are the bounded linear operators* $V_1^\Phi, ..., V_n^\Phi \in L(\mathcal{H}, \mathcal{H}_\Phi)$ *and* $W_1^\Phi, ..., W_n^\Phi \in L(\mathcal{K}, \mathcal{K}_\Phi)$ *such that:*

(1) $\varphi_{ij} (a) = \left(V_i^\Phi\right)^* \pi_{[\varphi]} (a) V_i^\Phi$*for all* $a \in A$ *and for all* $i, j \in \{1, 2, ..., n\}$;
(2) $\Phi_i (x) = \left(W_i^\Phi\right)^* \pi_\Phi (x) V_i^\Phi$ *for all* $x \in X$ *and for all* $i \in \{1, 2, ..., n\}$;
(3) $v_g^\Phi V_i^\Phi = V_i^\Phi u_g$ *for all* $g \in G$ *and for all* $i \in \{1, 2, ..., n\}$;
(4) $w_g^\Phi W_i^\Phi = W_i^\Phi v_g$ *for all* $g \in G$ *and for all* $i \in \{1, 2, ..., n\}$;
(5) $\overline{Span} \{\pi_{[\varphi]} (a) V_i^\Phi \xi; a \in A, \xi \in \mathcal{H}, i = 1, 2, ..., n\} = \mathcal{H}_\Phi$;
(6) $\overline{Span} \{\pi_\Phi (x) V_i^\Phi \xi; x \in X, \xi \in \mathcal{H}, i = 1, 2, ..., n\} = \mathcal{K}_\Phi$.

Proof. Let $(\pi_{[\varphi]}, v^{[\varphi]}, \mathcal{H}_1, V_1, ..., V_n)$ be the covariant Stinespring construction associated to $[\varphi]$ (see, for example, [4, Theorem 6.1.3]). Recall that $(A \otimes_{\text{alg}} \mathcal{H})^n / \mathcal{N}$, where

$$\mathcal{N} = \text{ Span}\{(a_i \otimes \xi_i)_{i=1}^n ; \sum_{i,j=1}^n \langle \xi_i, \varphi_{ij} (a_i^* a_j) \xi_j\rangle = 0\},$$

is a pre-Hilbert space with the inner product given by

$$\langle (a_i \otimes \xi_i)_{i=1}^n + \mathcal{N}, (b_i \otimes \eta_i)_{i=1}^n + \mathcal{N}\rangle = \sum_{i,j=1}^n \langle \xi_i, \varphi_{ij} (a_i^* b_j) \eta_j\rangle$$

and $\mathcal{H}_1$ is its completion, the representation $\pi_{[\varphi]}$ is given by

$$\pi_{[\varphi]} (a) ((a_i \otimes \xi_i)_{i=1}^n + \mathcal{N}) = (a a_i \otimes \xi_i)_{i=1}^n + \mathcal{N}$$

$V_j \xi = \lim\limits_{\rightarrow} \xi_{j,\lambda}$, where $\xi_{j,\lambda} = (e_\lambda \otimes \delta_{ij} \xi)_{i=1}^n + \mathcal{N}$ and $\{e_\lambda\}_\lambda$ is an approximate unit of A and $v^{[\varphi]}$ is a unitary representation of G on $\mathcal{H}_1$ given by $v_g^{[\varphi]} ((a_i \otimes \xi_i)_{i=1}^n + \mathcal{N}) = (\alpha_g (a_i) \otimes u_g (\xi_i))_{i=1}^n + \mathcal{N}$. Moreover, $\mathcal{H}_1$ is generated by $\{\pi_{[\varphi]}(a) V_i \xi; a \in A, \xi \in \mathcal{H}, i = 1, 2, ..., n\}$ and $v_g^{[\varphi]} V_i = V_i u_g$ for all $g \in G$ and for all $i \in \{1, 2, ..., n\}$.

Let $\mathcal{K}_1$ be the vector subspace of $\mathcal{K}$ generated by $\{\Phi_i(x)\xi; x \in X, \xi \in \mathcal{H}, i = 1, 2, ..., n\}$. From

$$v_g\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\Phi_i\left(x_{ij}\right)\xi_{ij}\right) = \sum_{j=1}^{m}\sum_{i=1}^{n}v_g\Phi_i\left(x_{ij}\right)\xi_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n}\Phi_i\left(\gamma_g\left(x_{ij}\right)\right)u_g\xi_{ij}$$

for all $g \in G$, we deduce that $\mathcal{K}_1$ is invariant under v. Then $w^\Phi = v|_{\mathcal{K}_1}$ is a unitary representation of G on $\mathcal{K}_1$.

For each $x \in X$, the linear map $\pi_\Phi(x)$ given by

$$\pi_\Phi(x)\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{[\varphi]}\left(a_{ij}\right)V_i\xi_{ij}\right) = \sum_{j=1}^{m}\sum_{i=1}^{n}\Phi_i\left(xa_{ij}\right)\xi_{ij}$$

defines a bounded linear map from $\mathcal{H}_1$ to $\mathcal{K}_1$ [7, Theoem 2.1]. Thus, we obtain a map $\pi_\Phi : X \to L(\mathcal{H}_1, \mathcal{K}_1)$. Moreover, since $\pi_\Phi(x)^*\pi_\Phi(y) = \pi_{[\varphi]}(\langle x, y\rangle)$ for all $x, y \in X$, and since

$$\begin{aligned}
& w_g^\Phi\pi_\Phi(x)\left(v_g^{[\varphi]}\right)^*\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{[\varphi]}\left(a_{ij}\right)V_i\xi_{ij}\right)\\
= {} & w_g^\Phi\pi_\Phi(x)\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{[\varphi]}\left(\alpha_{g^{-1}}a_{ij}\right)\left(v_g^{[\varphi]}\right)^*V_i\xi_{ij}\right)\\
= {} & w_g^\Phi\pi_\Phi(x)\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{[\varphi]}\left(\alpha_{g^{-1}}\left(a_{ij}\right)\right)V_iu_g^*\xi_{ij}\right)\\
= {} & w_g^\Phi\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\Phi_i\left(x\alpha_{g^{-1}}\left(a_{ij}\right)\right)u_g^*\xi_{ij}\right)\\
= {} & \sum_{j=1}^{m}\sum_{i=1}^{n}v_g\Phi_i\left(x\alpha_{g^{-1}}\left(a_{ij}\right)\right)u_g^*\xi_{ij}\\
= {} & \sum_{j=1}^{m}\sum_{i=1}^{n}\Phi_i\left(\gamma_g\left(x\alpha_{g^{-1}}\left(a_{ij}\right)\right)\right)\xi_{ij}\\
= {} & \sum_{j=1}^{m}\sum_{i=1}^{n}\Phi_i\left(\gamma_g(x)a_{ij}\right)\xi_{ij}\\
= {} & \pi_\Phi\left(\gamma_g(x)\right)\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{[\varphi]}\left(a_{ij}\right)V_i\xi_{ij}\right),
\end{aligned}$$

$\left(\left(\pi_\Phi, \pi_{[\varphi]}\right), v^{[\varphi]}, w^\Phi, \mathcal{H}_1, \mathcal{K}_1\right)$ is a representation of X, covariant with respect to (γ, α).

For each $i \in \{1, 2, ..., n\}$, W_i^Φ denotes the orthogonal projection of $\mathcal{K}$ on the vector subspace generated by $\{\Phi_i(x)\xi; x \in X, \xi \in \mathcal{H}\}$. Moreover, since $\overline{\text{Span}}\{\Phi_i(x)\xi; x \in X, \xi \in \mathcal{H}\}$ is invariant under the unitary representation v, $w_g^\Phi W_i^\Phi = v_gW_i^\Phi = W_i^\Phi v_g$ for all $g \in G$ and for all $i \in \{1, 2, ..., n\}$.

Let $\mathcal{H}_\Phi = \mathcal{H}_1$, $\mathcal{K}_\Phi = \mathcal{K}_1$, $v^\Phi = v^{[\varphi]}$ and $V_i^\Phi = V_i$, $i \in \{1, 2, ..., n\}$. Then $((\pi_\Phi, \pi_{[\varphi]}),$ $v^\Phi, w^\Phi, \mathcal{H}_\Phi, \mathcal{K}_\Phi)$ is a representation of (X, A), covariant with respect to (γ, α) such that

$v_g^{\Phi}V_i^{\Phi} = V_i^{\Phi}u_g$ and $w_g^{\Phi}W_i^{\Phi} = W_i^{\Phi}v_g$ for all $g \in G$ and for all $i \in \{1,2,...,n\}$. Clearly, $\overline{\text{Span}}\ \{\pi_{\Phi}(x)V_i\xi; x \in X, \xi \in \mathcal{H}, i = 1,2,...,n\} = \mathcal{K}_{\Phi}$, $\overline{\text{Span}}\{\pi_{[\varphi]}(a)V_i\xi; a \in A, \xi \in \mathcal{H}, i = 1,2,...,n\} = \mathcal{H}_{\Phi}$ and

$$\varphi_{ij}(a) = \left(V_i^{\Phi}\right)^* \pi_{[\varphi]}(a)\, V_i^{\Phi}$$

for all $a \in A$ and for all $i, j \in \{1,2,...,n\}$. It is easy to check that

$$\Phi_i(x) = \left(W_i^{\Phi}\right)^* \pi_{\Phi}(x)\, V_i^{\Phi}$$

for all $x \in X$ and for all $i \in \{1,2,...,n\}$. □

The $2n+5$-tuple $\left(\left(\pi_{\Phi}, \pi_{[\varphi]}\right), v^{\Phi},\ w^{\Phi}, \mathcal{H}_{\Phi}, \mathcal{K}_{\Phi}, V_1^{\Phi}, ..., V_n^{\Phi}, W_1^{\Phi}, ..., W_n^{\Phi}\right)$ constructed in the above theorem is called the minimal covariant Stinespring construction associated to the n-tuple $\Phi = (\Phi_1, \Phi_2, ..., \Phi_n)$ of completely positive maps from (X, A) to $L(\mathcal{H}, \mathcal{K})$, (u, v)-covariant with respect to the action (γ, α).

Theorem 3.2. *Let $\Phi = (\Phi_1, \Phi_2, ..., \Phi_n)$ be an n-tuple of completely positive maps from (X, A) to $L(\mathcal{H}, \mathcal{K})$, (u, v)-covariant with respect to the action (γ, α). If $((\pi_X, \pi_A), v', w', \mathcal{H}', \mathcal{K}')$ is a representation of (X, A), covariant with respect to (γ, α), $V_1, ..., V_n$ are elements in $L(\mathcal{H}, \mathcal{H}')$, $W_1, ..., W_n$ are elements in $L(\mathcal{K}, \mathcal{K}')$ and the following relations hold:*

1. $\varphi_{ij}(a) = V_i^*\pi_A(a)\, V_i$ *for all* $a \in A$ *and for all* $i, j \in \{1,2,...,n\}$;
2. $\Phi_i(x) = W_i^*\ \pi_X(x)\, V_i$ *for all* $x \in X$ *and for all* $i \in \{1,2,...,n\}$;
3. $v'_gV_i = V_iu_g$ *for all* $g \in G$ *and for all* $i \in \{1,2,...,n\}$;
4. $w'_gW_i = W_iv_g$ *for all* $g \in G$ *and for all* $i \in \{1,2,...,n\}$;
5. $\overline{\textit{Span}}\ \{\pi_A(a)\, V_i\xi; a \in A, \xi \in \mathcal{H}, i = 1,2,...,n\} = \mathcal{H}'$;
6. $\overline{\textit{Span}}\ \{\pi_X(x)\, V_i\xi; x \in X, \xi \in \mathcal{H}, i = 1,2,...,n\} = \mathcal{K}'$,
   then there are two unitary operators $U_1 \in L(\mathcal{H}_{\Phi}, \mathcal{H}')$ *and* $U_2 \in L(\mathcal{K}_{\Phi}, \mathcal{K}')$ *such that:*
   (a) $U_2\pi_{\Phi}(x) = \pi_X(x)\, U_1$ *for all* $x \in X$,
   (b) $U_1\pi_{[\varphi]}(a) = \pi_A(a)U_1$ *for all* $a \in A$,
   (c) $v'_gU_1 = U_1v_g^{\Phi}$, $w'_gU_2 = U_2w_g^{\Phi}$ *for all* $g \in G$,
   (d) $V_i = U_1V_i^{\Phi}$ *and* $W_i = U_2W_i^{\Phi}$ *for all* $i \in \{1,2,...,n\}$.

Proof. Since $(\pi_A, v', \mathcal{H}', V_1, ..., V_n)$ is unitarily equivalent to the covariant Stinespring construction associated to $[\varphi]$ (see, for example, [4, Theorem 4.1.8]), there is a unitary operator $U_1 \in L(\mathcal{H}_{\Phi}, \mathcal{H}')$ such that $U_1\pi_{[\varphi]}(a) = \pi_A(a)U_1$ for all $a \in A$, $V_i = U_1V_i^{\Phi}$, $v'_gU_1 = U_1v_g^{\Phi}$ for all $g \in G$, and for all $i \in \{1,2,...,n\}$. Moreover,

$$U_1\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{[\varphi]}(a_{ij})\, V_i^{\Phi}\xi_{ij}\right) = \sum_{j=1}^{m}\sum_{i=1}^{n}\pi_A(a_{ij})\, V_i\xi_{ij}.$$

Since $((\pi_X, \pi_A), w', v', \mathcal{H}', V_1, ..., V_n, W_1, ..., W_n)$ is unitarily equivalent to the Stinespring construction associated to Φ [7, Theorem 2.2], there is a unitary operator $U_2 : \mathcal{K}_{\Phi} \to \mathcal{K}'$ such that

$$U_2\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\pi_{\Phi}(x_{ij})\, V_i^{\Phi}\xi_{ij}\right) = \sum_{j=1}^{m}\sum_{i=1}^{n}\pi_X(x_{ij})\, V_i\xi_{ij}$$

for all $x_{ij} \in X$ and for all $\xi_{ij} \in \mathcal{H}, i \in \{1,2,...,n\}, j \in \{1,2,...,m\}$. Moreover, $U_2\pi_\Phi(x) = \pi_X(x)U_1$ for all $x \in X$ and $W_i' = U_2 W_i^\Phi$ for all $i \in \{1,2,...,n\}$. From

$$\begin{aligned} & w_g' U_2 \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \pi_\Phi(x_{ij}) V_i^\Phi \xi_{ij} \right) \\ = & \; w_g' \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \pi_X(x_{ij}) V_i \xi_{ij} \right) \\ = & \sum_{j=1}^{m} \sum_{i=1}^{n} \pi_X(\gamma_g(x_{ij})) v_g' V_i \xi_{ij} \\ = & \sum_{j=1}^{m} \sum_{i=1}^{n} \pi_X(\gamma_g(x_{ij})) V_i u_g \xi_{ij} \\ = & \; U_2 \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \pi_\Phi(\gamma_g(x_{ij})) V_i^\Phi u_g \xi_{ij} \right) \\ = & \; U_2 \left(\sum_{j=1}^{m} \sum_{i=1}^{n} w_g^\Phi \pi_\Phi(x_{ij}) V_i^\Phi \xi_{ij} \right) \\ = & \; U_2 w_g^\Phi \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \pi_\Phi(x_{ij}) V_i^\Phi \xi_{ij} \right) \end{aligned}$$

we deduce that $w_g' U_2 = U_2 w_g^\Phi$ for all $g \in G$. □

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