U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 1, 2016 ISSN 1223-7027

BEURLING AND MATRIX ALGEBRAS, (APROXIMATE) CONNES-AMENABILITY

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We characterize the approximate Connes-amenable Beurling algebras $\ell^1(S,\omega)$ through the existence of some specified nets in $\ell^\infty(S\times S)^*$, where S is a discrete, weakly cancellative semigroup. For a discrete group G, we prove that approximate Connes-amenability and approximate amenability of $\ell^1(G,\omega)$ are the same. We show that Connes-amenability of a dual Banach algebra A and that of $M_n(A)$ are equivalent.

2010 Mathematics Subject Classification. Primary: 22D15, 43A10, Secondary: 43A20, 46H25,47B47.

Keywords: Connes-amenability, approximate amenability, Beurling algebras, approximate σwc -diagonal.

1. Introduction

In [5], F. Ghahramani and R.J. Loy introduced the notion of approximate amenability for Banach algebras which modifies Johnson's original definition of amenability [7] by relaxing the structure of the derivations. Another modification of the concept of amenability was introduced by V. Runde in [10], where it had been studied previously under different names (see for instance [6,8]), that make sense for dual Banach algebras. We recall the definitions in Definitions 1.1 and 1.3 below. Before proceeding further we recall some terminology.

Let A be a Banach algebra. The projective tensor product $A\hat{\otimes}A$ is a Banach A-bimodule under the operations defined by

$$a.(x\otimes y):=ax\otimes y,\ (x\otimes y).a:=x\otimes ya\quad (a,x,y\in A),$$

and there is a continuous linear A-bimodule homomorphism $\pi: A\hat{\otimes}A\to A$ such that $\pi(a\otimes b)=\text{ab}$, for $a,b,\in A$. Throughout, we use the term *unital* for a semigroup (or an algebra) X with an identity element e_X, if it exists. Let E be a Banach space. The dual of E is denoted by E^*. In the case where E is a Banach A-bimodule, E^* is also a Banach A-bimodule. We then have the canonical map $\iota_E: E\to E^{**}$ defined by $\langle \mu,\iota_E(x)\rangle=\langle x,\mu\rangle$ for $\mu\in E^*$, $x\in E$. The closed unit ball of E is denoted by $ball\,E$. For Banach spaces E and F, we write

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$L(E,F)$ for the Banach space of bounded linear maps between E and F. It is standard that $\left(E\hat{\otimes}F\right)^{*}=L\left(F,E^{*}\right)$ with the duality

$$\langle x\otimes y,T\rangle=\langle x,Ty\rangle \ ; x\otimes y\in E\hat{\otimes}F, T\in L\left(F,E^{*}\right).$$

For a Banach algebra A, then we obtain a bimodule structure on $L(A,A^{*})=(A\hat{\otimes}A)^{*}$ through

$$(a.T)(b)=T(ba),(T.a)(b)=T(b).a\,(a,b,\in A,T\in L(A,A^{*})).$$

The reader may see [1] for more information.

Let A be a Banach algebra and let E be a Banach A-bimodule. A *derivation* is a bounded linear map $D:A\to E$ satisfying

$$D(ab)=Da.b+a.Db \ \ (a,b\in A).$$

For $x\in E$, set $ad_x:A\to E, a\to a.x-x.a$. Then ad_x is a derivation; these are the *inner* derivations. A derivation $D:A\to E$ is *approximately inner* if there exists a net $(x_\alpha)_\alpha\subseteq E$ such that $Da=\lim_\alpha(a.x_\alpha-x_\alpha.a)$ for every $a\in A$, the limit being in norm.

Definition 1.1 A Banach algebra A is *approximately amenable* if for each Banach A-bimodule E, every derivation $D:A\to E^{*}$ is a approximately inner.

For unital Banach algebras we may re-write $[5, \mathit{Theorem}\ 2.1]$ as follows.

Theorem 1.2 Let A be a unital Banach algebra. Then the following are equivalent:

(i) A is approximately amenable .

(ii) There is a net $(M_\alpha)_\alpha\subseteq\left(A\hat{\otimes}A\right)^{**}$ such that for every $a\in A$, $a.M_\alpha-M_\alpha.a\to 0$ and $\pi^{**}(M_\alpha)\to e_A$.

(iii) There is a net $(M'_\alpha)_\alpha\subseteq\left(A\hat{\otimes}A\right)^{**}$ such that for every $a\in A$, $a.M'_\alpha-M'_\alpha.a\to 0$ and $\pi^{**}(M'_\alpha)=e_A$.

Let A be a Banach algebra. A Banach A-bimodule E is *dual* if there is a closed submodule E_* of E^{*} such that $E=(E_*)^{*}$. We call E_* the *predual* of E. A dual Banach A-bimodule E is *normal* if the module actions of A on E are $w^{*}-$continuous. A Banach algebra A is *dual* if it is dual as a Banach A-bimodule. We write $A=(A_*)^{*}$ if we wish to stress that A is a dual Banach algebra with predual A_*.

Definition 1.3 A dual Banach algebra A is *Connes-amenable* if every w^*-continuous derivation from A into a normal dual Banach A-bimodule is inner.

The reader is referred to $[11]$ for basic properties of Connes-amenable dual algebras. Let $A=(A_*)^*$ be a dual Banach algebra and let E be a Banach A-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$A \to E, a \to \begin{cases} a\,.\;\; x \\ x\;\; .\, a \end{cases},$$

are w^*-weak continuous. The space $\sigma wc(E)$ is a closed submodule of E. It is shown in $[12, Corollary\ 4.6]$ that $\pi^*(A_*) \subseteq \sigma wc\left(A \hat{\otimes} A\right)$. Taking adjoint, we can extend π to an A-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc\left(\left(A \hat{\otimes} A\right)^*\right)^*$ to A. A σwc-*virtual diagonal* for a dual Banach algebra A is an element $U \in \sigma wc((A \hat{\otimes} A)^*)^*$ such that $a.U = U.a$ and $a\pi_{\sigma wc}(U) = a$ for $a \in A$. From $[12]$ we know that Connes-amenability of a dual Banach algebra A is equivalent to existence of a $\sigma wc-$virtual diagonal for A.

The concept of approximate Connes-amenability for dual Banach algebras, motivated by Definitions 1.1 and 1.3 was introduced and studied in $[4]$, see also $[9]$.

Definition 1.4 A dual Banach algebra A is *approximately Connes–amenable* if for each normal dual Banach A-bimodule E, every $w^*-continous$ derivation $D: A \to E$ is approximately inner.

We state the following, which is a combination of $[4, Propositions\ 2.3\ and\ 3.3]$.

Proposition 1.5 Let A be a unital dual Banach algebra. Then the following are equivalent:

(i) A is approximately Connes-amenable.

(ii) There is a net $(M_\alpha)_\alpha \subseteq \sigma wc\left(\left(A \hat{\otimes} A\right)^*\right)^*$ such that

$$a.M_\alpha - M_\alpha.a \to 0 \text{ and } \pi_{\sigma wc} M_\alpha \to e_A\, (a \in A).$$

(iii) There is a net $(M'_\alpha)_\alpha \subseteq \sigma wc\left(\left(A \hat{\otimes} A\right)^*\right)^*$ such that

$$a.M'_\alpha - M'_\alpha.a \to 0 \text{ and } \pi_{\sigma wc} M'_\alpha = e_A\ (a \in A).$$

In section 2, we briefly extend the Daws's result to the approximate case; M. Daws proved that Connes-amenability and amenability are the same notion for a Beurling algebra $\ell^1(G,w)$, where G is a discrete group $[3]$. For a discrete weakly cancellative semigroup S, we show that the approximate Connes-amenability of

Beurling algebra $\ell^1(S,w)$ is equivalent to existence of a net in $\ell^\infty(S\times S)^*$ which is an object analogous to $\sigma wc-$virtual diagonal for Connes-amenability.

In section 3, we first consider a kind of diagonal for a dual Banach algebra A and see that the existence of such a diagonal is equivalent to Connes-amenability of A. Then we study Connes-amenability of $M_n(A)$ with predual $M_n(A_*)$, where A_* is the predual of A. We show that $M_n(A)$ is Connes-amenable if and only if A is Connes-amenable. For comparison, we recall $[2, \textit{Theorem } 2.7]$ that a Banach algebra A is amenable if and only if $M_n(A)$ is amenable.

2. Approximate Connes-amenability of weighted semigroup algebras

Let S be a discrete semigroup. A function $\omega: S\to(0,\infty)$ is a *weight* if $\omega(st)\le\omega(s)\omega(t)$ for each $s,t\in S$. If S is unital then, without loss of generality, we put $\omega(e_S)=1$. The Banach space

$$\ell^1(S,\omega)=\left\{(a_g)_{g\in S}\subseteq C:\left\|(a_g)_{g\in S}\right\|=\sum_{g\in S}|a_g|\omega(g)<\infty\right\},$$

with the convolution product is a Banach algebra, called a *Beurling algebra*. Following $[3]$ we consider $\ell^1(S,\omega)$ as the Banach space $\ell^1(S)$ with the product $\delta_g *_\omega \delta_h := \delta_{gh}\Omega(g,h)$, where $\Omega(g,h):=\frac{\omega(gh)}{\omega(g)\omega(h)}(g,h\in S)$ and extend $*_\omega$ to $\ell^1(S)$ by linearity and continuity. We define the maps $L_s, R_s: S\to S$ by $L_s(t)=st$ and $R_s(t)=ts$. A semigroup S is *weakly cancellative* if for each $s\in S$, the maps L_s and R_s are finite-to-one. In this case $\ell^1(S,\omega)$ is a dual Banach algebra with predual $c_0(S)$, [3, Proposition 5.1].

Proposition 2.1 Let A be a unital dual Banach algebra. Then the following are equivalent:

(i) A is approximately Connes- amenable.

(ii) There is a net $(M_\alpha)_\alpha\subseteq(A\hat{\otimes}A)^{**}$ such that $\langle T, a.M_\alpha - M_\alpha.a\rangle\to 0$ for every $a\in A$ and uniformly for all $T\in \text{ball}\,\sigma wc(L(A,A^*))$, and $\iota_{A_*}{}^*\pi^{**}M_\alpha\to e_A$.

(iii) There is a net $(M'_\alpha)_\alpha\subseteq(A\hat{\otimes}A)^{**}$ such that

$\langle T, a.M'_\alpha - M'_\alpha.a\rangle \to 0$ for every $a \in A$ and uniformly for all $T \in \text{ball}\, \sigma wc(L(A, A^*))$, and $\iota_{A_*}{}^{*}\pi^{**}M'_\alpha = e_A$.

Proof. As $\sigma wc((A \hat{\otimes} A)^*)^*$ is a quotient of $(A \hat{\otimes} A)^{**}$, this is just a re-statement of Proposition 1.5.

For a set S, we recall that $\ell^1(S) \hat{\otimes} \ell^1(S) = \ell^1(S \times S)$, where $\delta_g \otimes \delta_h$ is identified with $\delta_{(g,h)}$ for $g, h \in S$. Thus we have $L(\ell^1(S), \ell^\infty(S)) = (\ell^1(S) \hat{\otimes} \ell^1(S))^* = \ell^1(S \times S)^* = \ell^\infty(S \times S)$, where $T \in L(\ell^1(S), \ell^\infty(S))$ is identified with $(T_{(g,h)})_{(g,h) \in S \times S} \in \ell^\infty(S \times S)$, where $T_{(g,h)} = \langle \delta_h, T(\delta_g)\rangle$.

Theorem 2.2 Let S be a discrete unital semigroup, let ω be a weight on S and let $A := \ell^1(S, \omega)$. Then the following are equivalent:

(i) A is approximately amenable.

(ii) There is a net $(M_\alpha)_\alpha \subseteq (A \hat{\otimes} A)^{**} = \ell^\infty(S \times S)^*$ such that

$$\langle (f(hk, g)\Omega(h,k) - f(h, kg)\Omega(k, g))_{(g,h) \in S \times S}, M_\alpha \rangle \to 0$$

for every $k \in S$, where the convergence is uniformly for all $f \in \text{ball}\ \ell^\infty(S \times S)$, and

$$\langle (f_{gh}\Omega(g,h))_{(g,h) \in S \times S}, M'_\alpha \rangle \to f_{e_S}$$

uniformly for all $f \in$ ball $\ell^\infty(S)$.

(iii) There is a net $(M'_\alpha)_\alpha \subseteq (A \hat{\otimes} A)^{**} = \ell^\infty(S \times S)^*$ such that

$$\langle (f(hk, g)\Omega(h,k) - f(h, kg)\Omega(k, g))_{(g,h) \in S \times S}, M'_\alpha \rangle \to 0$$

for every $k \in S$, where the convergence is uniformly for all $f \in \text{ball}\ \ell^\infty(S \times S)$, and

$$\langle (f_{gh}\Omega(g,h))_{(g,h) \in S \times S}, M'_\alpha \rangle = f_{e_S}$$

for all $f \in \ell^\infty(S)$.

Proof. First, we notice that for every $f = (f_g)_{g \in S} \in \ell^\infty(S)$

$$\pi^*(f) = (\langle \delta_{gh}, f \rangle \Omega(g,h))_{(g,h) \in S \times S} \in \ell^\infty(S \times S).$$

Next, for every $T \in L(A, A^*) = \ell^\infty(S \times S)$ and every $k \in S$ we have

$$\langle \delta_g \otimes \delta_h, \delta_k.T - T.\delta_k \rangle = \langle \delta_g, T(\delta_{hk}) \rangle \Omega(h,k) - \langle \delta_{kg}, T(\delta_h) \rangle \Omega(k, g).$$

We also observe that $e_A = \delta_{e_S}$ and therefore $\langle f, e_A \rangle = f_{e_S}$.

(i)$\to$(ii) We use Theorem 1.2. Suppose that A is approximately amenable and take the net $(M_\alpha)_\alpha \subseteq \ell^\infty(S\times S)^*$ as in Theorem 1.2 (ii). For every $f \in$ball $\ell^\infty(S\times S)^*$, then we have

$$\left|\left\langle \left(f_{gh}\Omega(g,h)\right)_{(g,h)\in S\times S}, M_\alpha \right\rangle - f_{e_S}\right| = \left|\left\langle f, \pi^{**}(M_\alpha) - e_A \right\rangle\right| \le \left\|\pi^{**}(M_\alpha) - e_A\right\|.$$

Take $f \in ball\ \ell^\infty(S\times S), k \in S$ and consider $T \in L(A, A^*) = \ell^\infty(S\times S)$ defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$. Then we see that

$$\left|\left\langle \left(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g)\right)_{(g,h)\in S\times S}, M_\alpha \right\rangle\right| \le \left\|\delta_k . M_\alpha - M_\alpha . \delta_k\right\|.$$

Hence, all in all, we have condition (ii).

Similarly, we may prove the implications (ii)$\to$(i) and (i)$\leftrightarrow$(iii).

The following is [3, Proposition 5.5].

Proposition 2.3 Let S be a weakly cancellative semigroup, let ω be a weight on S and let $A = \ell^1(S,\omega)$. Let $T \in L(A, A^*)$ be such that $T(A) \subseteq \iota_{c_0(S)}(c_0(S))$ and $T^*(\iota_A(A)) \subseteq \iota_{c_0(S)}(c_0(S))$. Then $T \in W(A, A^*)$ and $T \in WAP(W(A, A^*))$ if and only if, for each sequence (k_n) of distinct elements of S, and each sequence (g_m, h_m) of distinct elements of $S\times S$ such that the repeated limits

$$\lim_n \lim_m \left\langle \delta_{k_n g_m}, T(\delta_{h_m})\right\rangle, \ \lim_n \lim_m \Omega(k_n, g_m)$$

$$\lim_n \lim_m \left\langle \delta_{g_m}, T(\delta_{h_m} k_n)\right\rangle, \ \lim_n \lim_m \Omega(h_m, k_n)$$

all exist, we have at least one repeated limit in each row is zero.

Proposition 2.4 Let S be a discrete, weakly cancellative semigroup, let ω be a weight on S and let $A = \ell^1(S,\omega)$ be unital. Then the following are equivalent:

(i) A is approximately Connes-amenable, with respect to the predual $c_0(S)$.

(ii) There is a net $(M_\alpha)_\alpha \subseteq (A \hat{\otimes} A)^{**} = \ell^\infty(S\times S)^*$ such that

$$\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M_\alpha \right\rangle \to 0$$

for each $k \in$S and uniformly for all $f \in$ball $\ell^\infty(S\times S)$,which are such that the maps $T \in L(A, A^*)$ defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$, for $g,h \in S$, satisfy the conclusions of proposition 2.3, and

$$\left\langle \left(f_{gh}\Omega(g,h)\right)_{(g,h)\in S\times S}, M_\alpha \right\rangle \to \langle f, e_A \rangle$$

uniformly for all $f \in$ ball $c_0(S)$.

(iii) There is a net $(M'_\alpha)_\alpha \subseteq (A \widehat{\otimes} A)^{**} = \ell^\infty(S\times S)^*$ such that

$$\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M'_\alpha) \right\rangle \to 0$$

for each $k \in$ S and uniformly for all $f \in$ ball $\ell^\infty(S\times S)$, which are such that the maps $T \in L(A, A^*)$ defined by $\left\langle \delta_h, T(\delta_g) \right\rangle = f(g,h)$, for $g,h \in S$, satisfy the conclusions of proposition 2.3, and

$$\left\langle \left(f_{gh}\Omega(g,h)\right)_{(g,h)\in S\times S}, M'_\alpha \right\rangle = \langle f, e_A \rangle$$

for all $f \in c_0(S)$.

Proof. This follows as Theorem 2.2 but by using Proposition 2.1 in place of Theorem 1.2.

Let G be a discrete group and let $h \in G$. Following Daws as in [3], we define $J_h : \ell^\infty(G) \to \ell^\infty(G)$ by

$$J_h(f) := \left(f_{hg}\Omega(h,g)\omega(h)\Omega(g^{-1},h^{-1})\omega(h^{-1})\right)_{g\in G}; \ \left(f = \left(f_g\right)_g \in \ell^\infty(G)\right).$$

It is clear that $\|J_h(f)\| \le \omega(h)\omega(h^{-1})$, so that J_h is bounded.

Theorem 2.5 Let G be a discrete group, let ω be a weight on G and let $A = \ell^1(G,\omega)$. Then the following are equivalent:

(i) A is approximately Connes-–amenable, with respect to the predual $c_0(G)$.

(ii) A is approximately amenable .

(iii) There is a net $(N_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ such that for every $k \in G$, $J_K{}^*(N_\alpha) - N_\alpha \to 0$ and $\left\langle \left(\Omega(g,g^{-1})\right)_{g\in G}, N_\alpha \right\rangle \to 1$.

(iv) There is a net $(N'_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ such that for every $k \in G$, $J_K{}^*(N'_\alpha) - N'_\alpha \to 0$ and $\left\langle \left(\Omega(g,g^{-1})\right)_{g\in G}, N_\alpha \right\rangle = 1$.

Proof. The implications (ii)$\Rightarrow$(i) and (iv)$\Rightarrow$(iii) are clear .

(i)$\Rightarrow$(iv) Let the net $(M'_\alpha)_\alpha \subseteq \ell^\infty(G\times G)^*$ be given as in Proposition 2.4 (iii). Define $\phi : \ell^\infty(G) \to \ell^\infty(G\times G)$ by

$$\left\langle \delta_{(g,h)}, \phi(f) \right\rangle := \begin{cases} f_g & , g = h^{-1} \\ 0 & , g \ne h^{-1} \end{cases}.$$

Let $N'_\alpha := \phi^*(M'_\alpha)$. Then we have $\phi\left(\left(\Omega\left(g,g^{-1}\right)\right)_{g\in G}\right) = \left(\delta_{gh,e_G}\Omega(g,h)\right)_{(g,h)\in G\times G}$. Hence

$$\left\langle\left(\Omega\left(g,g^{-1}\right)\right)_{g\in G}, N'_\alpha\right\rangle = \left\langle\left(\delta_{gh,e_G}\Omega(g,h)\right)_{(g,h)\in G\times G}, M'_\alpha\right\rangle = \left\langle\left(\delta_{gh,e_G}\right)_{(g,h)\in G\times G}, \delta_{e_G}\right\rangle = \delta_{e_G,e_G} = 1,$$

by the second condition on $(M'_\alpha)_\alpha$ from Proposition 2.4 (iii).

Fix $k\in G$ and $f\in\ell^\infty(G)$. Define $F: G\times G\to C$ by

$$F(g,h) := \delta_{gh,k}\, f_g\, \omega(k)\,\omega(hk^{-1})\,\omega(h)^{-1}\ (g,h\in G).$$

It is clear that F is bounded and $\|F\|_\infty \le \|f\|_\infty\,\omega(k)\omega\left(k^{-1}\right)$. Let T be the operator associated with F. The same argument as in the proof [3, Theorem 5.11] shows that F satisfies the conditions of Proposition 2.3. Notice that

$$\left\langle\delta_{(g,h)}, \phi(J_k(f))\right\rangle = \delta_{gh,e}\ f_{kg}\ \omega(kg)\omega(g)^{-1}\omega(g^{-1}k^{-1})\omega((g)^{-1})^{-1}.$$

Thus we have

$$\begin{aligned}\left\|J_k^*(N'_\alpha)-N'_\alpha\right\| &= \sup\left\{\left|\left\langle f, J_k^*(N'_\alpha)-N'_\alpha\right\rangle\right| : f\in ball\ \ell^\infty(G)\right\}\\ &= \sup\left\{\left|\left\langle\phi(f)-\phi(J_k(f)), M'_\alpha\right\rangle\right| : f\in ball\ \ell^\infty(G)\right\}\\ &= \sup\left\{\left|\left\langle\left(F(hk,g)\Omega(h,k)-F(h,kg)\Omega(k,g)\right)_{(g,h)}, M'_\alpha\right\rangle\right| : f\in ball\ \ell^\infty(G)\right\}\end{aligned}$$

so that $J_k^*(N'_\alpha)-N'_\alpha\to 0$ by the first condition on $(M'_\alpha)_\alpha$ from Proposition 2.4 (iii).

(iii)$\Rightarrow$(ii): Let $(N_\alpha)_\alpha\subseteq\ell^\infty(G)^*$ be given as in (iii). Define $\psi:\ell^\infty(G\times G)\to\ell^\infty(G)$ by $\langle\delta g,\psi(F)\rangle := F(g,g^{-1})$, for each $F\in\ell^\infty(G\times G)$ and $g\in G$. Put $M_\alpha := \psi^*(N_\alpha)$ for every α. Then it suffices to show that the net $(M_\alpha)_\alpha$ has desired properties in Theorem 2.2 (ii). First, for every $f\in \mathrm{ball}\,\ell^\infty(G)$, we see that

$$\left|\left\langle\left(f_{gh}\Omega(g,h)\right)_{(g,h)}, M_\alpha\right\rangle - f_{e_G}\right| = \left|\left\langle\left(f_{e_G}\Omega\left(g,g^{-1}\right)\right)_g, N_\alpha\right\rangle - f_{e_G}\right| \le \left|\left\langle\left(\Omega\left(g,g^{-1}\right)\right)_g, N_\alpha\right\rangle - 1\right|.$$

Next for an arbitrary bounded function $f: G\times G\to C$ and an element $k\in G$, it is clear that

$$\psi\left(\left(f(hk,g)\Omega(h,k)-f(h,kg)\Omega(k,g)\right)_{(g,h)}\right) = \left(f\left(g^{-1}k,g\right)\Omega\left(g^{-1},k\right)-f\left(g^{-1},kg\right)\Omega(k,g)\right)_g.$$

Define $F: G\times G\to C$ by $F(g,h) = f(hk,g)\Omega(h,k)$, for each $g,h\in G$. Hence, it is readily seen that F is bounded and $\|F\|_\infty < \|f\|_\infty$. Therefore

$$\left|\left\langle(f(hk,g)\Omega(h,k)-f(h,kg)\Omega(k,g))_{(g,h)}, M_\alpha\right\rangle\right| = \left|\left\langle(f(g^{-1}k,g)\Omega(g^{-1},k)-f(g^{-1},kg)\Omega(k,g))_g, N_\alpha\right\rangle\right|$$

$$=\left|\left\langle \psi(F)-J_k(\psi(F)),N_\alpha\right\rangle\right| =\left|\left\langle \psi(F),N_\alpha - J_k^*(N_\alpha)\right\rangle\right| \le \left\|J_k{}^*(N_\alpha)-N_\alpha\right\|,$$

as required.

3. Connes-amenability for $M_n(A)$

We fix some matrix conventions from [2]. Let $m,n\in N=\{1,2,3...\}$ and let S be a set. We write $N_m=\{1,2,...,m\}$. The collection of all $m\times n$ matrices $(x_{i,j})$ with entries from S is denoted by $M_{m,n}(S)$, with $M_n(S)$ for $M_{n,n}(S)$ and $M_{m,n}$ for $M_{m,n}(C)$. If x is an arbitary element in S, then we denote by $(x)_{i,j}$ the element of $M_{m,n}(S)$ with x in the $(i,j)^{th}$ place and 0 elsewhere. In particular, M_n is a unital algebra with *matrix units* $\varepsilon_{i,j}$, so that $\varepsilon_{i,j}\varepsilon_{k,l}=\delta_{j,k}\varepsilon_{i,l}$, $(i,j,k,l\in N_n)$. The identity matrix in M_n is $I_n=(\delta_{i,j})=\sum_{i=1}^n\varepsilon_{i,i}$. Let E be a Banach space. We regard $M_{m,n}(E)$ as a Banach space by taking the norm to be specified by

$$\left\|(x_{i,j})\right\|=\sum\left\{\left\|x_{i,j}\right\|: i\in N_m,\ j\in N_n\right\},\ \left((x_{i,j})\in M_{m,n}(E)\right).$$

We identify $M_{m,n}(E)^*$ with $M_{m,n}(E^*)$, using the duality

$$\langle x,\Lambda\rangle=\sum\left\{\left\langle x_{i,j},\lambda_{i,j}\right\rangle: i\in N_m,\ j\in N_n\right\},$$

for $x=(x_{i,j})\in M_{m,n}(E)$ and $\Lambda=(\lambda_{i,j})\in M_{m,n}(E^*)$. Let A be an algebra. Then $M_n(A)$ is also an algebra in the obvious way. The matrix $(a_{i,j})$ is identified with $\sum_{i,j=1}^n\varepsilon_{i,j}\otimes a_{i,j}$ so that $M_n(A)$ is isomorphic to $M_n\otimes A$. In the case where A is a Banach algebra, the algebra $M_n(A)$ is a Banach algebra with respect to the norm defined as above. Let A be a Banach algebra and let E be a Banach A-bimodule. We shall regard $M_n(E)$ as a Banach $M_n(A)$-bimodule through

$$(a.x)_{i,j}=\sum_{k=1}^n a_{i,k}.x_{k,j} \text{ and } (x.a)_{i,j}=\sum_{k=1}^n x_{i,k}.a_{k,j}\ .$$

For $a=(a_{i,j})\in M_n(A)$ and $x=(x_{i,j})\in M_n(E)$. In particular $M_n(E^*)$ is a Banach $M_n(A)$-bimodule. For $a=(a_{i,j})\in M_n(A)$ and $\Lambda=(\lambda_{i,j})\in M_n(E^*)$ we notice that

$$(a.\Lambda)_{i,j}=\sum_{k=1}^n a_{j,k}.\lambda_{i,k} \text{ and } (\Lambda.a)_{i,j}=\sum_{k=1}^n \lambda_{k,j}.a_{k,i}\ .$$

Suppose that A is a dual Banach algebra. It is known that $A\hat{\otimes}A$ is canonically mapped into $\sigma wc\left(\left(A\hat{\otimes}A\right)^*\right)^*$, [12]. Hence we may consider the w^*-topology on $A\hat{\otimes}A$ inherited from $\sigma wc\left(\left(A\hat{\otimes}A\right)^*\right)^*$.

Definition 3.1 Suppose that A is a dual Banach algebra. A net (u_α) in $A\hat{\otimes}A$ is an *approximate* $\sigma wc-$*diagonal* for A if for every $a \in A$

(i) $a.u_\alpha - u_\alpha.a \xrightarrow{w^*} 0$ in $\sigma wc\left(\left(A\hat{\otimes}A\right)^*\right)^*$, and

(ii) $a\pi_{\sigma wc}(u_\alpha) \xrightarrow{w^*} a$ in A.

We may characterize a dual Banach algebra to be Connes-amenable in terms of diagonals as follows.

Proposition 3.2 Suppose that A is a dual Banach algebra. Then the following are equivalent:

(i) A is Connes-amenable .

(ii) There exists a $\sigma wc-$virtual diagonal for A.

(iii) There exists a bounded approximate $\sigma wc-$diagonal for A.

Proof . The equivalences of (i) and (ii) is just [12, Theorem 4.8].

(ii)$\Rightarrow$(iii): Let U be a $\sigma wc-$virtual diagonal for A. Since $A\hat{\otimes}A$ is w^*-dense in $\sigma wc\left(\left(A\hat{\otimes}A\right)^*\right)^*$, there is a net (u_α) in $A\hat{\otimes}A$ which tends to U in the w^*-topology. We know that $\sigma wc\left(\left(A\hat{\otimes}A\right)^*\right)^*$ is a closed submodule of $\left(A\hat{\otimes}A\right)^*$, and so restriction gives a quotient map $(A\hat{\otimes}A)^{**} \to \sigma wc((A\hat{\otimes}A)^*)^*$. This, together with Goldstein's theorem, shows that (u_α) can be chosen to be a bounded net. Then, it is easy to check that (u_α) is an approximate $\sigma wc-$diagonal for A.

(iii)$\Rightarrow$(ii): Let $U \in \sigma wc((A\hat{\otimes}A)^*)^*$ be a w^*-accumulation point of the given bounded approximate $\sigma wc-$diagonal (u_α) for A. Without loss of generality, we may suppose that $U = w^* - \lim_\alpha u_\alpha$. Then, it is readily seen that U is a $\sigma wc-$virtual diagonal for A.

We shall see the role of Proposition 3.2 in the proof of Theorem 3.7 below.

The following is easy to verify.

Lemma 3.3 Suppose that E is a Banach space and that $\Lambda=(\lambda_{i,j})$ and $\Lambda_\alpha=(\lambda_{i,j}^\alpha)$ are elements of $M_n(E^*)$. Then $\Lambda_\alpha \xrightarrow{w^*} \Lambda$ in $M_n(E^*)$ if and only if $\lambda_{i,j}^\alpha \xrightarrow{w^*} \lambda_{i,j}$ in E^*, for all $i,j\in N_n$.

Let $A=(A_*)^*$ be a dual Banach algebra and let $E=(E_*)^*$ be a normal, dual Banach A-bimodule. Then, using Lemma 3.3, it is not hard to see that $M_n(E)=M_n(E_*)^*$ is a normal, dual Banach $M_n(A)$-bimodule. In particular, $M_n(A)=M_n(A_*)^*$ is a dual Banach algebra.

Let A be a Banach algebra, E be a Banach A-bimodule and let $D:A\to E^*$ be a derivation. We may consider the derivation $\tilde{D}:M_n(A)\to M_n(E^*)$ by setting $\tilde{D}((a_{i,j}))=(D(a_{j,i}))$, where we note the transposition of i and j $[2]$. Further, if A is dual and D is w^*-continuous then it is easily seen that $\tilde{D}$ is also a w^*-continuous derivation.

Suppose that A is a Banach algebra. We shall identify $M_n(A)$ with $M_n\otimes A$, so that we can identify $M_n(A)\hat{\otimes}M_n(A)$ with $M_{n^2}\otimes(A\hat{\otimes}A)$.

Definition 3.4 Let A be a Banach algebra. For $u\in A\hat{\otimes}A$ and $r,s\in N_n$, we define elements

$$U=\frac{1}{n}\sum_{i,j=1}^{n}\varepsilon_{i,j}\otimes\varepsilon_{j,i}\otimes u \text{ and } V=\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{r,j}\otimes\varepsilon_{j,s}\otimes u$$

in $M_{n^2}\otimes(A\hat{\otimes}A)$. Moreover, for $\Omega\in(M_{n^2}\otimes(A\hat{\otimes}A))^*$ we define $\omega\in(A\hat{\otimes}A)^*$ by $\langle u,\omega\rangle=\langle V,\Omega\rangle$. Then for $a\in A$ we have

$$\langle u,a.\omega\rangle=\left\langle\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{r,j}\otimes\varepsilon_{j,s}\otimes(u.a),\Omega\right\rangle=\langle U.(\varepsilon_{r,s}\otimes a),\Omega\rangle=\langle U,(\varepsilon_{r,s}\otimes a).\Omega\rangle$$

and similarly $\langle u,\omega.a\rangle=\langle U,\Omega.(\varepsilon_{r,s}\otimes a)\rangle$. we also observe that

$$U.(\varepsilon_{r,s}\otimes a)=\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{r,j}\otimes\varepsilon_{j,s}\otimes(u.a)=\left(\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{r,j}\otimes\varepsilon_{j,s}\otimes u\right).(I_n\otimes a)=V.(I_n\otimes a)$$

and that $(\varepsilon_{r,s}\otimes a).U=(I_n\otimes a).V$.

Take $\phi\in(A\hat{\otimes}A)^{**}$ and take the net $(u_\alpha)\subseteq A\hat{\otimes}A$ such that $u_\alpha\to\phi$ in the w^*-topology on $(A\hat{\otimes}A)^{**}$. We consider the corresponding net (U_α) and (V_α) in $M_{n^2}\otimes(A\hat{\otimes}A)$, as Definition 3.4. We define the element $\Phi\in(M_{n^2}\otimes(A\hat{\otimes}A))^{**}$ (depends on ϕ) through $\langle\Omega,\Phi\rangle=\langle\omega,\phi\rangle$ for every $\Omega\in(M_{n^2}\otimes(A\hat{\otimes}A))^*$, where ω

is given by Definition 3.4. Then it is easy to see that $V_\alpha \xrightarrow{w^*} \Phi$ in $(M_{n^2} \otimes (A \hat{\otimes} A))^{**}$. Hence we see that $(\varepsilon_{r,s} \otimes a).U_\alpha \xrightarrow{w^*} (I_n \otimes a).\Phi$ in $(M_{n^2} \otimes (A \hat{\otimes} A))^{**}$. Therefore

$$\langle \phi, a.\omega \rangle = \lim_\alpha \langle u_\alpha, a.\omega \rangle = \lim_\alpha \langle U_\alpha, (\varepsilon_{r,s} \otimes a).\Omega \rangle = \lim_\alpha \langle U_\alpha.(\varepsilon_{r,s} \otimes a), \Omega \rangle$$
$$= \langle \Phi.(I_n \otimes a), \Omega \rangle = \langle \Phi, (I_n \otimes a).\Omega \rangle$$

and similarly $\langle \phi, \omega.a \rangle = \langle \Phi, \Omega.(I_n \otimes a) \rangle$.

We keep the notations of Definition 3.4 in the sequel.

Lemma 3.5 Suppose that A is a dual Banach algebra and that $\Omega \in \sigma wc(M_{n^2} \otimes (A \hat{\otimes} A))^*)$. Then $\omega \in \sigma wc((A \hat{\otimes} A)^*)$.

Proof. Suppose that $a_i \xrightarrow{w^*} a$ in A and that $\phi \in (A \hat{\otimes} A)^{**}$. By Lemma 3.3, $I_n \otimes a_i \xrightarrow{w*} I_n \otimes a$ in $M_n \otimes A$. Then by the assumption

$$\langle \Phi, (I_n \otimes a_i).\Omega \rangle \to \langle \Phi, (I_n \otimes a).\Omega \rangle,$$

and whence $\langle \phi, a_i.\omega \rangle \to \langle \phi, a.\omega \rangle$. A similar argument yields that $\langle \phi, \omega.a_i \rangle \to \langle \phi, \omega.a \rangle$, as required.

We denote by Π the corresponding diagonal operator for $M_n(A)$.

Lemma 3.6. Suppose that $u \in A \hat{\otimes} A$, $r, s \in N_n$ and that $a \in A$. Then

(i) $\Pi(U) = I_n \otimes \pi(u)$;

(ii) $\Pi(U)(\varepsilon_{r,s} \otimes a) = \varepsilon_{r,s} \otimes \pi(u)a$;

(iii) $(\varepsilon_{r,s} \otimes a)\Pi(U) = \varepsilon_{r,s} \otimes a\pi(u)$.

Proof. Take $u = \sum_{m=1}^{\infty} a_m \otimes b_m$ and then

$$U = \frac{1}{n}\sum_{i,j,m} \varepsilon_{i,j} \otimes \varepsilon_{j,i} \otimes a_m \otimes b_m = \frac{1}{n}\sum_{i,j,m} (\varepsilon_{i,j} \otimes a_m) \otimes (\varepsilon_{j,i} \otimes b_m).$$

Therefore

$$\Pi(U) = \frac{1}{n}\sum_{i,j,m} \varepsilon_{i,j}\varepsilon_{j,i} \otimes a_m b_m = \sum_{m=1}^{\infty} I_n \otimes a_m b_m = I_n \otimes \pi(u).$$

Then we obtain

$$\Pi(U)(\varepsilon_{r,s} \otimes a) = (I_n \otimes \pi(u))(\varepsilon_{r,s} \otimes a) = \varepsilon_{r,s} \otimes \pi(u)a,$$

and analogously (iii).

The following is our main result in this section.

Theorem 3.7. Suppose that $A=(A_*)^*$ is a dual Banach algebra and that $n\in N$. Then $M_n(A)=M_n(A_*)^*$ is Connes-amenable if and only if A is Connes-amenble.

Proof. Let $M_n(A)$ be Connes-amenable. Let E be a normal, dual Banach A-bimodule and $D:A\to E$ be a w^*-continuous derivation. We consider the w^*-continuous derivation $\tilde{D}:M_n(A)\to M_n(E)$ as before. By the assumption, there exists $x=(x_{i,j})\in M_n(E)$ for which $\tilde{D}(a)=a.x-x.a$, $a\in M_n(A)$. Take $a\in A$ and identify a with the matrix $(a)_{11}$. Then $x_{1,1}\in E$ and

$$D(a)=\left(\tilde{D}((a)_{11})\right)_{1,1}=((a)_{11}.x-x.(a)_{11})_{1,1}=a.x_{1,1}-x_{1,1}.a$$

so that D is an inner derivation as required.

Conversely, let A be Connes-amenable. Let $(u_\alpha)\subseteq A\hat{\otimes}A$ be a bounded approximate σwc-diagonal for A. We wish to show that the corresponding net (U_α), defined in Definiton 3.4, is a bounded approximate σwc-diagonal for $M_n(A)$. Take $r,s\in N_n, a\in A$ and $\Omega\in\sigma wc(M_{n^2}\otimes(A\hat{\otimes}A))^*)$. Then, using Lemma 3.5, we have

$$\left\langle \Omega,(\varepsilon_{r,s}\otimes a).U_\alpha-U_\alpha.(\varepsilon_{r,s}\otimes a)\right\rangle=\left\langle \Omega.(\varepsilon_{r,s}\otimes a)-(\varepsilon_{r,s}\otimes a).\Omega,U_\alpha\right\rangle$$

$$=\left\langle \omega.a-a.\omega,u_\alpha\right\rangle=\left\langle \omega,a.u_\alpha-u_\alpha.a\right\rangle\to 0.$$

It follows that

$$\left\langle \Omega,(a_{i,j}).U_\alpha-U_\alpha.(a_{i,j})\right\rangle\to 0,$$

for all $(a_{i,j})\in M_n(A)$ and $\Omega\in\sigma wc(M_{n^2}\otimes(A\hat{\otimes}A))^*)$.

Next for $\psi\in A_*$, by Lemma 3.6, we see that

$$\left\langle \varepsilon_{r,s}\otimes\psi,(\varepsilon_{r,s}\otimes a)\Pi(U_\alpha)\right\rangle=\left\langle \varepsilon_{r,s}\otimes\psi,\varepsilon_{r,s}\otimes a\pi(u_\alpha)\right\rangle=\left\langle \psi,a\pi(u_\alpha)\right\rangle$$

$$\to\left\langle \psi,a\right\rangle=\left\langle \varepsilon_{r,s}\otimes\psi,\varepsilon_{r,s}\otimes a\right\rangle,$$

and

$$\left\langle \varepsilon_{k,l}\otimes\psi,(\varepsilon_{r,s}\otimes a)\Pi(U_\alpha)\right\rangle=\left\langle \varepsilon_{k,l}\otimes\psi,\varepsilon_{r,s}\otimes a\right\rangle=0,\ \ (k,l\in N_n,(k,l)\neq(r,s)).$$

Hence for all $(a_{i,j})\in M_n(A)$ and $(\psi_{i,j})\in M_n(A_*)$ we have

$$\left\langle (\psi_{i,j}),(a_{i,j})\Pi(U_\alpha)\right\rangle\to\left\langle (\psi_{i,j}),(a_{i,j})\right\rangle,$$

which proves the claim.

4. Conclusions

We briefly point out the original results obtained in this work. We first, regarding a discrete weakly concellative semigroup S, Considered the Beurling algebras $\ell^1(S,\omega)$, where ω is a weight function on S. We showed that the existence of some specified nets in $l^\infty(S\times S)^*$ is equivalent to the approximate Connes-amenatility of $\ell^1(S,\omega)$. Next, for a discrete group G, we proved that approximate Connes-amenability and approximate amenability are the same notion for the Beurling algebra $\ell^1(G,\omega)$. Finally, for a dual Banach algebra A, we showed that the matrix algebra $M_n(A)$ is a dual Banach algebra as well. We proved that $M_n(A)$ is Connes-amenable if and only if A is Connes-amenable, which is our last result in this paper.

Acknowledgement

This paper was supported by the Islamic Azad University Central Tehran Branch and the author acknowledges this with thanks.

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