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KKM MAPPINGS IN PMT SPACES WITH APPLICATIONS

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In this paper, we study some topological properties of PMT spaces, next we obtain KKM mapping in these spaces, as an application, we get some fixed point existence results for set-valued mappings and a new version of Fan's best approximation theorem on such spaces.

Keywords: KKM property, Set-valued mappings, Probabilistic metric space, Metric type space, Admissible set, Approximate fixed point property.

MSC2010: 54H10, 54E20.

1. Introduction

Recently, Khamsi and Hussain [7] introduced the concept of metric type space and discussed a natural topology defined in any metric type space, which this topology enjoys most of the metric like properties (see also [2, 5, 6]). In this paper, we introduce probabilistic metric type space and establish some topological properties of these spaces. We study the class of KKM type mappings on probabilistic metric type space and apply it for getting some fixed point existence results for set-valued mappings and a new version of Fan's best approximation theorem on such spaces.

2. Basic definitions and results

First, let us start by making some basic definitions.

Definition 2.1 ([4, 9, 10])**.** mapping $F : (-\infty, \infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$. If in addition $F(0) = 0$, then F is called a distance distribution function. The set of all distance distribution functions ($d.d.f$) is denoted by Δ^+. The maximal element for Δ^+ in this order is the $d.d.f$, ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \le 0\,, \\ 1 & \text{if } t > 0\,. \end{cases}$$

Definition 2.2 ([1, 3, 8])**.** A triangular norm (shorter t-norm) is a binary operation T on $[0, 1]$, which satisfies the following conditions:

(1) T is associative and commutative;
(2) $T(a, 1) = a$ for all $a \in [0, 1]$;
(3) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

The operations $T_L(a, b) = \max(a + b - 1, 0)$, $T_M(a, b) = \min\{a, b\}$ and $T_p(a, b) = ab$ on $[0, 1]$ are t- norms.

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Definition 2.3. A probabilistic metric type space (PMT space) is a triple (M, F, T), where M is a nonempty set, T is a continuous t-norm and F is a mapping from $M \times M$ into Δ^+ such that, if $F_{x,y}$ denote the value of F at the pair (x, y), the following conditions hold:

(PMT1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PMT2) $F_{x,y}(t) = F_{y,x}(t)$;
(PMT3) $F_{x,y}(K(s+t)) \geq T(F_{x,z}(s), F_{z,y}(t))$ for any $x, y, z \in M$, $t, s \geq 0$ for some constant $K > 1$.

Observe that if $K = 1$, then the PMT space is a probabilistic metric space, however it does not hold true when $K > 1$. Thus the class of PMT spaces is effectively larger than that of the ordinary probabilistic metric spaces. That is, every probabilistic metric space is a PMT space, but the converse need not be true.

Example 2.1. Let (M, D) be a metric type space with constant $K \geq 1$. Define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ \frac{t}{t+D(x,y)} & \text{if } t > 0\,. \end{cases}$$

Then (M, F, T_p) is a PMT space with constant K. (PMT1) and (PMT2) are obvious and we show (PMT3).

$$\begin{aligned} T_p(F_{x,z}(t), F_{z,y}(s)) &= \frac{t}{t+D(x,z)} . \frac{s}{s+D(z,y)} \\ &= \frac{1}{1+\frac{D(x,z)}{t}} . \frac{1}{1+\frac{D(z,y)}{s}} \\ &\leq \frac{1}{1+\frac{D(x,z)}{(t+s)}} . \frac{1}{1+\frac{D(z,y)}{(t+s)}} \\ &\leq \frac{1}{1+\frac{(D(x,z)+D(z,y))}{(t+s)}} \\ &\leq \frac{1}{1+\frac{D(x,y)}{K(t+s)}} \\ &= \frac{K(t+s)}{K(t+s)+D(x,y)} \\ &= F_{x,y}(K(t+s))\,. \end{aligned}$$

Remark 2.1. Let L_p $(0 < p < 1)$ be the set of all real functions $f(x)$, $x \in [0, 1]$ such that $\int_0^1 |f(x)|^p dx < \infty$. Define

$$D(x, y) = (\int_0^1 |f(x) - g(x)|^p dx)^{\frac{1}{p}}\,,$$

for each $f, g \in L_p$. Then D is a metric type space with $K = 2^{\frac{1}{p}}$.

Example 2.2. Let M be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^p dx < \infty$, where $p > 0$ is a real number. Define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ \frac{t}{t+(\int_0^1 |f(x)-g(x)|^p dx)^{\frac{1}{p}}} & \text{if } t > 0\,. \end{cases}$$

Then by Example 2.4 and Remark 2.5, (M, F, T_p) is a PMT space with $K = 2^{\frac{1}{p}}$.

Example 2.3. Let (M, D) be a metric type spaces with constant $K \geq 1$. Define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ e^{\frac{-(D(x,y))}{t}} & \text{if } t > 0\,. \end{cases}$$

Then (M, F, T_p) is a PMT space with constant K. (PMT1) and (PMT2) are obvious and we show (PMT3).

$$\begin{aligned} T_p(F_{x,z}(t), F_{z,y}(s)) &= \frac{t}{t + D(x,z)}.\frac{s}{s + D(z,y)} \\ &= e^{\frac{-(D(x,z))}{t}}.e^{\frac{-(D(z,y))}{s}} \\ &\leq e^{-(\frac{D(x,y)}{K(t+s)})} \\ &= F_{x,y}(K(t+s))\,. \end{aligned}$$

Remark 2.2. Let (M, d) be a metric space and $D(x,y) = (d(x,y))^n$, where $n > 1$ is a real number. Then D is a metric type space with $K = 2^{n-1}$. The triangle inequality follows easily from the convexity of the function $f(x) = x^n$ $(x > 0)$.

Example 2.4. Let M be a nonempty set. Define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ e^{-\frac{|x-y|^n}{t}} & \text{if } t > 0\,. \end{cases}$$

Then by Example 2.7 and Remark 2.8 (M, F, T_p) is a PMT space with $K = 2^{n-1}$.

3. **Topology induced by probabilistic metric type**

We continue to present some concepts and results from probabilistic metric space theory, in the context of PMT spaces. Let (M, F, T) be a PMT space. We define the open ball $B_x(r,t)$ and the closed ball $B_x[r,t]$ with center $x \in M$ and radius $0 < r < 1$, $t > 0$ as follows:

$$B_x(r,t) = \{y \in M \ : \ F_{x,y}(t) > 1 - r\},$$
$$B_x[r,t] = \{y \in M \ : \ F_{x,y}(t) \geq 1 - r\}.$$

Definition 3.1. Let (M, F, T) be a PMT space. A subset $A \subset M$ is said to be open if and only if, for any $x \in A$ there exists $t > 0$ and $0 < r < 1$ such that $B_x(r,t) \subset A$.

Proposition 3.1. *Let (M, F, T) be a PMT space. Define*

$$\begin{aligned} \tau_F \;\; = \;\; &\{A \subset M : x \in A \textit{ if and only if there exists} \\ &t > 0 \textit{ and } 0 < r < 1\,, \textit{ such that } B_x(r,t) \subset A\}. \end{aligned}$$

Then τ_F is a topology on M.

Proof. (i) Clearly $\emptyset$ and M belong to τ_F.
(ii) Let $A_1, A_2, ..., A_i \in \tau_F$, and put

$$U = \cup_{i \in I} A_i.$$

We shall show that $U \in \tau_F$. If $a \in U$, then $a \in \cup_{i \in I} A_i$ which implies that $a \in A_i$ for some $i \in I$. Since $A_i \in \tau_F$, there exists $0 < r < 1$, $t > 0$, such that $B_a(r,t) \subset A_i$. Hence

$$B_a(r,t) \subset A_i \subset \cup_{i \in I} A_i = U\,.$$

This shows that $U \in \tau_F$.
(iii) Let $A_1, A_2, ..., A_n \in \tau_F$, and $U = \cap_{i=1}^{n} A_i$. We shall show that $U \in \tau_F$. Let $a \in U$. Then $a \in A_i$ for all $1 \leq i \leq n$. Hence, for each $1 \leq i \leq n$, there exists $0 < r_i < 1$, $t_i > 0$ such that $B_a(r_i, t_i) \subset A_i$. Let

$$r = \min\{r_i\,,\, 1 \leq i \leq n\}$$

and

$$t = \max\{t_i\,,\ 1 \le i \le n\}.$$

Thus $r \le r_i$ for all $1 \le i \le n$, $1 - r \ge 1 - r_i$ for all $1 \le i \le n$. Also, $t > 0$. So, $B_a(r, t) \subseteq A_i$ for all $1 \le i \le n$. Therefore

$$B_a(r, t) \subset \cap_{i=1}^{n} A_i = U\,.$$

This shows that $U \in \tau_F$. □

Proposition 3.2. *Every PMT space with constant K is Hausdorff.*

Proof. Let (M, F, T) be a PMT space. Let x, y be two distinct points of M. Then $0 < F_{x,y}(t) < 1$. Let $F_{x,y}(t) = r$, for some r, $0 < r < 1$. For each r_0, $r < r_0 < 1$, we can find an r_1 such that $T(r_1, r_1) \ge r_0$. Now consider the open balls $B_x(1-r_1, \frac{t}{2K})$ and $B_y(1-r_1, \frac{t}{2K})$. Clearly

$$B_x(1 - r_1, \frac{t}{2K}) \cap B_y(1 - r_1, \frac{t}{2K}) = \emptyset.$$

Otherwise, if there exists $z \in B_x(1 - r_1, \frac{t}{2K}) \cap B_y(1 - r_1, \frac{t}{2K})$. Then

$$\begin{aligned} r &= F_{x,y}(t) \\ &\ge T(F_{x,z}(\frac{t}{2K}), F_{z,y}(\frac{t}{2K})) \\ &\ge T(r_1, r_1) \ge r_0 \\ &> r\,, \end{aligned}$$

which is a contradiction. Therefore (M, F, T) is Hausdorff. □

Proposition 3.3. *Let (M, D) be a metric type space and $F_{x,y}(t) = \frac{t}{t+D(x,y)}$ be the corresponding standard PMT on M. Then the topology τ_D induced by the metric D and the topology τ_F induced by the F are the same. That is, $\tau_D = \tau_F$.*

Proof. Suppose that $A \in \tau_D$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$, for every $x \in A$. For a fixed $t > 0$, we obtain that

$$F_{x,y}(t) = \frac{t}{t + D(x, y)} > \frac{t}{t + \epsilon}\,.$$

Let

$$1 - r = \frac{t}{t + \epsilon}\,.$$

Then

$$F_{x,y}(t) > 1 - r\,.$$

It follows that, $B_x(r, t) \subset A$. Hence $A \in \tau_F$. This shows that $\tau_D \subseteq \tau_F$. Conversely, suppose that $A \in \tau_F$. Then there exists $0 < r < 1$ and $t > 0$ such that $B_x(r, t) \subset A$ for every $x \in A$. We obtain that

$$F_{x,y}(t) = \frac{t}{t + D(x, y)} > 1 - r$$

$$t > (1 - r)t + (1 - r)D(x, y)$$

$$D(x, y) < \frac{rt}{1 - r}.$$

Let $\epsilon = \frac{rt}{1-r}$ where $0 < \epsilon < 1$. Then $D(x, y) < \epsilon$, and therefore $B(x, \epsilon) \subset A$. Hence $A \in \tau_D$. This implies that $\tau_F \subseteq \tau_D$. Therefore $\tau_D = \tau_F$. □

Definition 3.2. Let (M, F, T) be a PMT space. A subset X of M is said to be p-bounded if there exists $t > 0$ and $0 < r < 1$ such that $F_{x,y}(t) > 1 - r$ for all $x, y \in X$.

Remark 3.1. Let (M, F, T) be a PMT space induced by a metric type D on M. Then $X \subseteq M$ is p-bounded if an only if it is bounded.

Definition 3.3. Let (M, F, T) be a PMT space. We say that $\{x_n\}$ is:

(1) Convergent sequence, if for $0 < r < 1$ and $t > 0$ there exists $n_0 \in N$ such that $F_{x_n,x}(t) > 1 - r$ for all $n \geq n_0$ and for some fixed $x \in M$;

(2) Cauchy sequence, if for every $0 < r < 1$ and $t > 0$ there exists $n_0 \in N$ such that $F_{x_n,x_m}(t) > 1 - r$ for all $n, m \geq n_0$. A PMT space is said to be complete if every Cauchy sequence is convergent in M.

Theorem 3.1. *Let (M, F, T) be a PMT space and τ_F be the topology induced by the PMT. Then for a sequence $\{x_n\}$ in M, the sequence $\{x_n\}$ converges to x if and only if $F_{x_n,x}(t)$ converges to 1 as $n \to \infty$.*

Proof. Fix $t > 0$. Suppose that the sequence $\{x_n\}$ converges to x. Then for $0 < r < 1$, there exists $n_0 \in N$ such that $x_n \in B_x(r, t)$ for all $n \geq n_0$. It follows that $F_{x_n,x}(t) > 1 - r$ and hence $1 - F_{x_n,x}(t) < r$. Therefore $F_{x_n,x}(t)$ converges to 1 as $n \to \infty$.
Conversely, if for each $t > 0$, $F_{x_n,x}(t)$ converges to 1 as $n \to \infty$ then for $0 < r < 1$, there exists $n_0 \in N$ such that $1 - F_{x_n,x}(t) < r$ for all $n \geq n_0$. It follows that $F_{x_n,x}(t) > 1 - r$ for all $n \geq n_0$. Thus $x_n \in B_x(r, t)$ for all $n \geq n_0$, and hence the sequence$\{x_n\}$ converges to x. □

Remark 3.2. Let (M, F, T) be a PMT space induced by a metric type D on M. Then $\{x_n\}$ is convergent in τ_F if and only if $\{x_n\}$ is convergent in (M, D).

Theorem 3.2. *Let (X, F, T) be a PMT space and τ_F be the topology induced by the PMT. Then for a sequence $\{x_n\}$ in X, the sequence $\{x_n\}$ is Cauchy if and only if $F_{x_n,x_m}(t)$ converges to 1 as $n, m \to \infty$.*

Proof. Fix $t > 0$. Suppose that the sequence $\{x_n\}$ is Cauchy. Then for $0 < r < 1$, there exists $n_0 \in N$ such that $x_n \in B_{x_m}(r, t)$ for all $n, m \geq n_0$. It follows that $F_{x_n,x_m}(t) > 1 - r$ and hence $1 - F_{x_n,x_m}(t) < r$. Therefore $F_{x_n,x_m}(t)$ converges to 1 as $n, m \to \infty$.
Conversely, if for each $t > 0$, $F_{x_n,x}(Kt)$ converges to 1 as $n, m \to \infty$ then for $0 < r < 1$, there exists $n_0 \in N$ such that $1 - F_{x_n,x_m}(t) < r$ for all $n, m \geq n_0$. It follows that $F_{x_n,x_m}(t) > 1 - r$ for all $n, m \geq n_0$. Thus $x_n \in B_{x_m}(r, t)$ for all $n, m \geq n_0$, and hence the sequence$\{x_n\}$ is Cauchy. □

Remark 3.3. Let (M, F, T) be a PMT space induced by a metric type D on M. Then $\{x_n\}$ is Cauchy in τ_F if and only if $\{x_n\}$ is Cauchy in (M, D).

Proposition 3.4. *Let (M, F, T) be a PMT space and τ_F be the topology induced by PMT. Then for any nonempty subset $X \subset M$ we have*

(1) *X is closed if and only if for any sequence $\{x_n\}$ in X which converges to x, we have $x \in X$;*

(2) *if we define $\bar{X}$ to be the intersection of all closed subsets of M which contain X, then for any $x \in \bar{X}$ and for any $0 < r < 1$ and $t > 0$, we have $B_x(r, t) \cap X \neq \emptyset$.*

Proof. Let us prove (1) first. Assume that X is closed and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Let us prove that $x \in X$. Assume not, i.e. $x \notin X$. Since X is closed, then there exists $0 < r < 1$ and $t > 0$ such that $B_x(r, t) \cap X = \emptyset$. Since $\{x_n\}$ converges to x, then there exists $N \geq 1$ such that for any $n \geq N$ we have $x_n \in B_x(r, t)$. Hence $x_n \in B_x(r, t) \cap X$, which leads to a contradiction. Conversely assume that for any sequence $\{x_n\}$ in X which converges to x, we have $x \in X$. Let us prove that X is closed. Let $x \notin X$. We need to prove that there exists $0 < r < 1$ and $t > 0$ such that $B_x(r, t) \cap X = \emptyset$. Assume not, i.e. for any $0 < r < 1$ and $t > 0$, we have $B_x(r, t) \cap X \notin \emptyset$. So for any $n \geq 1$, choose $x_n \in B_x(\frac{1}{n}, t) \cap A$. Clearly we have $\{x_n\}$ converges to x. Our assumption on X implies $x \in X$, a contradiction.

Let us prove (2). Clearly $\bar{X}$ is the smallest closed subset which contains X. Set

$$X^* = \{x \in M; \text{ for any } 0 < r < 1, \text{ there exists } a \in X \text{ such that} : F_{x,a}(t) > 1 - r\}.$$

We have $X \subset X^*$. Next we prove that X^* is closed. For this we use property (1). Let $\{x_n\}$ be a sequence in X^* such that $\{x_n\}$ converges to x. Let $0 < r < 1$ and $t > 0$. Since $\{x_n\}$ converges to x, there exists $N \geq 1$ such that for any $n \geq N$ we have

$$F_{x,x_n}(\frac{t}{2K}) > 1 - r\,,$$

where K is the constant to the condition (PMT3). Let $r_0 = F_{x,x_n}(\frac{t}{2K}) > 1 - r$. Since $r_0 > 1 - r$, we can find an s, $0 < s < 1$, such that $r_0 > 1 - s > 1 - r_0$. Now for a given r_0 and s such that $r_0 > 1 - s$ we can find r_1, $0 < r_1 < 1$, such that

$$T(r_0, (1 - r_1)) \geq 1 - s\,.$$

Now, since $x_n \in X^*$, there exists $a \in X$ such that

$$F_{x_n,a}(\frac{t}{2K}) > 1 - r_1\,.$$

Hence

$$F_{x,a}(t) \geq T(F_{x,x_n}(\frac{t}{2K}), F_{x_n,a}(\frac{t}{2K})) > T(r_0, (1 - r_1)) \geq 1 - s > 1 - r\,,$$

which implies $x \in X^*$. Therefore X^* is closed and contains X. The definition of $\bar{X} \subset X^*$, which implies the conclusion of (2). □

Proposition 3.5. *Every compact subset X of a PMT space M is p-bounded.*

Proof. Given X a compact subset of M. Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B_x(r,t) : x \in X\}$ of X. Since X is compact, there exists $x_1, x_2, ..., x_n \in X$ such that

$$X \subseteq \cup_{i=1}^{n} B_{x_i}(r,t)\,.$$

Let $x, y \in X$. Then $x \in B_{x_i}(r,t)$ and $y \in B_{x_j}(r,t)$ for some i, j. Therefore $F_{x,x_i}(t) > 1 - r$ and $F_{y,x_j}(t) > 1 - r$. Now, let $\alpha = \min\{F_{x_i,x_j}(t) : 1 \leq i, j \leq n\}$. Then $\alpha > 0$. Now

$$F_{x,y}(K(2Kt + t)) \geq T(T(F_{x,x_i}(t), F_{x_i,x_j}(t)), F_{x_j,y}(t) \geq T(T((1-r),(1-r)),\alpha)\,,$$

where K is the constant in the condition (PMT3). Taking $t^{'} = K(2Kt + t)$ and $T(T((1 - r),(1 - r)),\alpha) > 1 - s$, $0 < s < 1$, we have $F_{x,y}(t^{'}) > 1 - s$ for all $x, y \in X$. Hence X is p-bounded. □

Every compact subset of a Hausdorff topological space is closed. Then:

Remark 3.4. In a PMT space every compact subset is closed and p-bounded.

Proposition 3.6. *Let (M, F, T) be a PMT space and τ_F the topology defined above. Let X be a nonempty subset of M. The following properties are equivalent*

(1) *X is compact.*
(2) *For any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and if $\{x_{n_k}\}$ converges to x then $x \in X$.*

Proof. Assume that X is a nonempty compact subset of M. It is easy to see that any decreasing sequence of nonempty closed subsets of X has a nonempty intersection. Let $\{x_n\}$ be a sequence in X. Set $C_n = \{x_m : m \geq n\}$. Then we have $\bigcap_{n\geq 1} \overline{C_n} \neq \emptyset$. Let $x \in \bigcap_{n\geq 1} \overline{C_n}$. Then for $0 < r < 1$, $t > 0$ and for any $n \geq 1$, there exists $m_n \geq n$ such that $F_{x,x_{m_n}}(t) > 1 - r$. This clearly implies the existence of a subsequence of $\{x_n\}$ which converges to x. Since X is closed, then we must have $x \in X$.

Conversely, let X be a nonempty subset of M such that the conclusion of (2) is true. Let us prove that X is compact. First, note that for any $0 < r < 1$, $t > 0$, there exists $x_1, x_2, ..., x_n \in A$ such that

$$X \subseteq \bigcup_{i=1}^{n} B_{x_i}(r,t)\,.$$

Assume not, then there exists $0 < r_0 < 1$, such that for any finite number of points $x_1, x_2, ..., x_n \in X$, we have

$$X \nsubseteq \bigcup_{i=1}^{n} B_{x_i}(r_0,t)\,.$$

Fix $x_1 \in X$. Since $X \nsubseteq B_{x_1}(r_0,t)$, there exists $x_2 \in X \setminus B_{x_1}(r_0,t)$. By induction we build a sequence $\{x_n\}$ such that

$$x_{n+1} \in X \setminus (B_{x_1}(r_0,t) \cup ... \cup B_{x_n}(r_0,t))$$

for all $n \geq 1$. Clearly we have $F_{x_n,x_m}(t) < 1-r_0$, for all $n, m \geq 1$, with $n \neq m$. This condition implies that no subsequence of $\{x_n\}$ will be Cauchy or convergent. This contradicts our assumption on X. Next let $\{O_\alpha\}_{\alpha\in J}$ be an open cover of X. Let us prove that only finitely many O_α cover X. Fix $t > 0$, First, note that there exists $0 < r_0 < 1$ such that for any $x \in X$, there exists $\alpha \in J$ such that $B_x(r_0,t) \subset O_\alpha$. Assume not, then for any $0 < r < 1$, there exists $x_r \in X$ such that for any $\alpha \in J$, we have$B_{x_r}(r,t) \nsubseteq O_\alpha$. In particular, for any $n \geq 1$, there exists $x_n \in X$ such that for any $\alpha \in J$, we have $B_{x_n}(\frac{1}{n},t) \nsubseteq O_\alpha$. By our assumption on X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some point $x \in X$. Since the family $\{O_\alpha\}_{\alpha\in J}$ covers X, there exists $\alpha_0 \in J$ such that $x \in O_{\alpha_0}$. Since O_{α_0} is open, there exists $0 < r_0 < 1$, and $t_0 > 0$ such that $B_x(r_0,t) \subset O_{\alpha_0}$. Fix $t > 0$ and let $t_1 = tK$, for any $n_K \geq 1$ and $a \in B_{x_{n_K}}(\frac{1}{n_K},t) = B_{x_{n_K}}(\frac{1}{n_K},\frac{t_1}{K})$, we have

$$F_{x,a}(t_0) \geq T(F_{x,x_{n_k}}(\frac{t_0-t_1}{K}), F_{x_{n_k},a}(\frac{t_1}{K})) > T(F_{x,x_{n_k}}(\frac{t_0-t_1}{K}), 1-\frac{1}{n_k})$$

for n_k large enough, we will get $F_{x,a}(t) > 1 - r_0$ for any $a \in B_{x_{n_k}}(\frac{1}{n_k},t)$. In the other words, we have $B_{x_{n_k}}(\frac{1}{n_k},t) \subset B_x(r_0,t_0)$, which implies $B_{x_{n_k}}(\frac{1}{n_k},t) \subset O_{\alpha_0}$. This is in clear contradiction with the way the sequence $\{x_n\}$ was constructed. Therefore, there exists $0 < r_0 < 1$ such that for any $x \in X$, there exists $\alpha \in J$ such that $B_x(r_0,t) \subset O_\alpha$. For such r_0, there exist $x_1, x_2, ..., x_n \in X$ such that

$$X \subset B_{x_1}(r_0,t) \cup ... \cup B_{x_n}(r_0,t).$$

But for any $i = 1, ..., n$, there exists $\alpha \in J$ such that $B_{x_i}(r_0,t) \subset O_{\alpha_i}$, i.e., $X \subset O_{\alpha_1} \cup ... \cup O_{\alpha_n}$. This completes the proof that X is compact. □

Definition 3.4. The subset X is called sequentially compact if and only if for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{n_k\to\infty} x_{n_k} \in X$. Also X is called totally bounded if for any $0 < r < 1$ and $t > 0$, there exist $x_1, x_2, ..., x_n \in X$ such that

$$X \subset B_{x_1}(r,t) \cup ... \cup B_{x_n}(r,t).$$

In the above proof we showed the following result.

Theorem 3.3. *Let (M, F, T) be a PMT space and τ_F the topology defined above. Let X be a nonempty subset of M.*

(1) *X is compact if and only if X is sequentially compact.*
(2) *If X is compact, then X is totally bounded.*

4. **KKM maps in PMT spaces**

For a set X, we denote the set of all nonempty finite subsets of X by $\langle X\rangle$. Let A be a nonempty p-bounded subset of PMT space (M,F,T). Then

(1) $co\,(A) = \cap\{B \subset M,\, B$ is a closed ball in M such that $A \subset B\}$.
(2) $\mathbb{A}(M) = \{A \subset M,\, A = co\,(A)\}$, i.e. $A \in \mathbb{A}(M)$ if and only if A is an intersection of all closed balls containing A. In this case, we say that A is an admissible set in M.
(3) A is called subadmissible, if for each $D \subset \langle A\rangle$, $co(D) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

Let (M,F,T) be a PMT space and X a subadmissible subset of M. A set-valued mapping $G : X \to 2^M$ is called a KKM mapping, if for each $A \in \langle X\rangle$, we have $co\,(A) \subset G(A) = \cup\{G(a), a \in A\}$. More generally, if Y is a topological space and $G : X \to 2^Y$, $S : X \to 2^Y$ are two set-valued mappings such that for any $A \in \langle X\rangle$, we have $S(co\,(A)) \subset G(A)$, then G is called a generalized KKM mapping with respect to S. If the set-valued mapping $S : X \to 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \to 2^Y$ with respect to S the family $\{\overline{G(x)}, x \in X\}$ has the finite intersection property, then S is said to have the KKM property. We define

$$KKM(X,Y) = \{S : X \to 2^Y,\ S \text{ has the } KKM \text{ property }\}.$$

Let X be a nonempty subset of a PMT space M. Then $S : X \to 2^M$ is said to have the approximate fixed point property if for any $0 < r < 1$ and $t > 0$, there exists an $x \in X$ such that $S(x) \cap B_x(r,t) \neq \emptyset$, i.e. there exists $y \in S(x)$ such that $F_{x,y}(t) > 1 - r$. We now establish the approximate fixed point property of KKM-type mapping on a subadmissible subset of a PMT space.

Theorem 4.1. *Let (M,F,T) be a PMT space and X a nonempty subadmissible subset of M. Let $S \in KKM(X,X)$ be such that $\overline{S(X)}$ is totally bounded. Then S has the approximate fixed point property.*

Proof. Set $Y = \overline{S(X)} \subset \overline{X}$. Since Y is totally bounded, fix $t > 0$ then for any $0 < r < 1$ and $t > 0$, there exists a finite subset $A \subset X$ such that $Y \subseteq \bigcup_{x\in A} B_x(r, \frac{t}{2})$. Define $G : X \to 2^X$ by

$$G(x) = Y \bigcap \overline{B_x(r,Kt)^c}$$

where Z^c is the complement of Z in M. Clearly $G(x)$ is closed. Note that for any $x \in M$, we have

$$B_x(r, \frac{t}{2}) \subset \overline{B_x(r,Kt)^c}^{\,c} \subset B_x(r,Kt).$$

Indeed, let $y \in B_x(r, \frac{t}{2})$. Assume that $y \notin \overline{B_x(r,Kt)^c}^{\,c}$, i.e., $y \in \overline{B_x(r,Kt)^c}$. From the properties of the closure in PMT spaces, there exists a sequence $\{y_n\} \in B_x(r,Kt)^c$ such that $\lim_{n\to\infty} y_n = y$. Hence

$$1 - r \geq F_{x,y_n}(Kt) \geq T(F_{x,y}(\frac{t}{2}), F_{y,y_n}(\frac{t}{2}))$$

If we let $n \to \infty$, we get $1 - r \geq F_{x,y}(\frac{t}{2})$. This is a contradiction to $y \in B_x(r, \frac{t}{2})$. Hence

$$B_x(r, \frac{t}{2}) \subset \overline{B_x(r,Kt)^c}^{\,c} \quad .$$

Next let $y \in \overline{B_x(r,Kt)^c}^{\,c}$. Let us prove that $y \in B_x(r,Kt)$. Assume not, i.e., $y \notin B_x(r,Kt)$. Hence $y \in B_x(r,Kt)^c$, which implies $y \in \overline{B_x(r,Kt)^c}$. This is a contradiction with $y \in \overline{B_x(r,Kt)^c}^{\,c}$. Therefore, we have

$$\overline{B_x(r,Kt)^c}^{\,c} \subset B_x(r,Kt).$$

On the other hand, since $Y \subset \bigcup_{x\in A} B_x(r, \frac{t}{2})$, then we have $\bigcap_{x\in A} G(x) = \emptyset$. So G is not a generalized KKM mapping with respect to S. Since $S \in KKM(X,X)$, there exists a finite nonempty subset $B \subset X$ such that

$$S(co(B)) \nsubseteq \bigcup_{x\in B} G(x).$$

So there exists $x_0 \in S(co(B))$ such that $x_0 \notin G(x)$ for any $x \in B$.

In other words, we have $x_0 \in \overline{B_x(r,Kt)^c}^c$, for any $x \in B$. Hence $x_0 \in B_x(r, Kt)$ for any $x \in B$ or $B \subset B_x(r, Kt)$. By the definition of $co(B)$ we deduce that $co(B) \subset B_x(r, Kt)$. Since $x_0 \in S(co(B))$, there exists $x_r \in co(B)$ such that $x_0 \in S(x_r)$. But $x_r \in co(B) \subset B_{x_0}(r, Kt)$, gives $F_{x_0,x_r}(Kt) \geq 1 - r$. Therefore, we have proved

$$S(x_r)\bigcap B_{x_r}(r, Kt) \neq \emptyset.$$

Since $0 < r < 1$ and $t > 0$ were arbitrary, the proof of the theorem is complete. □

As a direct consequence of this result, we get the following fixed point result.

Theorem 4.2. *Let (M, F, T) be a PMT space and X a nonempty subadmissible subset of M. Let $S \in KKM(X,X)$ be closed and compact. Then S has a fixed point, i.e. there exists $x \in X$ such that $x \in S(x)$.*

Proof. Since S is compact, then $\overline{S(X)}$ is compact. Hence $\overline{S(X)}$ is totally bounded. The previous theorem implies the existence of $x_r \in X$ such that

$$S(x_r)\bigcap B_{x_r}(r, Kt) \neq \emptyset \quad ,$$

for any $0 < r < 1$ and $t > 0$. In particular, for any $n \geq 1$, there exists $x_n \in X$ such that

$$S(x_n)\bigcap B_{x_n}(\frac{1}{n}, Kt) \neq \emptyset \quad .$$

Hence there exists $y_n \in S(x_n)$ such that $F_{x_n,y_n}(Kt) > 1 - \frac{1}{n}$, for any $n \geq 1$. Since S is compact, there exists a subsequence $\{y_{n_k}\}$ which is convergent to y. Clearly we have $\{x_{n_k}\}$ is also convergent to y. Since $\{(x_n, y_n)\} \in Gr(S)$ and $Gr(S)$ is closed, then $(y, y) \in Gr(S)$, i.e. $y \in S(y)$ where $Gr(S)$ denotes the graph of the mapping S. □

Before we obtain on further results, we would like to give an example to support Theorem 4.2.

Example 4.1. *Assume that $M := \Re$ and $(\Re, F, T_p)$ be a PMT space similar to Example 2.4. Let $X = [0, 1]$ and define a map $S : X \to 2^X$ by*

$$S(x) = \begin{cases} [1-x, 1] & \text{if } x \in [0, \frac{1}{2}), \\ \{1\} & \text{if } x = \frac{1}{2}, \\ [0, 1-x] \cup \{1\} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, we have X being subadmissible and S being closed and compact. Now, let $G : X \to 2^X$ be a given generalized KKM map with respect to S. It is clear that $S(x) \subset G(x)$ for all $x \in X$. Since S has the finite intersection property, so does G. Therefore, we have $S \in KKM(X,X)$. In view of Theorem 4.2, S has a fixed point.

The following lemma will be useful to prove Schauder's type fixed point theorem for PMT spaces.

Lemma 4.1. *Let (M, F, T) be a PMT space and X a nonempty subadmissible subset of M. Suppose that Y is a topological space, $S \in KKM(X, Y)$ and $f : Y \to X$ is continuous, then $f \circ S \in KKM(X,X)$.*

Proof. Let $G : X \to 2^X$ be generalized KKM mappings with respect to $f \circ S$ such that $G(x)$ is closed for each $x \in X$. Then, for any finite subset $\{x_1, x_2, ..., x_n\}$ of X, since G is a generalized KKM mapping with respect to $f \circ S$ we have $f \circ S(co\{x_1, x_2, ..., x_n\}) \subset \bigcup_{1 \le i \le n} G(x_i)$. Hence

$$S(co\{x_1, x_2, ..., x_n\}) \subset \bigcup_{1 \le i \le n} f^{-1}(G(x_i)) \quad .$$

Therefore, $f^{-1}(G)$ is a generalized KKM mapping with respect to S. Since $S \in KKM(X, Y)$, then the family $\{f^{-1}(G(x)),\, x \in X\}$ has the finite intersection property since f is continuous. This will imply that the family $\{G(x),\, x \in X\}$ has the finite intersection property. This shows that $f \circ S \in KKM(X, X)$. □

Corollary 4.1. *Let (M, F, T) be a PMT space and X a nonempty subadmissible subset of M. Suppose that the identity mapping $I : X \to X$ belongs to $KKM(X, X)$, then any continuous mapping $f : X \to X$ such that $\overline{f(X)}$ is compact, has a fixed point.*

Proof. Since $I \in KKM(X, X)$, and f is continuous, then by Lemma 4.4, $f \in KKM(X, X)$. Using that $\overline{f(X)}$ is compact and every continuous map is closed, we conclude by Theorem 4.2 that f has a fixed point. □

5. **Applications**

In this section as an application of the PMT-KKM principle, we give the version of Fan's best approximation in nonexpansive retraction probabilistic metric type spaces ($\mathcal{NR}$-PMT spaces).

Definition 5.1. A PMT space (M, F, T_M) is called $\mathcal{NR}$-PMT space if there exists a closed convex subset (W, μ, T_M) of a completely probabilistic metrizable topological vector space (V, μ, T_M), in which

$$\mu_{\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2}(t) \ge T_M(\mu_{x_1, y_1}(t), \mu_{x_2, y_2}(t))$$

for all $x_1, x_2, y_1, y_2 \in W$, $\alpha + \beta = 1$, $\alpha, \beta \ge 0$, and $t > 0$ such that (M, F, T_M) is isometrically embedded into (W, μ, T_M) and there exists a nonexpensive retraction $r : W \to M$.

Lemma 5.1. *Let (M, F, T_M) be an $\mathcal{NR}$-PMT space, then $r(convA) \subseteq co(A)$ for any $A \in \langle M \rangle$, where $convA$ means the convex hull of A.*

Proof. Since each closed ball in (W, μ, T_M) is convex, then $convA \subset \cap\{B^W_{x_\alpha}[r_\alpha, t] \,:\, A \subset B^W_{x_\alpha}[r_\alpha, t]\}$. Therefore,

$$\begin{aligned} r(convA) &\subset r(\cap\{B^W_{x_\alpha}[r_\alpha, t] \,:\, A \subset B^W_{x_\alpha}[r_\alpha, t]\,,\, x_\alpha \in M\}) \\ &\subseteq \cap\{B^W_{x_\alpha}[r_\alpha, t] \,:\, A \subset B^W_{x_\alpha}[r_\alpha, t]\,,\, x_\alpha \in M\} = co(A). \end{aligned}$$

□

The above Lemma tells that in every $\mathcal{NR}$-PMT space (M, F, T_M) and for any subadmissible subset X of M, the identity mapping belongs to $KKM(X, X)$. This result will be called Fan's Lemma.

Theorem 5.1. *Let $X \in \mathbb{A}(M)$ be compact subset of an $\mathcal{NR}$-PMT space (M, F, T_M). Suppose that $S : X \to 2^M$ is continuous with admissible values, then there exists an $x_0 \in X$, such that $F_{x_0, S(x_0)}(t) = \sup_{x \in X} F_{x, S(x_0)}(t)$ for $t > 0$. In particular, if $S(x_0)$ is compact and $x_0 \notin S(x_0)$, then x_0 must be a boundary point of X.*

Proof. Let $r : W \to M$ be a nonexpansive retraction as in Definition 5.1. Define $G : X \to 2^X$ by $G(y) = \{x \in X : F_{x,S(x)}(t) \geq F_{y,S(x)}(t)\}$. Since S is continuous then $G(y)$ is closed. (In fact, let $x_n \in G(y)$ such that $x_n \to x$, we want to show that $x \in G(y)$. Since $x_n \to x$ and S is continuous then $S(x_n) \to S(x)$. Since $x_n \in G(y)$ then $F_{x_n,S(x_n)}(t) \geq F_{y,S(x_n)}(t)$. Take limit for both sides, we get $F_{x,S(x)}(t) \geq F_{y,S(x)}(t)$ which implies that $x \in G(y)$). Since X is compact and $G(y)$ is closed subset of X, then $G(y)$ is compact.

Now we will show that $r^{-1}G : X \subseteq W \to 2^W$ is a KKM mapping (i.e $co(A) \subset r^{-1}G(A)$, $\forall A \in \langle X\rangle$ or $r(co(A)) \subset G(A)$). Let $A = \{y_1, y_2, ..., y_n\} \in< X >$ and $y_0 \notin \cup_{k=1}^n G(y_k)$. If $G(A) = X$ then $co(A) \subset X = G(A)$ and hence, there is nothing to prove. Let $y_0 \notin G(A) = \cup_{k=1}^n G(y_k)$. Then we have by definition of G, $F_{y_0,S(y_0)}(t) < F_{y_k,S(y_0)}(t)$, $\forall k = 1, 2, ..., n$. Let $J(y_0) = \{y \in X : F_{y_0,S(y_0)}(t) < F_{y,S(y_0)}(t)\}$. In particular $A \subset J(y_0)$. Take $z_k \in S(y_0)$ such that for $k = 1, 2, ..., n$, $F_{y_0,S(y_0)}(t) < F_{y_k,z_k}(t)$. This is possible by using the definition $F_{y_k,S(y_0)}(t) = \sup_{z\in S(x_0)} F_{y_k,z}(t)$ and that $F_{y_0,S(y_0)}(t) < F_{y_k,S(y_0)}(t)$. Let $\lambda_k > 0$ and $\sum_{k=1}^n \lambda_k = 1$. Then we have

$$\begin{aligned} F_{r(\sum_{k=1}^n \lambda_k y_k),r(\sum_{k=1}^n \lambda_k z_k)}(t) &\geq \mu_{\sum_{k=1}^n \lambda_k y_k,\sum_{k=1}^n \lambda_k z_k}(t) \\ &\geq \min_{1\leq k\leq n} F_{y_n,z_k}(t) > F_{y_0,S(y_0)}(t). \qquad (1) \end{aligned}$$

By Lemma 5.2, $r(\sum_{k=1}^n \lambda_k y_k) \in co(\{z_1, ..., z_n\})$ and since $S(y_0)$ is subadmissible $co(\{z_1, ..., z_n\}) \subset S(y_0)$, we have $r(\sum_{k=1}^n \lambda_k y_k) \in S(y_0)$ and from (1) $F_{r(\sum_{k=1}^n \lambda_k y_k),S(y_0)}(t) > F_{y_0,S(y_0)}(t)$. Hence, we deduce that $r(\sum_{k=1}^n \lambda_k y_k) \in J(y_0)$. As $y_0 \notin J(y_0)$, we have $y_0 \notin r(conv(\{y_1, ..., y_n\}))$. Consequently, $r(conv(\{y_1, ..., y_n\})) \subset \cup_{k=1}^n (y_k)$ implies $conv(\{y_1, ..., y_n\}) \subset \cup_{k=1}^n r^{-1}G(y_k)$. This implies that $r^{-1}G$ is a KKM mapping. By Fan's Lemma mentioned after Lemma 5.2, which says that $I \in KKM(X, X)$, the $family\{\overline{r^{-1}G(x)} : x \in X\} = \{r^{-1}G(x) : x \in X\}$ has the finite intersection property, and therefore the $family\{G(x) : x \in X\}$. has the finite intersection property. The compactness of $G(x)$ for each $x \in X$ implies that there exists an $x_0 \in \cap_{y\in X} G(y)$. Hence, $F_{x_0,S(x_0)}(t) \geq F_{y,S(x_0)}(t)$, for all $y \in X$. Which implies $F_{x_0,S(x_0)}(t) = \sup_{y\in X} F_{y,S(x_0)}(t)$ for $t > 0$.

If $x_0 \notin S(x_0)$ and $S(x_0)$ is compact, then there exists $u_0 \in S(x_0)$ such that $F_{x_0,u_0}(t) = F_{x_0,S(x_0)}(t)$. In this case we will show that $x_0 \in \partial X$. Suppose that $x_0 \in IntX$. Then there exists $0 < r < 1$, such that $B_x(r, t) \subset IntX \subset X$ and $0 < r < F_{y,S(x_0)}(t) \leq F_{x_0,S(x_0)}(t)$, for all $y \in B_{x_0}(r, t)$ for $t > 0$. Then, it is clear that$B^W_{x_0}(r, t) \cap B^W_{u_0}(F_{x_0,S(x_0)}(t) - r, t) \neq \emptyset$.

$$\begin{aligned} \emptyset &\neq r(B^W_{x_0}(r, t) \cap B^W_{u_0}(F_{x_0,S(x_0)}(t) - r, t)) \\ &\subseteq r(B^W_{x_0}(r, t)) \cap r(B^W_{u_0}(F_{x_0,S(x_0)}(t) - r, t)) \\ &\subseteq B^M_{x_0}(r, t) \cap B^M_{u_0}(F_{x_0,S(x_0)}(t) - r, t). \end{aligned}$$

And hence, $B^M_{x_0}(r, t) \cap B^M_{u_0}(F_{x_0,S(x_0)}(t) - r, t) \neq \emptyset$. If y is any element of this intersection, then $y \in X$ and since $u_0 \in S(x_0)$ implies $F_{y,S(x_0)}(t) > F_{x_0,S(x_0)}(t)$ which is a contradiction. Therefore x_0 must be a boundary point of X. □

Theorem 5.2. *Let $X \in \mathbb{A}(M)$ be a compact subset of an $\mathcal{NR}$-PMT space (M, F, T_M). Suppose that $S : X \to \mathbb{A}(M)$ is continuous. Then S has a fixed point if one of the following conditions holds for all $x \in \partial X$ such that $x \notin S(x)$:*

(1) *There exists a $y \in X$ such that $F_{y,S(x)}(t) > F_{x,S(x)}(t)$.*
(2) *There exists an $\alpha > 1$ such that $\alpha F_{x,S(x)}(t) < 1$ and $X \cap B_{S(x)}(1 - \alpha F_{x,S(x)}(t), t) \neq \emptyset$.*
(3) *$S(x) \cap X \neq \emptyset$.*

Proof. (1) Suppose S has no fixed point. Then by Theorem 5.3, there exist an $x_0 \in \partial X$ such that $0 < F_{y,S(x_0)}(t) \leq F_{x_0,S(x_0)}(t)$ for all $y \in X$ which contradicts condition.
(2) For any $x \in X$ such that $x \notin S(x)$, there exists a $y \in X$ such that $y \in X \cap B_{S(x)}(1 - \alpha F_{x,S(x)}(t), t)$ which implies $F_{y,S(x)}(t) \geq 1 - (1 - \alpha F_{x,S(x)}(t)) > 1 - (1 - F_{x,S(x)}(t)) =$

$F_{x,S(x)}(t)$ which (2) implies (1).
(3) (3) $\Rightarrow$ (2) trivial. $\square$

6. **Conclusions**

In this paper, we introduced a notion of KKM mapping in probabilistic metric type spaces. As application, some existence theorems of solutions for fixed point theorem are obtained. Also, we defined $\mathcal{NR}$-probabilistic metric type spaces and we obtained a version of Fan's best approximation theorem on these spaces.

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