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BRINGING TOGETHER DUAL-GENERALIZED COMPLEX NUMBERS AND DUAL QUATERNIONS VIA FIBONACCI AND LUCAS NUMBERS

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In this study, the main target is to construct a bridge between dual quaternions and dual-generalized complex numbers via Fibonacci and Lucas numbers for $\mathfrak{p} \in \mathbb{R}$. For this purpose, the algebraic structures and the well-known recurrence relations are investigated. Different matrix representations are improved, and examples are presented.

Keywords: Dual quaternion, Dual-generalized complex number, Fibonacci number, Lucas number.

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1. **Introduction**

Dual quaternions, as an extension of real quaternions ([17]), are expressed as $Q = a + be_1 + ce_2 + de_3\,(a, b, c, d \in \mathbb{R})$ with quaternionic units $\{e_1, e_2, e_3\}$ which satisfy the conditions ([25]):

$$e_1{}^2 = e_2{}^2 = e_3{}^2 = 0,\; e_1e_2 = -e_2e_1 = e_2e_3 = -e_3e_2 = e_3e_1 = -e_1e_3 = 0.$$

In addition, as an extension of Hamilton's idea ([20]), *nth* Fibonacci and Lucas quaternions are introduced in [15, 19, 21, 22]. In [33], *nth* dual Fibonacci and Lucas quaternions are studied, while, complex and dual numbers with the Fibonacci quaternion coefficients are defined in [16, 27].

In other respects, generalized complex ($\mathcal{GC}$) numbers have the form [5, 18]:

$$\mathbb{C}_{\mathfrak{p}} = \left\{ z_1 = x_1 + x_2J \middle| \; x_1, x_2 \in \mathbb{R},\; J^2 = \mathfrak{p},\; -\infty < \mathfrak{p} < \infty \right\},$$

which is referred to as a $\mathfrak{p}$-complex plane. $\mathbb{C}_{\mathfrak{p}}$ (vector space over $\mathbb{R}$) is an *elliptic*, *parabolic* and *hyperbolic complex number system* for $\mathfrak{p} < 0, \mathfrak{p} = 0$, and $\mathfrak{p} > 0$, respectively. The set of complex numbers [32], hyperbolic numbers [7,11,30] and dual numbers [28,31] are obtained for specific values of $\mathfrak{p} = -1, \mathfrak{p} = 0$, and $\mathfrak{p} = 1$, respectively. For constructing new systems, many pieces of research have been conducted using these numbers as coefficients, [1–3, 6, 8–10, 12, 23, 24, 26, 29]. So, the set of dual-generalized complex numbers is defined in [14] as:

$$\mathbb{DC}_{\mathfrak{p}} = \left\{ a = z_1 + z_2\varepsilon \middle| \; z_1 = x_1 + x_2J, z_2 = x_3 + x_4J \in \mathbb{C}_{\mathfrak{p}},\; \varepsilon^2 = 0,\; \varepsilon \neq 0 \right\},$$

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where J denotes the generalized complex unit ($J^2 = \mathfrak{p}$), ε represents the pure dual unit, and $J\varepsilon = \varepsilon J$ represents the generalized complex-dual unit. $\mathbb{DC}_{\mathfrak{p}}$[1] is analogous to the dual-complex numbers for $\mathfrak{p} = -1$ [6, 26], the dual-hyperbolic numbers for $\mathfrak{p} = 1$ [1], and the hyper-dual numbers for $\mathfrak{p} = 0$ [9, 10]. In [13], the *nth* dual-generalized complex Fibonacci/Lucas numbers are defined as:

$$\widetilde{\mathfrak{F}}_n = F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon, \tag{1}$$

$$\widetilde{\mathfrak{L}}_n = L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon \tag{2}$$

and the recurrence relationships between of them are introduced.

This paper concerns dual quaternions with dual-generalized complex ($\mathcal{DGC}$) Fibonacci and Lucas numbers for $\mathfrak{p} \in \mathbb{R}$. Algebraic structures in the form of, Binet's formulas, and D'Ocagne's, Catalan's and Cassini's identities are taken into account. Moreover, different matrix representations are examined and examples are given to enhance intelligibility. The outstanding part of this paper is that for

$\star$ $\mathfrak{p} = -1$, dual-complex $\star$ $\mathfrak{p} = 0$, hyper-dual $\star$ $\mathfrak{p} = 1$, dual-hyperbolic Fibonacci and Lucas numbers-based dual quaternions can be obtained.

2. Dual-Generalized Complex Fibonacci and Lucas Numbers- Based Dual Quaternions

Definition 2.1. *The $\mathcal{DGC}$ numbers with dual Fibonacci and Lucas quaternion coefficients are defined, respectively, as follows:*

$$\begin{aligned} \widetilde{\mathfrak{Q}}_n &= Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon, \\ \widetilde{\mathcal{K}}_n &= K_n + K_{n+1}J + K_{n+2}\varepsilon + K_{n+3}J\varepsilon, \end{aligned}$$

where $Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$ is the nth dual Fibonacci quaternion, $K_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$ is the nth dual Lucas quaternion, $F_n = F_{n+1} - F_{n-1}$ is the nth Fibonacci number, and $L_n = L_{n+1} - L_{n-1}$ is the nth Lucas number ($n \geq 1, F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1$). The base elements $\{1, J, \varepsilon, J\varepsilon\}$ and the quaternionic units $\{e_1, e_2, e_3\}$ satisfy the conditions given in Table 1.

Table 1. Multiplication scheme

$\triangle$	1	J	ε	$J\varepsilon$	e_1	e_2	e_3
1	1	J	ε	$J\varepsilon$	e_1	e_2	e_3
J	J	$\mathfrak{p}$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	Je_1	Je_2	Je_3
ε	ε	$J\varepsilon$	0	0	εe_1	εe_2	εe_3
$J\varepsilon$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	0	0	$J\varepsilon e_1$	$J\varepsilon e_2$	$J\varepsilon e_3$
e_1	e_1	Je_1	εe_1	$J\varepsilon e_1$	0	0	0
e_2	e_2	Je_2	εe_2	$J\varepsilon e_2$	0	0	0
e_3	e_3	Je_3	εe_3	$J\varepsilon e_3$	0	0	0

[1] $\mathbb{DC}_{\mathfrak{p}}$ is a vector space over $\mathbb{R}$. For $a_1 = z_{11} + z_{12}\varepsilon, a_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $\lambda \in \mathbb{R}$, the operations are given as follows [14]:

$$\begin{aligned} \text{Equality}: &\quad a_1 = a_2 &&\Leftrightarrow z_{11} + z_{12}\varepsilon = z_{21} + z_{22}\varepsilon \Leftrightarrow z_{11} = z_{21}, z_{12} = z_{22}, \\ \text{Addition}: &\quad a_1 \oplus a_2 &&= (z_{11} + z_{12}\varepsilon) \oplus (z_{21} + z_{22}\varepsilon) = (z_{11} + z_{21}) + (z_{12} + z_{22})\varepsilon, \\ \text{Scalar multiplication}: &\quad \lambda \odot a_1 &&= \lambda \odot (z_{11} + z_{12}\varepsilon) = (\lambda z_{11}) + (\lambda z_{12})\varepsilon, \\ \text{Multiplication}: &\quad a_1 a_2 &&= (z_{11} + z_{12}\varepsilon)(z_{21} + z_{22}\varepsilon) = (z_{11}z_{21}) + (z_{11}z_{22} + z_{12}z_{21})\varepsilon. \end{aligned}$$

TABLE 2. Structures for $\widetilde{\mathfrak{Q}}_n$ as a $\mathfrak{DGC}$ number

Addition		$\widetilde{\mathfrak{Q}}_n \pm \widetilde{\mathfrak{Q}}_m = (Q_n \pm Q_m) + (Q_{n+1} \pm Q_{m+1})\, J + (Q_{n+2} \pm Q_{m+2})\,\varepsilon + (Q_{n+3} \pm Q_{m+3})\, J\varepsilon$
Multiplication		$\widetilde{\mathfrak{Q}}_n \widetilde{\mathfrak{Q}}_m = (Q_n Q_m + \mathfrak{p} Q_{n+1} Q_{m+1}) + (Q_n Q_{m+1} + Q_{n+1} Q_m)\, J + (Q_n Q_{m+2} + \mathfrak{p} Q_{n+1} Q_{m+3} + Q_{n+2} Q_m + \mathfrak{p} Q_{n+3} Q_{m+1})\,\varepsilon + (Q_n Q_{m+3} + Q_{n+1} Q_{m+2} + Q_{n+2} Q_{m+1} + Q_{n+3} Q_m)\, J\varepsilon$
Equality		$\widetilde{\mathfrak{Q}}_n = \widetilde{\mathfrak{Q}}_m \Leftrightarrow Q_n = Q_m \wedge Q_{n+1} = Q_{m+1} \wedge Q_{n+2} = Q_{m+2} \wedge Q_{n+3} = Q_{m+3}$
Conjugates	$\mathfrak{GC}$	$\widetilde{\mathfrak{Q}}_n^{\dagger_1} = Q_n - Q_{n+1} J + Q_{n+2}\varepsilon - Q_{n+3} J\varepsilon$
	Dual	$\widetilde{\mathfrak{Q}}_n^{\dagger_2} = Q_n + Q_{n+1} J - Q_{n+2}\varepsilon - Q_{n+3} J\varepsilon$
	Coupled	$\widetilde{\mathfrak{Q}}_n^{\dagger_3} = Q_n - Q_{n+1} J - Q_{n+2}\varepsilon + Q_{n+3} J\varepsilon$
	$\mathfrak{DGC}$	$\widetilde{\mathfrak{Q}}_n^{\dagger_4} = (Q_n - Q_{n+1} J)\left(1 - \frac{Q_{n+2} + Q_{n+3} J}{Q_n + Q_{n+1} J}\varepsilon\right)$
	Anti-dual	$\widetilde{\mathfrak{Q}}_n^{\dagger_5} = Q_{n+2} + Q_{n+3} J - Q_n\varepsilon - Q_{n+1} J\varepsilon$
Modules	$\mathfrak{GC}$	$\lvert\widetilde{\mathfrak{Q}}_n\rvert^2_{\dagger_1} = \widetilde{\mathfrak{Q}}_n \widetilde{\mathfrak{Q}}_n^{\dagger_1}$
	Dual	$\lvert\widetilde{\mathfrak{Q}}_n\rvert^2_{\dagger_2} = \widetilde{\mathfrak{Q}}_n \widetilde{\mathfrak{Q}}_n^{\dagger_2}$
	Coupled	$\lvert\widetilde{\mathfrak{Q}}_n\rvert^2_{\dagger_3} = \widetilde{\mathfrak{Q}}_n \widetilde{\mathfrak{Q}}_n^{\dagger_3}$
	$\mathfrak{DGC}$	$\lvert\widetilde{\mathfrak{Q}}_n\rvert^2_{\dagger_4} = \widetilde{\mathfrak{Q}}_n \widetilde{\mathfrak{Q}}_n^{\dagger_4}$

Remark 2.1. *Every $\widetilde{\mathfrak{Q}}_n$ and $\widetilde{\mathfrak{K}}_n$ can also be rewritten as, respectively:*

$$\begin{aligned} \widetilde{\mathfrak{Q}}_n &= \widetilde{\mathfrak{F}}_n + \widetilde{\mathfrak{F}}_{n+1} e_1 + \widetilde{\mathfrak{F}}_{n+2} e_2 + \widetilde{\mathfrak{F}}_{n+3} e_3, \\ \widetilde{\mathfrak{K}}_n &= \widetilde{\mathcal{L}}_n + \widetilde{\mathcal{L}}_{n+1} e_1 + \widetilde{\mathcal{L}}_{n+2} e_2 + \widetilde{\mathcal{L}}_{n+3} e_3, \end{aligned}$$

where $\widetilde{\mathfrak{F}}_n$ is nth $\mathfrak{DGC}$ Fibonacci number and $\widetilde{\mathcal{L}}_n$ is nth $\mathfrak{DGC}$ Lucas number given in Eqs. (1) *and* (2)*, respectively ([13]). It is obvious that, there is no difference between $\mathfrak{DGC}$ numbers with dual Fibonacci/Lucas quaternions (Table 2) and dual quaternion with $\mathfrak{DGC}$ Fibonacci/Lucas numbers (Table 3). The analog of Table 2 and Table 3 can given similarly for Lucas numbers.*

Theorem 2.1. *Let $\widetilde{\mathfrak{Q}}_n \in \mathbb{QDC}_{\mathfrak{p}}$ and $\widetilde{\mathfrak{K}}_n \in \mathbb{KDC}_{\mathfrak{p}}$. The following additional recurrence relationships then hold for $n, r \geq 0$:*

1. $\widetilde{\mathfrak{Q}}_n + \widetilde{\mathfrak{Q}}_{n+1} = \widetilde{\mathfrak{Q}}_{n+2}$,
2. $\widetilde{\mathfrak{Q}}_{n+r} + \widetilde{\mathfrak{Q}}_{n-r} = \begin{cases} L_r \widetilde{\mathfrak{Q}}_n, & r = 2k \\ F_r \widetilde{\mathfrak{K}}_n, & r = 2k+1, \end{cases}$
3. $\widetilde{\mathfrak{Q}}_{n+r} - \widetilde{\mathfrak{Q}}_{n-r} = \begin{cases} F_r \widetilde{\mathfrak{K}}_n, & r = 2k \\ L_r \widetilde{\mathfrak{Q}}_n, & r = 2k+1, \end{cases}$
4. $\widetilde{\mathfrak{Q}}_n - \widetilde{\mathfrak{Q}}_{n+1} e_1 - \widetilde{\mathfrak{Q}}_{n+2} e_2 - \widetilde{\mathfrak{Q}}_{n+3} e_3 = \widetilde{\mathfrak{F}}_n$,
5. $\widetilde{\mathfrak{K}}_n + \widetilde{\mathfrak{K}}_{n+1} = \widetilde{\mathfrak{K}}_{n+2}$,

Nomenclature

Fibonacci numbers	$\mathbb{F} = \{F_n \mid F_n \text{ is } nth \text{ Fibonacci number}\}$
Lucas numbers	$\mathbb{L} = \{L_n \mid L_n \text{ is } nth \text{ Lucas number}\}$
$\mathfrak{GC}$ Fibonacci numbers	$\mathbb{C}_{\mathfrak{p}}\mathbb{F} = \{F_n + F_{n+1} J \mid F_n \in \mathbb{F}\}$
$\mathfrak{GC}$ Lucas numbers	$\mathbb{C}_{\mathfrak{p}}\mathbb{L} = \{L_n + L_{n+1} J \mid L_n \in \mathbb{L}\}$
$\mathfrak{DGC}$ Fibonacci numbers [13]	$\mathbb{DC}_{\mathfrak{p}}\mathbb{F} = \{\widetilde{\mathfrak{F}}_n = F_n + F_{n+1} J + F_{n+2}\varepsilon + F_{n+3} J\varepsilon \mid F_n \in \mathbb{F}\}$
$\mathfrak{DGC}$ Lucas numbers [13]	$\mathbb{DC}_{\mathfrak{p}}\mathbb{L} = \{\widetilde{\mathcal{L}}_n = L_n + L_{n+1} J + L_{n+2}\varepsilon + L_{n+3} J\varepsilon \mid L_n \in \mathbb{L}\}$
Fibonacci dual quaternions [33]	$\mathbb{Q} = \{Q_n = F_n + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3 \mid F_n \in \mathbb{F}\}$
Lucas dual quaternions [33]	$\mathbb{K} = \{K_n = L_n + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3 \mid L_n \in \mathbb{L}\}$
$\mathfrak{GC}$ numbers with Fibonacci dual quaternion	$\mathbb{C}_{\mathfrak{p}}\mathbb{Q} = \{Q_n + Q_{n+1} J \mid Q_n \in \mathbb{Q}\}$
$\mathfrak{GC}$ numbers with Lucas dual quaternion	$\mathbb{C}_{\mathfrak{p}}\mathbb{K} = \{K_n + K_{n+1} J \mid K_n \in \mathbb{K}\}$
$\mathfrak{DGC}$ numbers with Fibonacci dual quaternion	$\mathbb{DC}_{\mathfrak{p}}\mathbb{Q} = \{\widetilde{\mathfrak{Q}}_n = Q_n + Q_{n+1} J + Q_{n+2}\varepsilon + Q_{n+3} J\varepsilon \mid Q_n \in \mathbb{Q}\}$
$\mathfrak{DGC}$ numbers with Lucas dual quaternion	$\mathbb{DC}_{\mathfrak{p}}\mathbb{K} = \{\widetilde{\mathfrak{K}}_n = K_n + K_{n+1} J + K_{n+2}\varepsilon + K_{n+3} J\varepsilon \mid K_n \in \mathbb{K}\}$
Dual quaternions with $\mathfrak{DGC}$ Fibonacci number	$\mathbb{QDC}_{\mathfrak{p}} = \{\widetilde{\mathfrak{Q}}_n = \widetilde{\mathfrak{F}}_n + \widetilde{\mathfrak{F}}_{n+1} e_1 + \widetilde{\mathfrak{F}}_{n+2} e_2 + \widetilde{\mathfrak{F}}_{n+3} e_3 \mid \widetilde{\mathfrak{F}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{F}\}$
Dual quaternions with $\mathfrak{DGC}$ Lucas number	$\mathbb{KDC}_{\mathfrak{p}} = \{\widetilde{\mathfrak{K}}_n = \widetilde{\mathcal{L}}_n + \widetilde{\mathcal{L}}_{n+1} e_1 + \widetilde{\mathcal{L}}_{n+2} e_2 + \widetilde{\mathcal{L}}_{n+3} e_3 \mid \widetilde{\mathcal{L}}_n \in \mathbb{DC}_{\mathfrak{p}}\mathbb{L}\}$

TABLE 3. Structures for $\widetilde{\mathbb{Q}}_n$ as a dual quaternion

Scalar and Vector parts	$S_{\widetilde{\mathbb{Q}}_n} = \widetilde{\mathfrak{F}}_n,\ V_{\widetilde{\mathbb{Q}}_n} = \widetilde{\mathfrak{F}}_{n+1}e_1 + \widetilde{\mathfrak{F}}_{n+2}e_2 + \widetilde{\mathfrak{F}}_{n+3}e_3$
Conjugate	$\overline{\widetilde{\mathbb{Q}}}_n = \widetilde{\mathfrak{F}}_n - \widetilde{\mathfrak{F}}_{n+1}e_1 - \widetilde{\mathfrak{F}}_{n+2}e_2 - \widetilde{\mathfrak{F}}_{n+3}e_3 = S_{\widetilde{\mathbb{Q}}_n} - V_{\widetilde{\mathbb{Q}}_n}$
Addition, Subtraction	$\widetilde{\mathbb{Q}}_n \pm \widetilde{\mathbb{Q}}_m = \left(S_{\widetilde{\mathbb{Q}}_n} \pm S_{\widetilde{\mathbb{Q}}_m}\right) + \left(V_{\widetilde{\mathbb{Q}}_n} \pm V_{\widetilde{\mathbb{Q}}_m}\right)$
Multiplication	$\begin{aligned} \widetilde{\mathbb{Q}}_n\widetilde{\mathbb{Q}}_m &= \widetilde{\mathfrak{F}}_n\widetilde{\mathfrak{F}}_m + \left(\widetilde{\mathfrak{F}}_n\widetilde{\mathfrak{F}}_{m+1} + \widetilde{\mathfrak{F}}_{n+1}\widetilde{\mathfrak{F}}_m\right)e_1 \\ &+ \left(\widetilde{\mathfrak{F}}_n\widetilde{\mathfrak{F}}_{m+2} + \widetilde{\mathfrak{F}}_{n+2}\widetilde{\mathfrak{F}}_m\right)e_2 \\ &+ \left(\widetilde{\mathfrak{F}}_n\widetilde{\mathfrak{F}}_{m+3} + \widetilde{\mathfrak{F}}_{n+3}\widetilde{\mathfrak{F}}_m\right)e_3 \\ &= S_{\widetilde{\mathbb{Q}}_n}S_{\widetilde{\mathbb{Q}}_m} + S_{\widetilde{\mathbb{Q}}_n}V_{\widetilde{\mathbb{Q}}_m} + S_{\widetilde{\mathbb{Q}}_m}V_{\widetilde{\mathbb{Q}}_n} \end{aligned}$
Equality	$\widetilde{\mathbb{Q}}_n = \widetilde{\mathbb{Q}}_m \Leftrightarrow S_{\widetilde{\mathbb{Q}}_n} = S_{\widetilde{\mathbb{Q}}_m} \wedge V_{\widetilde{\mathbb{Q}}_n} = V_{\widetilde{\mathbb{Q}}_m}$
Module	$\left\Vert\widetilde{\mathbb{Q}}_n\overline{\widetilde{\mathbb{Q}}}_n\right\Vert^2 = \widetilde{\mathfrak{F}}_n^2$
Inverse	$\widetilde{\mathbb{Q}}_n^{-1} = \frac{\overline{\widetilde{\mathbb{Q}}}_n}{\left\Vert\widetilde{\mathbb{Q}}_n\overline{\widetilde{\mathbb{Q}}}_n\right\Vert^2}$, for $\left\Vert\widetilde{\mathbb{Q}}_n\overline{\widetilde{\mathbb{Q}}}_n\right\Vert^2 \neq 0$

6. $\widetilde{\mathfrak{K}}_{n+r} + \widetilde{\mathfrak{K}}_{n-r} = \begin{cases} L_r\widetilde{\mathfrak{K}}_n, & r = 2k \\ 5F_r\widetilde{\mathbb{Q}}_n, & r = 2k+1, \end{cases}$
7. $\widetilde{\mathfrak{K}}_{n+r} - \widetilde{\mathfrak{K}}_{n-r} = \begin{cases} 5F_r\widetilde{\mathbb{Q}}_n, & r = 2k \\ L_r\widetilde{\mathfrak{K}}_n, & r = 2k+1, \end{cases}$
8. $\widetilde{\mathfrak{K}}_n - \widetilde{\mathfrak{K}}_{n+1}e_1 - \widetilde{\mathfrak{K}}_{n+2}e_2 - \widetilde{\mathfrak{K}}_{n+3}e_3 = \widetilde{\mathfrak{L}}_n$.

Proof. The proof is conducted by the following relationships ([13]):

$$\widetilde{\mathfrak{F}}_n + \widetilde{\mathfrak{F}}_{n+1} = \widetilde{\mathfrak{F}}_{n+2}, \qquad \widetilde{\mathfrak{L}}_n + \widetilde{\mathfrak{L}}_{n+1} = \widetilde{\mathfrak{L}}_{n+2},$$

$$\widetilde{\mathfrak{F}}_{n+r} + \widetilde{\mathfrak{F}}_{n-r} = \begin{cases} L_r\widetilde{\mathfrak{F}}_n, & r = 2k \\ F_r\widetilde{\mathfrak{L}}_n, & r = 2k+1, \end{cases} \qquad \widetilde{\mathfrak{L}}_{n+r} + \widetilde{\mathfrak{L}}_{n-r} = \begin{cases} L_r\widetilde{\mathfrak{L}}_n, & r = 2k \\ 5F_r\widetilde{\mathfrak{F}}_n, & r = 2k+1, \end{cases}$$

$$\widetilde{\mathfrak{F}}_{n+r} - \widetilde{\mathfrak{F}}_{n-r} = \begin{cases} F_r\widetilde{\mathfrak{L}}_n, & r = 2k \\ L_r\widetilde{\mathfrak{F}}_n, & r = 2k+1, \end{cases} \qquad \widetilde{\mathfrak{L}}_{n+r} - \widetilde{\mathfrak{L}}_{n-r} = \begin{cases} 5F_r\widetilde{\mathfrak{F}}_n, & r = 2k \\ L_r\widetilde{\mathfrak{L}}_n, & r = 2k+1. \end{cases}$$

□

Theorem 2.2. *Let $\widetilde{\mathbb{Q}}_n \in \mathbb{QDC}_\mathfrak{p}$, $\widetilde{\mathfrak{K}}_n \in \mathbb{KDC}_\mathfrak{p}$, $\overline{\widetilde{\mathbb{Q}}}_n$ be a conjugate of $\widetilde{\mathbb{Q}}_n$, $\overline{\widetilde{\mathfrak{K}}}_n$ be a conjugate of $\widetilde{\mathfrak{K}}_n$, $\widetilde{\mathfrak{F}}_n \in \mathbb{DC}_\mathfrak{p}\mathbb{F}$ and $\widetilde{\mathfrak{L}}_n \in \mathbb{DC}_\mathfrak{p}\mathbb{L}$. The following additional recurrence relationships then hold for $n \geq 0$:*

1. $\widetilde{\mathbb{Q}}_n + \overline{\widetilde{\mathbb{Q}}}_n = 2\widetilde{\mathfrak{F}}_n$,
2. $\widetilde{\mathbb{Q}}_n\overline{\widetilde{\mathbb{Q}}}_n = \widetilde{\mathfrak{F}}_n^2$,
3. $\widetilde{\mathbb{Q}}_n\overline{\widetilde{\mathbb{Q}}}_n + \widetilde{\mathbb{Q}}_{n+1}\overline{\widetilde{\mathbb{Q}}}_{n+1} = \widetilde{\mathfrak{F}}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J + (F_{2n+3} + 2\mathfrak{p}F_{2n+5})\varepsilon + 3F_{2n+4}J\varepsilon$,
4. $\widetilde{\mathbb{Q}}_n^2 = 2\widetilde{\mathfrak{F}}_n\widetilde{\mathbb{Q}}_n - \widetilde{\mathfrak{F}}_n^2$,
5. $\widetilde{\mathfrak{K}}_n + \overline{\widetilde{\mathfrak{K}}}_n = 2\widetilde{\mathfrak{L}}_n$,
6. $\widetilde{\mathfrak{K}}_n\overline{\widetilde{\mathfrak{K}}}_n = \widetilde{\mathfrak{L}}_n^2$,
7. $\widetilde{\mathfrak{K}}_n\overline{\widetilde{\mathfrak{K}}}_n + \widetilde{\mathfrak{K}}_{n+1}\overline{\widetilde{\mathfrak{K}}}_{n+1} = \widetilde{\mathfrak{L}}_{2n+1} + \mathfrak{p}L_{2n+3} + L_{2n+2}J + (L_{2n+3} + 2\mathfrak{p}L_{2n+5})\varepsilon + 3L_{2n+4}J\varepsilon$,
8. $\widetilde{\mathfrak{K}}_n^2 = 2\widetilde{\mathfrak{L}}_n\widetilde{\mathfrak{K}}_n - \widetilde{\mathfrak{L}}_n^2$.

Proof. The proof is clear by considering the following relationships ([13]):

$$\widetilde{\mathfrak{F}}_n^2 + \widetilde{\mathfrak{F}}_{n+1}^2 = \widetilde{\mathfrak{F}}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J + (F_{2n+3} + 2\mathfrak{p}F_{2n+5})\varepsilon + 3F_{2n+4}J\varepsilon,$$

$$\widetilde{\mathfrak{L}}_n^2 + \widetilde{\mathfrak{L}}_{n+1}^2 = \widetilde{\mathfrak{L}}_{2n+1} + \mathfrak{p}L_{2n+3} + L_{2n+2}J + (L_{2n+3} + 2\mathfrak{p}L_{2n+5})\varepsilon + 3L_{2n+4}J\varepsilon,$$

□

Theorem 2.3. *Let $\widetilde{\mathfrak{Q}}_n, \widetilde{\mathfrak{Q}}_m \in \mathbb{QDC}_{\mathfrak{p}}$ and $\widetilde{\mathfrak{K}}_n, \widetilde{\mathfrak{K}}_m \in \mathbb{KDC}_{\mathfrak{p}}$. The following multiplication recurrence relationships then hold for $n, m, r \geq 0$:*

1. $$\begin{aligned}\widetilde{\mathfrak{Q}}_m\widetilde{\mathfrak{Q}}_n - \widetilde{\mathfrak{Q}}_{m+r}\widetilde{\mathfrak{Q}}_{n-r} &= (-1)^{n-r}F_{m-n+r}F_r\left[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon\right] \\ &\quad [1 + e_1 + 3e_2 + 4e_3],\end{aligned} \tag{3}$$

2. $$\begin{aligned}\widetilde{\mathfrak{Q}}_n\widetilde{\mathfrak{Q}}_m + \widetilde{\mathfrak{Q}}_{n+1}\widetilde{\mathfrak{Q}}_{m+1} = \; & \widetilde{\mathfrak{Q}}_{n+m+1} + \mathfrak{p}Q_{n+m+3} + Q_{n+m+2}J \\ & + (Q_{n+m+3} + 2\mathfrak{p}Q_{n+m+5})\varepsilon + 3Q_{n+m+4}J\varepsilon \\ & + V_{\widetilde{\mathfrak{Q}}_{n+m+1}} + \mathfrak{p}V_{Q_{n+m+3}} + V_{Q_{n+m+2}}J \\ & + \left(V_{Q_{n+m+3}} + 2\mathfrak{p}V_{Q_{n+m+5}}\right)\varepsilon + 3V_{Q_{n+m+4}}J\varepsilon,\end{aligned} \tag{4}$$

3. $$\begin{aligned}\widetilde{\mathfrak{Q}}_n^2 + \widetilde{\mathfrak{Q}}_{n+1}^2 = \; & \widetilde{\mathfrak{Q}}_{2n+1} + \mathfrak{p}Q_{2n+3} + Q_{2n+2}J + (Q_{2n+3} + 2\mathfrak{p}Q_{2n+5})\varepsilon \\ & + 3Q_{2n+4}J\varepsilon + V_{\widetilde{\mathfrak{Q}}_{2n+1}} + \mathfrak{p}V_{Q_{2n+3}} + V_{Q_{2n+2}}J \\ & + \left(V_{Q_{2n+3}} + 2\mathfrak{p}V_{Q_{2n+5}}\right)\varepsilon + 3V_{Q_{2n+4}}J\varepsilon,\end{aligned} \tag{5}$$

4. $$\begin{aligned}\widetilde{\mathfrak{K}}_m\widetilde{\mathfrak{K}}_n - \widetilde{\mathfrak{K}}_{m+r}\widetilde{\mathfrak{K}}_{n-r} = \; & 5(-1)^{n-r+1}F_{m-n+r}F_r\left[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon\right] \\ & [1 + e_1 + 3e_2 + 4e_3],\end{aligned} \tag{6}$$

5. $$\begin{aligned}\widetilde{\mathfrak{K}}_n\widetilde{\mathfrak{K}}_m + \widetilde{\mathfrak{K}}_{n+1}\widetilde{\mathfrak{K}}_{m+1} = \; & 5\left[\widetilde{\mathfrak{Q}}_{n+m+1} + \mathfrak{p}Q_{n+m+3} + Q_{n+m+2}J\right. \\ & \left. + (Q_{n+m+3} + 2\mathfrak{p}Q_{n+m+5})\varepsilon + 3Q_{n+m+4}J\varepsilon\right] \\ & + 5\left[V_{\widetilde{\mathfrak{Q}}_{n+m+1}} + \mathfrak{p}V_{Q_{n+m+3}} + V_{Q_{n+m+2}}J\right. \\ & \left. + \left(V_{Q_{n+m+3}} + 2\mathfrak{p}V_{Q_{n+m+5}}\right)\varepsilon + 3V_{Q_{n+m+4}}J\varepsilon\right],\end{aligned} \tag{7}$$

6. $$\begin{aligned}\widetilde{\mathfrak{K}}_n^2 + \widetilde{\mathfrak{K}}_{n+1}^2 = \; & 5\left[\widetilde{\mathfrak{Q}}_{2n+1} + \mathfrak{p}Q_{2n+3} + Q_{2n+2}J\right. \\ & \left. + (Q_{2n+3} + 2\mathfrak{p}Q_{2n+5})\varepsilon + 3Q_{2n+4}J\varepsilon\right] \\ & + 5\left[V_{\widetilde{\mathfrak{Q}}_{2n+1}} + \mathfrak{p}V_{Q_{2n+3}} + V_{Q_{2n+2}}J\right. \\ & \left. + \left(V_{Q_{2n+3}} + 2\mathfrak{p}V_{Q_{2n+5}}\right)\varepsilon + 3V_{Q_{2n+4}}J\varepsilon\right].\end{aligned} \tag{8}$$

Proof. Write the following relationships for Fibonacci/Lucas numbers ([4]):

$$F_{n+r} + F_{n-r} = \begin{cases} L_rF_n, & r = 2k \\ F_rL_n, & r = 2k+1, \end{cases} \tag{9}$$

$$F_{n+r} - F_{n-r} = \begin{cases} L_rF_n, & r = 2k+1 \\ F_rL_n, & r = 2k, \end{cases} \tag{10}$$

and for $\mathfrak{DGC}$ Fibonacci/Lucas numbers ([13]):

$$\widetilde{\mathfrak{F}}_m\widetilde{\mathfrak{F}}_n - \widetilde{\mathfrak{F}}_{m+r}\widetilde{\mathfrak{F}}_{n-r} = (-1)^{n-r}F_{m-n+r}F_r\left[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon\right], \tag{11}$$

$$\begin{aligned}\widetilde{\mathfrak{F}}_n\widetilde{\mathfrak{F}}_m + \widetilde{\mathfrak{F}}_{n+1}\widetilde{\mathfrak{F}}_{m+1} &= \widetilde{\mathfrak{F}}_{n+m+1} + \mathfrak{p}F_{n+m+3} + F_{n+m+2}J \\ &\quad + (F_{n+m+3} + 2\mathfrak{p}F_{n+m+5})\varepsilon + 3F_{n+m+4}J\varepsilon,\end{aligned} \tag{12}$$

$$\widetilde{\mathfrak{L}}_m\widetilde{\mathfrak{L}}_n - \widetilde{\mathfrak{L}}_{m+r}\widetilde{\mathfrak{L}}_{n-r} = 5(-1)^{n-r+1}F_{m-n+r}F_r\left[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon\right], \tag{13}$$

$$\begin{aligned}\widetilde{\mathfrak{L}}_n\widetilde{\mathfrak{L}}_m + \widetilde{\mathfrak{L}}_{n+1}\widetilde{\mathfrak{L}}_{m+1} &= 5\left[\widetilde{\mathfrak{F}}_{n+m+1} + \mathfrak{p}F_{n+m+3} + F_{n+m+2}J\right. \\ &\quad \left. + (F_{n+m+3} + 2\mathfrak{p}F_{n+m+5})\varepsilon + 3F_{n+m+4}J\varepsilon\right].\end{aligned} \tag{14}$$

The proof of the first relationship starts with multiplication (see Table 3):

$$\begin{aligned}\widetilde{\mathfrak{Q}}_m\widetilde{\mathfrak{Q}}_n - \widetilde{\mathfrak{Q}}_{m+r}\widetilde{\mathfrak{Q}}_{n-r} = \; & \left(\widetilde{\mathfrak{F}}_m\widetilde{\mathfrak{F}}_n - \widetilde{\mathfrak{F}}_{m+r}\widetilde{\mathfrak{F}}_{n-r}\right) \\ & + \left[\left(\widetilde{\mathfrak{F}}_m\widetilde{\mathfrak{F}}_{n+1} - \widetilde{\mathfrak{F}}_{m+r}\widetilde{\mathfrak{F}}_{n-r+1}\right) + \left(\widetilde{\mathfrak{F}}_{m+1}\widetilde{\mathfrak{F}}_n - \widetilde{\mathfrak{F}}_{m+r+1}\widetilde{\mathfrak{F}}_{n-r}\right)\right]e_1 \\ & + \left[\left(\widetilde{\mathfrak{F}}_m\widetilde{\mathfrak{F}}_{n+2} - \widetilde{\mathfrak{F}}_{m+r}\widetilde{\mathfrak{F}}_{n-r+2}\right) + \left(\widetilde{\mathfrak{F}}_{m+2}\widetilde{\mathfrak{F}}_n - \widetilde{\mathfrak{F}}_{m+r+2}\widetilde{\mathfrak{F}}_{n-r}\right)\right]e_2 \\ & + \left[\left(\widetilde{\mathfrak{F}}_m\widetilde{\mathfrak{F}}_{n+3} - \widetilde{\mathfrak{F}}_{m+r}\widetilde{\mathfrak{F}}_{n-r+3}\right) + \left(\widetilde{\mathfrak{F}}_{m+3}\widetilde{\mathfrak{F}}_n - \widetilde{\mathfrak{F}}_{m+r+3}\widetilde{\mathfrak{F}}_{n-r}\right)\right]e_3.\end{aligned}$$

From Eqs. (9), (10), and (11), the proof is straightforward. The proof of the second and third relationships are completed by using Eq. (12) and writing $m \to n$ in Eq. (4), respectively. The other parts can be proved similarly.

□

Theorem 2.4. *Let $\widetilde{\mathfrak{Q}}_{-n}$ and $\widetilde{\mathfrak{K}}_{-n}$ be nega$\mathfrak{DGC}$ dual Fibonacci and Lucas quaternions, respectively. Then, the following identities can be given for $n \geqslant 0$:*

- $\widetilde{\mathcal{Q}}_{-n}=(-1)^{n+1}\widetilde{\mathcal{Q}}_n+(-1)^n\widetilde{\mathcal{L}}_n\left(e_1+e_2+2e_3\right)+(-1)^{n+1}K_n\left(J+\varepsilon+2J\varepsilon\right)e_3$,
- $\widetilde{\mathcal{K}}_{-n}=(-1)^n\widetilde{\mathcal{K}}_n+5(-1)^{n-1}\widetilde{\mathcal{F}}_n\left(e_1+e_2+2e_3\right)+5(-1)^nK_n\left(J+\varepsilon+2J\varepsilon\right)e_3$.

Proof. The proof is clear with the help of the following identities ([13]):

$$\begin{aligned}\widetilde{\mathcal{F}}_{-n}&=(-1)^{n+1}\widetilde{\mathcal{F}}_n+(-1)^nL_n\left(J+\varepsilon+2J\varepsilon\right),\\ \widetilde{\mathcal{L}}_{-n}&=(-1)^n\widetilde{\mathcal{L}}_n+5(-1)^{n-1}L_n\left(J+\varepsilon+2J\varepsilon\right).\end{aligned}$$

□

Theorem 2.5. ***(Binet's Formula)*** *Let $\widetilde{\mathcal{Q}}_n\in\mathbb{DC}_{\mathfrak{p}}\mathbb{Q}$ and $\widetilde{\mathcal{K}}_n\in\mathbb{DC}_{\mathfrak{p}}\mathbb{K}$. Then, for $n\geq 1$, the following Binet formulas can be given:*

$$\widetilde{\mathcal{Q}}_n=\frac{\tilde{\alpha}^*\alpha^n-\tilde{\beta}^*\beta^n}{\alpha-\beta}\quad\text{and}\quad\widetilde{\mathcal{K}}_n=\tilde{\alpha}^*\alpha^n+\tilde{\beta}^*\beta^n,$$

where $\alpha^*=1+\alpha J+\alpha^2\varepsilon+\alpha^3J\varepsilon$, $\tilde{\alpha}^*=\alpha^*\left(1+\alpha e_1+\alpha^2e_2+\alpha^3e_3\right)$, $\beta^*=1+\beta J+\beta^2\varepsilon+\beta^3J\varepsilon$ *and* $\tilde{\beta}^*=\beta^*\left(1+\beta e_1+\beta^2e_2+\beta^3e_3\right)$.

Proof. The proof is completed by considering the Binet formulas given in [13]:

$$\widetilde{\mathcal{F}}_n=\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta}\quad\text{and}\quad\widetilde{\mathcal{L}}_n=\alpha^*\alpha^n+\beta^*\beta^n.$$

□

Theorem 2.6. ***(D'Ocagne's Identity)*** *Let $\widetilde{\mathcal{Q}}_n,\widetilde{\mathcal{Q}}_m\in\mathbb{QDC}_{\mathfrak{p}}$ and $\widetilde{\mathcal{K}}_n,\widetilde{\mathcal{K}}_m\in\mathbb{KDC}_{\mathfrak{p}}$. Then, for $n,m\geq 0$, the D'Ocagne's identities can be given as follows:*

- $\widetilde{\mathcal{Q}}_m\widetilde{\mathcal{Q}}_{n+1}-\widetilde{\mathcal{Q}}_{m+1}\widetilde{\mathcal{Q}}_n=\quad(-1)^nF_{m-n}\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right]\left[1+e_1+3e_2+4e_3\right]$,
- $\widetilde{\mathcal{K}}_m\widetilde{\mathcal{K}}_{n+1}-\widetilde{\mathcal{K}}_{m+1}\widetilde{\mathcal{K}}_n=\quad5(-1)^{n+1}F_{m-n}\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right]\left[1+e_1+3e_2+4e_3\right]$.

Proof. It is clear by taking $n\to n+1$, $r\to 1$ in Eqs. (3) and (6). □

Theorem 2.7. ***(Catalan's Identity)*** *Let $\widetilde{\mathcal{Q}}_n\in\mathbb{QDC}_{\mathfrak{p}}$ and $\widetilde{\mathcal{K}}_n\in\mathbb{KDC}_{\mathfrak{p}}$. Then, the Catalan's identities can be given such that:*

- $\widetilde{\mathcal{Q}}_n^2-\widetilde{\mathcal{Q}}_{n+r}\widetilde{\mathcal{Q}}_{n-r}=\quad(-1)^{n-r}F_r^2\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right][1+e_1+3e_2+4e_3]$,
- $\widetilde{\mathcal{K}}_n^2-\widetilde{\mathcal{K}}_{n+r}\widetilde{\mathcal{K}}_{n-r}=\quad5(-1)^{n-r+1}F_r^2\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right][1+e_1+3e_2+4e_3]$.

Proof. By taking $m\to n$ in Eqs. (3) and (6), the proof is clear. □

Theorem 2.8. ***(Cassini's Identity)*** *Let $\widetilde{\mathcal{Q}}_n\in\mathbb{QDC}_{\mathfrak{p}}$ and $\widetilde{\mathcal{K}}_n\in\mathbb{KDC}_{\mathfrak{p}}$. Then, the Cassini's identities of these dual quaternions are as follows:*

- $\widetilde{\mathcal{Q}}_n^2-\widetilde{\mathcal{Q}}_{n+1}\widetilde{\mathcal{Q}}_{n-1}=(-1)^{n-1}\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right][1+e_1+3e_2+4e_3]$,
- $\widetilde{\mathcal{K}}_n^2-\widetilde{\mathcal{K}}_{n+1}\widetilde{\mathcal{K}}_{n-1}=5(-1)^{n}\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right][1+e_1+3e_2+4e_3]$.

Proof. By taking $r=1$ in Theorem 2.7, the proof is obvious. □

Example 2.1. *Find the following identities:*
D'Ocagne's identities for $m=3,n=1$ and $\mathfrak{p}=-\frac{1}{3}$:

- $\widetilde{\mathcal{Q}}_3\widetilde{\mathcal{Q}}_2-\widetilde{\mathcal{Q}}_4\widetilde{\mathcal{Q}}_1=-\left(\frac{4}{3}+J+4\varepsilon+3J\varepsilon\right)\left(1+e_1+3e_2+4e_3\right)$,
- $\widetilde{\mathcal{K}}_3\widetilde{\mathcal{K}}_2-\widetilde{\mathcal{K}}_4\widetilde{\mathcal{K}}_1=5\left(\frac{4}{3}+J+4\varepsilon+3J\varepsilon\right)\left(1+e_1+3e_2+4e_3\right)$.

Catalan's identities for $n=2,r=2$ and $\mathfrak{p}=0$:

- $\widetilde{\mathcal{Q}}_2^2-\widetilde{\mathcal{Q}}_4\widetilde{\mathcal{Q}}_0=\left(1+J+3\varepsilon+3J\varepsilon\right)\left(1+e_1+3e_2+4e_3\right)$,
- $\widetilde{\mathcal{K}}_2^2-\widetilde{\mathcal{K}}_4\widetilde{\mathcal{K}}_0=-5\left(1+J+3\varepsilon+3J\varepsilon\right)\left(1+e_1+3e_2+4e_3\right)$.

Cassini's identities for $n=2$ and $\mathfrak{p}=\frac{1}{5}$:

- $\widetilde{\mathcal{Q}}_2^2-\widetilde{\mathcal{Q}}_3\widetilde{\mathcal{Q}}_1=-\left(\frac{4}{5}+J+\frac{12}{5}\varepsilon+3J\varepsilon\right)\left(1+e_1+3e_2+4e_3\right)$,

- $\widetilde{\mathcal{K}}_2^2 - \widetilde{\mathcal{K}}_3\widetilde{\mathcal{K}}_1 = 5\left(\frac{4}{5} + J + \frac{12}{5}\varepsilon + 3J\varepsilon\right)(1 + e_1 + 3e_2 + 4e_3)$.

2.1. Examination of matrix representations

Theorem 2.9. *Every* $\widetilde{\mathbb{Q}}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon$ *can be represented by the following* 2×2 *matrix*[2]*:*

$$\mathcal{X}_{\widetilde{\mathbb{Q}}_n} = \begin{bmatrix} Q_n + Q_{n+1}J & 0 \\ Q_{n+2} + Q_{n+3}J & Q_n + Q_{n+1}J \end{bmatrix}.$$

Remark 2.2. $\mathcal{X}$ *is a linear transformation between* $\mathbb{DC}_{\mathfrak{p}}\mathbb{Q}$ *and the matrices*

$$\left\{ \begin{bmatrix} Q_n + Q_{n+1}J & 0 \\ Q_{n+2} + Q_{n+3}J & Q_n + Q_{n+1}J \end{bmatrix} \mid Q_n, Q_{n+1}, Q_{n+2}, Q_{n+3} \in \mathbb{Q} \right\}.$$

The columns of the matrix $\mathcal{X}_{\widetilde{\mathbb{Q}}_n}$ *are represented by the coefficients of the elements* $\{\widetilde{\mathbb{Q}}_n, \widetilde{\mathbb{Q}}_n\varepsilon\}$*, considering the basis* $\{1, \varepsilon\}$*. Hence,* $\mathbb{DC}_{\mathfrak{p}}\mathbb{Q}$ *is the subset of* $\mathcal{M}_2(\mathbb{C}_{\mathfrak{p}}\mathbb{Q})$*.*

Theorem 2.10. *Every* $\widetilde{\mathbb{Q}}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon$ *can be represented by a matrix*[3] *in* $\mathcal{M}_4(\mathbb{Q})$*:*

$$\mathcal{A}_{\widetilde{\mathbb{Q}}_n} = \begin{bmatrix} Q_n & \mathfrak{p}Q_{n+1} & 0 & 0 \\ Q_{n+1} & Q_n & 0 & 0 \\ Q_{n+2} & \mathfrak{p}Q_{n+3} & Q_n & \mathfrak{p}Q_{n+1} \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \end{bmatrix}.$$

The columns of the matrix $\mathcal{A}_{\widetilde{\mathbb{Q}}_n}$ are represented by the coefficients of the elements $\{\widetilde{\mathbb{Q}}_n, \widetilde{\mathbb{Q}}_nJ, \widetilde{\mathbb{Q}}_n\varepsilon, \widetilde{\mathbb{Q}}_nJ\varepsilon\}$, considering the basis $\{1, J, \varepsilon, J\varepsilon\}$. Moreover, $\mathbb{DC}_{\mathfrak{p}}\mathbb{Q}$ is the subset of $\mathcal{M}_4(\mathbb{Q})$.

Theorem 2.11. *Every* $\widetilde{\mathbb{Q}}_n = \widetilde{\mathcal{F}}_n + \widetilde{\mathcal{F}}_{n+1}e_1 + \widetilde{\mathcal{F}}_{n+2}e_2 + \widetilde{\mathcal{F}}_{n+3}e_3$ *can also be represented by a matrix in* $\mathcal{M}_4(\mathbb{DC}_{\mathfrak{p}}\mathbb{F})$*.*

Proof. Define the linear transformation $f_{\widetilde{\mathbb{Q}}_n}(\widetilde{\mathbb{Q}}_m) = \widetilde{\mathbb{Q}}_n\widetilde{\mathbb{Q}}_m$ for every $\widetilde{\mathbb{Q}}_n \in \mathbb{QDC}_{\mathfrak{p}}$ and calculate the image of the elements $\{1, e_1, e_2, e_3\}$ as follows:

$$\begin{array}{lllll} f_{\widetilde{\mathbb{Q}}_n}(1) & = & \widetilde{\mathbb{Q}}_n & = & \widetilde{\mathcal{F}}_n + \widetilde{\mathcal{F}}_{n+1}e_1 + \widetilde{\mathcal{F}}_{n+2}e_2 + \widetilde{\mathcal{F}}_{n+3}e_3, \\ f_{\widetilde{\mathbb{Q}}_n}(e_1) & = & \widetilde{\mathbb{Q}}_ne_1 & = & \widetilde{\mathcal{F}}_ne_1, \\ f_{\widetilde{\mathbb{Q}}_n}(e_2) & = & \widetilde{\mathbb{Q}}_ne_2 & = & \widetilde{\mathcal{F}}_ne_2, \\ f_{\widetilde{\mathbb{Q}}_n}(e_3) & = & \widetilde{\mathbb{Q}}_ne_3 & = & \widetilde{\mathcal{F}}_ne_3. \end{array}$$

Therefore, the matrix of f with respect to the basis $\{1, e_1, e_2, e_3\}$ is as follows:

$$\mathcal{B}_{\widetilde{\mathbb{Q}}_n} = \begin{bmatrix} \widetilde{\mathcal{F}}_n & 0 & 0 & 0 \\ \widetilde{\mathcal{F}}_{n+1} & \widetilde{\mathcal{F}}_n & 0 & 0 \\ \widetilde{\mathcal{F}}_{n+2} & 0 & \widetilde{\mathcal{F}}_n & 0 \\ \widetilde{\mathcal{F}}_{n+3} & 0 & 0 & \widetilde{\mathcal{F}}_n \end{bmatrix}.$$

The set of $\mathbb{QDC}_{\mathfrak{p}}$ is the subset of $\mathcal{M}_4(\mathbb{DC}_{\mathfrak{p}}\mathbb{F})$. □

Corollary 2.1. *In particular, consider the following statements:*

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[2]The set of dual numbers is isomorphic to a subset of 2×2 real matrices, $\lambda(a + b\varepsilon) = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, where $z = a + b\varepsilon$, $\varepsilon^2 = 0, \varepsilon \neq 0$, [32].

[3]The set of generalized complex numbers is isomorphic to a subset of 2×2 real matrices, $\lambda(a + bJ) = \begin{bmatrix} a & \mathfrak{p}b \\ b & a \end{bmatrix}$, where $z = a + bJ$, $J^2 = \mathfrak{p} \in \mathbb{R}$, [18].

- $\mathcal{A}_{\widetilde{\mathfrak{Q}}_n}$ *is also in the form* $\mathcal{A}_{\widetilde{\mathfrak{Q}}_n} = Q_n I_4 + Q_{n+1}\mathfrak{J} + Q_{n+2}\mathfrak{E} + Q_{n+3}\mathfrak{J}\mathfrak{E}$, *where*

$$J \leftrightarrow \mathfrak{J} = \begin{bmatrix} 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{p} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \varepsilon \leftrightarrow \mathfrak{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, J\varepsilon \leftrightarrow \mathfrak{J}\mathfrak{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

- $\mathcal{B}_{\widetilde{\mathfrak{Q}}_n}$ *is also in the form* $\mathcal{B}_{\widetilde{\mathfrak{Q}}_n} = \widetilde{\mathfrak{F}}_n + \widetilde{\mathfrak{F}}_{n+1}E_1 + \widetilde{\mathfrak{F}}_{n+2}E_2 + \widetilde{\mathfrak{F}}_{n+3}E_3$, *where*

$$e_1 \leftrightarrow E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_2 \leftrightarrow E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_3 \leftrightarrow E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 2.3. *With respect to the basis* $\{1, e_1, e_2, e_3\}$, *the column matrix representation of* $\widetilde{\mathfrak{Q}}_m = \widetilde{\mathfrak{F}}_m + \widetilde{\mathfrak{F}}_{m+1}e_1 + \widetilde{\mathfrak{F}}_{m+2}e_2 + \widetilde{\mathfrak{F}}_{m+3}e_3$ *is given by:*

$$\widetilde{\mathfrak{Q}}_m = \begin{bmatrix} \widetilde{\mathfrak{F}}_m & \widetilde{\mathfrak{F}}_{m+1} & \widetilde{\mathfrak{F}}_{m+2} & \widetilde{\mathfrak{F}}_{m+3} \end{bmatrix}^T.$$

By using $\mathcal{B}_{\widetilde{\mathfrak{Q}}_n}$, *the product of* $\widetilde{\mathfrak{Q}}_n, \widetilde{\mathfrak{Q}}_m \in \mathbb{QDC}_{\mathfrak{p}}$ *can also be expressed by:*

$$\widetilde{\mathfrak{Q}}_n\widetilde{\mathfrak{Q}}_m = \begin{bmatrix} \widetilde{\mathfrak{F}}_n & 0 & 0 & 0 \\ \widetilde{\mathfrak{F}}_{n+1} & \widetilde{\mathfrak{F}}_n & 0 & 0 \\ \widetilde{\mathfrak{F}}_{n+2} & 0 & \widetilde{\mathfrak{F}}_n & 0 \\ \widetilde{\mathfrak{F}}_{n+3} & 0 & 0 & \widetilde{\mathfrak{F}}_n \end{bmatrix} \begin{bmatrix} \widetilde{\mathfrak{F}}_m \\ \widetilde{\mathfrak{F}}_{m+1} \\ \widetilde{\mathfrak{F}}_{m+2} \\ \widetilde{\mathfrak{F}}_{m+3} \end{bmatrix} = \widetilde{\mathfrak{Q}}_m\widetilde{\mathfrak{Q}}_n.$$

Similarly, by using $\widetilde{\mathfrak{Q}}_m = \begin{bmatrix} Q_m & Q_{m+1} & Q_{m+2} & Q_{m+3} \end{bmatrix}^T$ *and* $\mathcal{A}_{\widetilde{\mathfrak{Q}}_n}$, *we have:*

$$\widetilde{\mathfrak{Q}}_n\widetilde{\mathfrak{Q}}_m = \begin{bmatrix} Q_n & \mathfrak{p}Q_{n+1} & 0 & 0 \\ Q_{n+1} & Q_n & 0 & 0 \\ Q_{n+2} & \mathfrak{p}Q_{n+3} & Q_n & \mathfrak{p}Q_{n+1} \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \end{bmatrix} \begin{bmatrix} Q_m \\ Q_{m+1} \\ Q_{m+2} \\ Q_{m+3} \end{bmatrix}.$$

Theorem 2.12. *Let* $\widetilde{\mathfrak{Q}}_n = \widetilde{\mathfrak{F}}_n + \widetilde{\mathfrak{F}}_{n+1}e_1 + \widetilde{\mathfrak{F}}_{n+2}e_2 + \widetilde{\mathfrak{F}}_{n+3}e_3 \in \mathbb{QDC}_{\mathfrak{p}}$ *and* $\overline{\widetilde{\mathfrak{Q}}}_n$ *be the quaternion conjugate of* $\widetilde{\mathfrak{Q}}_n$. *Then, we have*

$$\sigma\mathcal{A}_{\widetilde{\mathfrak{Q}}_n}\sigma = \mathcal{A}_{\overline{\widetilde{\mathfrak{Q}}}_n}, \quad \textit{where } \sigma = \operatorname{diag}(1, -1, -1, -1).$$

Theorem 2.13. *For any* $\widetilde{\mathcal{Q}}_n, \widetilde{\mathcal{Q}}_m$ *and* $\lambda \in \mathbb{R}$*, the following properties hold:*

- $\mathcal{X}_{\lambda\widetilde{\mathcal{Q}}_n} = \lambda\mathcal{X}_{\widetilde{\mathcal{Q}}_n}$,
- $\mathcal{A}_{\lambda\widetilde{\mathcal{Q}}_n} = \lambda\mathcal{A}_{\widetilde{\mathcal{Q}}_n}$,
- $\mathcal{B}_{\lambda\widetilde{\mathcal{Q}}_n} = \lambda\mathcal{B}_{\widetilde{\mathcal{Q}}_n}$,
- $\mathcal{X}_{\widetilde{\mathcal{Q}}_n\widetilde{\mathcal{Q}}_m} = \mathcal{X}_{\widetilde{\mathcal{Q}}_n}\mathcal{X}_{\widetilde{\mathcal{Q}}_m}$,
- $\mathcal{A}_{\widetilde{\mathcal{Q}}_n\widetilde{\mathcal{Q}}_m} = \mathcal{A}_{\widetilde{\mathcal{Q}}_n}\mathcal{A}_{\widetilde{\mathcal{Q}}_m}$,
- $\det(\mathcal{X}_{\widetilde{\mathcal{Q}}_n}) = (Q_n + Q_{n+1}J)^2$,
- $\det(\mathcal{A}_{\widetilde{\mathcal{Q}}_n}) = \left(Q_n^2 - \mathfrak{p}Q_{n+1}^2\right)^2 = \left\|\widetilde{\mathcal{Q}}_n\right\|_{\dagger_4}^4$,
- $\det(\mathcal{B}_{\widetilde{\mathcal{Q}}_n}) = \widetilde{\mathcal{F}}_n^4 = \left\|\widetilde{\mathcal{Q}}_n\right\|^4$.

Theorem 2.14. *Let* $\widetilde{\mathcal{Q}}_n \in \mathbb{QDC}_\mathfrak{p}$ *and* $\widetilde{\mathcal{Q}}_n^{-1}$ *be the inverse of* $\widetilde{\mathcal{Q}}_n$*. Then, we have:*

$$\mathcal{B}_{\widetilde{\mathcal{Q}}_n^{-1}} = \frac{1}{\sqrt{\det \mathcal{B}_{\widetilde{\mathcal{Q}}_n}}}\mathcal{B}_{\overline{\widetilde{\mathcal{Q}}}_n}.$$

Proof. The proof strongly depends on the inverse of $\widetilde{\mathcal{Q}}_n$ and its matrix representation. □

Definition 2.2. *Let* $\widetilde{\mathcal{Q}}_n = Q_n+Q_{n+1}J+Q_{n+2}\varepsilon+Q_{n+3}J\varepsilon \in \mathbb{DC}_\mathfrak{p}\mathbb{Q}$*. The vector representation of* $\widetilde{\mathcal{Q}}_n$ *is defined as*

$$\vec{\widetilde{\mathcal{Q}}}_n = \begin{bmatrix} \vec{Q}_n & \vec{Q}_{n+1} & \vec{Q}_{n+2} & \vec{Q}_{n+3} \end{bmatrix}^T = \begin{bmatrix} \vec{Q}_n \\ \vec{Q}_{n+1} \\ \vec{Q}_{n+2} \\ \vec{Q}_{n+3} \end{bmatrix} \in \mathcal{M}_{16\times 1}(\mathbb{F}),$$

where $Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$ *is the nth dual Fibonacci quaternion and* $\vec{Q}_n = (F_n, F_{n+1}, F_{n+2}, F_{n+3}) = [F_n\ F_{n+1}\ F_{n+2}\ F_{n+3}]^T$ *is the nth Fibonacci vector (matrix).*

Theorem 2.15. *Let* $\widetilde{\mathcal{Q}}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon \in \mathbb{DC}_\mathfrak{p}\mathbb{Q}$*. Then,*

- $\alpha\vec{\widetilde{\mathcal{Q}}}_n = \vec{\widetilde{\mathcal{Q}}}_n^{\dagger_1}$,
- $\beta\vec{\widetilde{\mathcal{Q}}}_n = \vec{\widetilde{\mathcal{Q}}}_n^{\dagger_2}$,
- $\gamma\vec{\widetilde{\mathcal{Q}}}_n = \vec{\widetilde{\mathcal{Q}}}_n^{\dagger_3}$,

where

$$\begin{cases} \alpha = \operatorname{diag}(1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1) \in \mathcal{M}_{16}(\mathbb{R}), \\ \beta = \operatorname{diag}(1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1) \in \mathcal{M}_{16}(\mathbb{R}), \\ \gamma = \operatorname{diag}(1,1,1,1,-1,-1,-1-1,-1,-1,-1,-1,1,1,1,1) \in \mathcal{M}_{16}(\mathbb{R}). \end{cases}$$

Theorem 2.16. *Let* $\widetilde{\mathcal{Q}}_n = \widetilde{\mathcal{F}}_n+\widetilde{\mathcal{F}}_{n+1}e_1+\widetilde{\mathcal{F}}_{n+2}e_2+\widetilde{\mathcal{F}}_{n+3}e_3$*. Then, the matrix representations* $\mathcal{C}_{\widetilde{\mathcal{Q}}_n}$ *with respect to the basis* $\{1, \varepsilon, e_1, \varepsilon e_1, e_2, \varepsilon e_2, e_3, \varepsilon e_3\}$ *and* $\mathcal{D}_{\widetilde{\mathcal{Q}}_n}$ *according to the base*

$$\{1, J, \varepsilon, J\varepsilon, e_1, Je_1, \varepsilon e_1, J\varepsilon e_1, e_2, Je_2, \varepsilon e_2, J\varepsilon e_2, e_3, Je_3, \varepsilon e_3, J\varepsilon e_3\}$$

of $\widetilde{\mathcal{Q}}_n$*, are as shown below. Moreover, it can be deduced that* $\mathbb{QDC}_\mathfrak{p}$ *is the subset of* $\mathcal{M}_8(\mathbb{C}_\mathfrak{p}\mathbb{F})$*, and* $\mathbb{QDC}_\mathfrak{p}$ *is the subset of* $\mathcal{M}_{16}(\mathbb{F})$*.*

Also, $\det(\mathcal{C}_{\widetilde{\mathcal{Q}}_n}) = (F_n + F_{n+1}J)^8$ *and* $\det(\mathcal{D}_{\widetilde{\mathcal{Q}}_n}) = \left(F_n^2 - \mathfrak{p}F_{n+1}^2\right)^8$.

$$\mathcal{C}_{\widetilde{\mathbb{Q}}_n} = \begin{bmatrix}
F_n + F_{n+1}J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+2} + F_{n+3}J & F_n + F_{n+1}J & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+1} + F_{n+2}J & 0 & F_n + F_{n+1}J & 0 & 0 & 0 & 0 & 0 \\
F_{n+3} + F_{n+4}J & F_{n+1} + F_{n+2}J & F_{n+2} + F_{n+3}J & F_n + F_{n+1}J & 0 & 0 & 0 & 0 \\
F_{n+2} + F_{n+3}J & 0 & 0 & 0 & F_n + F_{n+1}J & 0 & 0 & 0 \\
F_{n+4} + F_{n+5}J & F_{n+2} + F_{n+3}J & 0 & 0 & F_{n+2} + F_{n+3}J & F_n + F_{n+1}J & 0 & 0 \\
F_{n+3} + F_{n+4}J & 0 & 0 & 0 & 0 & 0 & F_n + F_{n+1}J & 0 \\
F_{n+5} + F_{n+6}J & F_{n+3} + F_{n+4}J & 0 & 0 & 0 & 0 & F_{n+2} + F_{n+3}J & F_n + F_{n+1}J
\end{bmatrix},$$

$$\mathcal{D}_{\widetilde{\mathbb{Q}}_n} = \begin{bmatrix}
F_n & \mathfrak{p}F_{n+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+1} & F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+2} & \mathfrak{p}F_{n+3} & F_n & \mathfrak{p}F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+3} & F_{n+2} & F_{n+1} & F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+1} & \mathfrak{p}F_{n+2} & 0 & 0 & F_n & \mathfrak{p}F_{n+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+2} & F_{n+1} & 0 & 0 & F_{n+1} & F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+3} & \mathfrak{p}F_{n+4} & F_{n+1} & \mathfrak{p}F_{n+2} & F_{n+2} & \mathfrak{p}F_{n+3} & F_n & \mathfrak{p}F_{n+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+4} & F_{n+3} & F_{n+2} & F_{n+1} & F_{n+3} & F_{n+2} & F_{n+1} & F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+2} & \mathfrak{p}F_{n+3} & 0 & 0 & 0 & 0 & 0 & 0 & F_n & \mathfrak{p}F_{n+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+3} & F_{n+2} & 0 & 0 & 0 & 0 & 0 & 0 & F_{n+1} & F_n & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{n+4} & \mathfrak{p}F_{n+5} & F_{n+2} & \mathfrak{p}F_{n+3} & 0 & 0 & 0 & 0 & F_{n+2} & \mathfrak{p}F_{n+3} & F_n & \mathfrak{p}F_{n+1} & 0 & 0 & 0 & 0 \\
F_{n+5} & F_{n+4} & F_{n+3} & F_{n+2} & 0 & 0 & 0 & 0 & F_{n+3} & F_{n+2} & F_{n+1} & F_n & 0 & 0 & 0 & 0 \\
F_{n+3} & \mathfrak{p}F_{n+4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_n & \mathfrak{p}F_{n+1} & 0 & 0 \\
F_{n+4} & F_{n+3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{n+1} & F_n & 0 & 0 \\
F_{n+5} & \mathfrak{p}F_{n+6} & F_{n+3} & \mathfrak{p}F_{n+4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{n+2} & \mathfrak{p}F_{n+3} & F_n & \mathfrak{p}F_{n+1} \\
F_{n+6} & F_{n+5} & F_{n+4} & F_{n+3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{n+3} & F_{n+2} & F_{n+1} & F_n
\end{bmatrix}.$$

Remark 2.4. *The matrix representations of $\mathcal{DGC}$ Lucas dual quaternion $\widetilde{\mathcal{K}}_n$ can be examined similarly. Moreover, the following statements are clear:*

- $\mathbb{DC}_{\mathfrak{p}}\mathbb{K}$ *is the subset of* $\mathcal{M}_2(\mathbb{C}_{\mathfrak{p}}\mathbb{K})$ *and* $\mathcal{M}_4(\mathbb{K})$,
- $\mathbb{KDC}_{\mathfrak{p}}$ *is the subset of* $\mathcal{M}_4(\mathbb{DC}_{\mathfrak{p}}\mathbb{L})$,
- $\mathbb{KDC}_{\mathfrak{p}}$ *is the subset of* $\mathcal{M}_8(\mathbb{C}_{\mathfrak{p}}\mathbb{L})$ *and* $\mathcal{M}_{16}(\mathbb{L})$.

Example 2.2. *Consider*

$$\begin{aligned}\widetilde{\mathbb{Q}}_2 &= (1+2e_1+3e_2+5e_3)+(2+3e_1+5e_2+8e_3)\,J\\ &+(3+5e_1+8e_2+13e_3)\,\varepsilon+(5+8e_1+13e_2+21e_3)\,J\varepsilon.\end{aligned}$$

Then, for $\mathfrak{p}\in\mathbb{R}$*:*

$$\mathcal{X}_{\widetilde{\mathbb{Q}}_n}=\begin{bmatrix}(1+2e_1+3e_2+5e_3)+(2+3e_1+5e_2+8e_3)\,J & 0\\ (3+5e_1+8e_2+13e_3)+(5+8e_1+13e_2+21e_3)\,J & (1+2e_1+3e_2+5e_3)+(2+3e_1+5e_2+8e_3)\,J\end{bmatrix}$$

$$\mathcal{A}_{\widetilde{\mathbb{Q}}_2}=\begin{bmatrix}1+2e_1+3e_2+5e_3 & \mathfrak{p}\,(2+3e_1+5e_2+8e_3) & 0 & 0\\ 2+3e_1+5e_2+8e_3 & 1+2e_1+3e_2+5e_3 & 0 & 0\\ 3+5e_1+8e_2+13e_3 & \mathfrak{p}\,(5+8e_1+13e_2+21e_3) & 1+2e_1+3e_2+5e_3 & \mathfrak{p}\,(2+3e_1+5e_2+8e_3)\\ 5+8e_1+13e_2+21e_3 & 3+5e_1+8e_2+13e_3 & 2+3e_1+5e_2+8e_3 & 1+2e_1+3e_2+5e_3\end{bmatrix}$$

$$\mathcal{B}_{\widetilde{\mathbb{Q}}_2}=\begin{bmatrix}1+2J+3\varepsilon+5J\varepsilon & 0 & 0 & 0\\ 2+3J+5\varepsilon+8J\varepsilon & 1+2J+3\varepsilon+5J\varepsilon & 0 & 0\\ 3+5J+8\varepsilon+13J\varepsilon & 0 & 1+2J+3\varepsilon+5J\varepsilon & 0\\ 5+8J+13\varepsilon+21J\varepsilon & 0 & 0 & 1+2J+3\varepsilon+5J\varepsilon\end{bmatrix}$$

$$\mathcal{C}_{\widetilde{\mathbb{Q}}_2}=\begin{bmatrix}1+2J & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 3+5J & 1+2J & 0 & 0 & 0 & 0 & 0 & 0\\ 2+3J & 0 & 1+2J & 0 & 0 & 0 & 0 & 0\\ 5+8J & 2+3J & 3+5J & 1+2J & 0 & 0 & 0 & 0\\ 3+5J & 0 & 0 & 0 & 1+2J & 0 & 0 & 0\\ 8+13J & 3+5J & 0 & 0 & 3+5J & 1+2J & 0 & 0\\ 5+8J & 0 & 0 & 0 & 0 & 0 & 1+2J & 0\\ 13+21J & 5+8J & 0 & 0 & 0 & 0 & 3+5J & 1+2J\end{bmatrix}$$

$$\mathcal{D}_{\widetilde{\mathcal{Q}}_2} = \begin{bmatrix} 1 & 2\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5\mathfrak{p} & 1 & 2\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3\mathfrak{p} & 0 & 0 & 1 & 2\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8\mathfrak{p} & 2 & 3\mathfrak{p} & 3 & 5\mathfrak{p} & 1 & 2\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 5 & 3 & 2 & 5 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 13\mathfrak{p} & 3 & 5\mathfrak{p} & 0 & 0 & 0 & 0 & 3 & 5\mathfrak{p} & 1 & 2\mathfrak{p} & 0 & 0 & 0 & 0 \\ 13 & 8 & 5 & 3 & 0 & 0 & 0 & 0 & 5 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 8\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\mathfrak{p} & 0 & 0 \\ 8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 13 & 21\mathfrak{p} & 5 & 8\mathfrak{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5\mathfrak{p} & 1 & 2\mathfrak{p} \\ 21 & 13 & 8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 3 & 2 & 1 \end{bmatrix}.$$

For $\mathfrak{p} = 1$, i.e. for Fibonacci and Lucas dual quaternions with dual-hyperbolic number coefficients, the following determinants are computed:

$$\begin{aligned} \det(\mathcal{X}_{\widetilde{\mathcal{Q}}_n}) &= (2+2J) + (8+8J)e_1 + (12+12J)e_2 + (20+20J)e_3, \\ \det(\mathcal{A}_{\widetilde{\mathcal{Q}}_n}) &= 0, \\ \det(\mathcal{B}_{\widetilde{\mathcal{Q}}_n}) &= 41 + 40J + 436\varepsilon + 428J\varepsilon, \\ \det(\mathcal{C}_{\widetilde{\mathcal{Q}}_n}) &= 3281 + 3280J, \\ \det(\mathcal{D}_{\widetilde{\mathcal{Q}}_n}) &= 6561. \end{aligned}$$

In the same vein, for $\mathfrak{p} = 0$ and $\mathfrak{p} = -1$, the calculations are conducted for Fibonacci and Lucas dual quaternions with hyper-dual number coefficients and dual-complex number coefficients, respectively. Moreover for $\mathfrak{p} \in \mathbb{R}$, we have:

$$\mathcal{B}_{\widetilde{\mathcal{Q}}_2^{-1}} = \frac{1}{\sqrt{41 + 40J + 436\varepsilon + 428J\varepsilon}} \mathcal{B}_{\overline{\widetilde{\mathcal{Q}}}_2}.$$

Finally, the vector representation of $\vec{\widetilde{\mathcal{Q}}}_n^{\dagger_1}$ is

$$\begin{aligned} \vec{\widetilde{\mathcal{Q}}}_n^{\dagger_1} = \alpha \vec{\widetilde{\mathcal{Q}}}_n &= \begin{bmatrix} I_4 & 0 & 0 & 0 \\ 0 & -I_4 & 0 & 0 \\ 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & -I_4 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix}^T \\ \begin{bmatrix} 2 & 3 & 5 & 8 \end{bmatrix}^T \\ \begin{bmatrix} 3 & 5 & 8 & 13 \end{bmatrix}^T \\ \begin{bmatrix} 5 & 8 & 13 & 21 \end{bmatrix}^T \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 & 5 & 2 & 3 & 5 & 8 & 3 & 5 & 8 & 13 & 5 & 8 & 13 & 21 \end{bmatrix}^T. \end{aligned}$$

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