A NEW STRONG CONVERGENCE ALGORITHM WITH MOMENTUM FOR SOLVING THE SPLIT COMMON FIXED POINT PROBLEMS

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In this paper, we introduce a novel iterative algorithm for solving the split common fixed point problem with demicontractive operators. Strong convergence theorem of the proposed algorithm is given. The step-sizes of our algorithm are chosen such that they do not depend on operator norms. The main results proven in this paper extend and improve some results in the literature.

Keywords: Split common fixed point; demicontractive operator; self-adaptive algorithm; strong convergence.

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1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is formulated as finding a point x satisfying the property

$$x \in C \text{ such that } Ax \in Q,$$

where $A : H_1 \to H_2$ is a bounded linear operator. The split feasibility problem (SFP) has been shown to have broad applicability across various fields, including computer tomography, image restoration, radiation therapy treatment, and numerous other impactful real-world applications, for instance, see [1, 6, 9, 5, 11, 26]. Due to it has applications across various fields, recently, the SFP has been widely studied by many authors (see [5, 12, 13, 19, 21, 22, 23, 24, 25, 27, 29, 30, 31, 32]). Due to application in signal processing, Byrne [5] introduced the so-called CQ algorithm. For any $x_0 \in H_1$ and define $\{x_n\}$ as

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n),$$

where $0 < \gamma < \dfrac{2}{\rho(A^*A)}$ and where P_C denotes the projection onto C and $\rho(A^*A)$ is the spectral radius of the operator A^*A. It is known that the CQ algorithm converges weakly to a solution of the SFP if such a solution exists.

In the case where both C and Q consist of fixed point sets of some nonlinear operators, the SFP is known as the split common fixed point problem (SCFP). More specifically, the SCFP is to find

$$x \in Fix(S) \text{ such that } Ax \in Fix(T),$$

where $Fix(S)$ and $Fix(T)$ are the fixed point sets of $S : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively. We denote the solution set of the SCFP by

$$\Omega := \{x \in H_1 : x \in Fix(S) \text{ and } Ax \in Fix(T)\}. \tag{1}$$

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When S and T are directed operators, Censor and Segal [8] proposed and proved the convergence of the following algorithm in the setting of the finite dimensional spaces:

$$x_{n+1} = S(I - \gamma A^*(I - T)A)x_n.$$

Note that a class of directed operators include the metric projection. So the results of Censor and Segal recover Byrne's CQ algorithm.

In the case that S and T are demicontractive mappings with constants $\beta \in [0, 1)$, $\mu \in [0, 1)$ respectively, Moudafi [14] introduced the following algorithm for solving the SCFP (1) as follows:

$$\begin{cases} x_0 \in H_1 \\ u_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S u_n, \end{cases} \tag{2}$$

where $\alpha_n \in (\delta, 1-\beta-\delta)$ for a small enough $\delta > 0$ and $\gamma \in (0, \frac{1-\mu}{\rho})$ with ρ being the spectral radius of A^*A and he presented the weak convergence of the sequence generated by algorithm (2). It is obvious that to solve the SCFP (1) for demicontractive operators by the sequence generated by (2) requires the norm of the linear mapping A. This is quite challenging in practice. To overcome this difficulty recently, some authors considered alternative ways of constructing variable step sizes.

In [10], Cui and Wang combined algorithm (2) with a self-adaptive step size and introduced the following algorithm for solving the SCFP (1):

$$\begin{cases} x_0 \in H_1 \\ u_n = x_n - \tau_n A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \lambda)u_n + \lambda S u_n, \end{cases}$$

where $\lambda \in (0, 1 - \beta)$ and

$$\tau_n = \begin{cases} \dfrac{(1-\mu)\|(I - T)Ax_n\|^2}{\|A^*(I - T)Ax_n\|^2}, & \text{if } Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

They obtained a weak convergence result for demicontractive operators provided that $\Omega \neq \emptyset$.

The split common fixed point problem (SCFP) (1) for demicontractive operators with weak convergent results has since garnered significant attention and has been extensively investigated by numerous researchers (see [20, 21, 28]).

A natural question that arises in the case of infinite dimensional Hilbert spaces is how to design an algorithm which provides strong convergence for solving SCFP (1). Based on the algorithm of Cui and Wang, Boikanyo [3] developed the following Halpern-type algorithm for demicontractive operators, which generates sequences that consistently converge strongly to a solution of the SCFPP (1) with step sizes that do not depend on the norm of the operator A.

$$\begin{cases} x_0, u \in H_1 \\ u_n = x_n - \tau_n A^*(I - T)Ax_n, \\ y_n = (1 - \lambda)u_n + \lambda S u_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases}$$

and the step size τ_n is chosen as in (3).

Now, let us mention an inertial type algorithm. We know that the problem of finding a zero of a maximal monotone operator A on a real Hilbert H is formulated as

$$\text{find } x \in H \text{ such that } 0 \in A(x). \tag{4}$$

One of the fundamental approaches to solving it is the proximal method, which generates the next iteration x_{n+1} by solving the subproblem

$$0 \in \lambda_n A(x) + (x - x_n), \tag{5}$$

where x_n is the current iteration and λ_n is a regularization parameter, see [4, **?**, 16]. In 2001, Attouch and Alvarez [2] applied an inertial technique to the algorithm (5) to contruct an inertial proximal method for solving (4). It works as follows: given $x_{n-1},\ x_n \in H$ and two parameters $\theta_n \in [0,1), \lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}),$$

which can be written equivalently to the following

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})),$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$ and it can be regarded as procedure of seeding up the convergence properties (see, e.g., [2, 15]).

Motivated by Boikanyo's work [3], we use the inertial technique in this examined direction, we propose a new algorithm for solving SCFP of demicontractive operators that converge strongly to a solution of the problem (1). The aim of our work in this study is as follows:

(1) First, we introduce a new iterative algorithm that combine the algorithm (2) with the inertial technique to improve the convergence rate of the algorithm for solving SCFP.
(2) Second, under the suitable conditions, we prove strong convergence result of the iterative sequence generated by our algorithm without prior knowledge of operator norm A.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with analyzing the convergence of the proposed algorithm. Finally, in Sect. 4 conclusions are given.

2. **Preliminaries**

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$.

For each $x, y \in H$, we have the following:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle. \tag{6}$$

Definition 2.1. *Assume that $T : H \to H$ is a nonlinear operator with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:*

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \Longrightarrow x \in Fix(T).$$

Definition 2.2. *Let $T : H \to H$ be an operator with $Fix(T) \neq \emptyset$. Then*

- *$T : H \to H$ is called directed if*

$$\langle z - Tx, x - Tx\rangle \le 0 \quad \forall z \in Fix(T), x \in H,$$

or equivalently

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H, \tag{7}$$

- $T : H \to H$ *is called quasi-nonexpansive if*

$$\|Tx - z\| \leq \|x - z\| \quad \forall z \in Fix(T), x \in H;$$

- $T : H \to H$ *is called β-demicontractive with $0 \leq \beta < 1$ if*

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|(I - T)x\|^2 \quad \forall z \in Fix(T), x \in H,$$

or equivalently

$$\langle Tx - x, x - z\rangle \leq \frac{\beta - 1}{2}\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H,$$

or equivalently

$$\langle Tx - z, x - z\rangle \leq \|x - z\|^2 + \frac{\beta - 1}{2}\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H.$$

Lemma 2.1. *Let $U : H \to H$ is β- demicontractive with $F(U) \neq \emptyset$ and set $U_\lambda = (1 - \lambda)I + \lambda U$, $\lambda \in (0, 1 - \beta)$. Then:*

$$\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2 \quad \forall x \in H, z \in Fix(U).$$

Proof. We have

$$\begin{aligned}
\|U_\lambda x - z\|^2 &= \|(1 - \lambda)x + \lambda Ux - z\|^2 \\
&= \|(x - z) + \lambda(Ux - x)\|^2 \\
&= \|x - z\|^2 + 2\lambda\langle x - z, Ux - x\rangle + \lambda^2\|Ux - x\|^2 \\
&\leq \|x - z\|^2 + \lambda(\beta - 1)\|Ux - x\|^2 + \lambda^2\|Ux - x\|^2 \\
&= \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2 \\
&= \|x - z\|^2 - \frac{1 - \beta - \lambda}{\lambda}\|(I - U_\lambda)x\|^2.
\end{aligned}$$

□

More information on quasi-nonexpansive mappings and demicontractive mappings can be found, for example [7, 18].

Lemma 2.2. ([17]) *Let $\{a_n\}$ be sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ then $\lim_{n\to\infty} a_n = 0$.

3. **Strong convergence result**

Our strong convergence theorem is established under the following conditions:

Condition 3.1. *The solution set $\Omega \neq \emptyset$.*

Condition 3.2. *$S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two demicontractive operators with constants $\beta \in [0, 1)$ and $\mu \in [0, 1)$, respectively such that $I - S$ and $I - T$ are demiclosed at zero.*

Condition 3.3. *$A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*.*

Now, we introduce our algorithm:

Algorithm 3.1.
Initialization: *Let* $\lambda \in (0, 1-\beta)$, $\gamma \in (0,1)$, $\alpha > 0$, *and* $v_0, v_1 \in H$ *be arbitrary. We assume that* $\{\beta_n\}, \{\epsilon_n\}$ *are two positive sequences such that* $\epsilon_n = 0(\beta_n)$, *that is* $\lim_{n\to\infty} \dfrac{\epsilon_n}{\beta_n} = 0$, *where* $\{\beta_n\} \subset (0,1)$ *satisfies the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty.$$

Iterative Steps: *Calculate* v_{n+1} *as follows:*
Step 1. *Given the current iterates* v_{n-1} *and* v_n $(n \geq 1)$, *compute*

$$\begin{cases} q_n = (1-\beta_n)[v_n + \alpha_n(v_n - v_{n-1})], \\ u_n = q_n - \tau_n A^*(I-T)Aq_n, \end{cases}$$

where

$$\tau_n = (1-\mu)\gamma \frac{\|(I-T)Aq_n\|^2}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2},$$

and

$$\alpha_n = \begin{cases} \min\{\alpha, \dfrac{\epsilon_n}{\|v_n - v_{n-1}\|}\} & \text{if } v_n \neq v_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{8}$$

If $u_n = 0$ *then stop and* q_n *is a solution of problem* (1). *Otherwise, go to* **Step 2**.
Step 2. *Compute*

$$v_{n+1} = U_\lambda u_n,$$

where $U_\lambda := (1-\lambda)I + \lambda S$.
Let $n := n+1$ *and return to* **Step 1**.

Remark 3.1. *From* (8), *the definition of* $\{\alpha_n\}$ *we have* $\lim_{n\to+\infty} \dfrac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| = 0$.

Theorem 3.1. *Assume that the Conditions* 3.1–3.3 *hold. Then the sequence* $\{v_n\}$ *generated by Algorithm 3.1 converges strongly to an element* $x^* \in \Omega$, *where* $\|x^*\| = \min\{\|u\| : u \in \Omega\}$.

Proof. **Claim 1.** The sequence $\{v_n\}$ is bounded. Indeed, from $x^* \in \Omega$ and using inequality (7) we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|q_n - \tau_n A^*(I-T)Aq_n - x^*\|^2 \\
&= \|q_n - x^*\|^2 + \tau_n^2\|A^*(I-T)Aq_n\|^2 - 2\tau_n\langle A^*(I-T)Aq_n, q_n - x^*\rangle \\
&= \|q_n - x^*\|^2 + \tau_n^2\|A^*(I-T)At_n\|^2 - 2\tau_n\langle (I-T)Aq_n, Aq_n - Ax^*\rangle \\
&\leq \|q_n - x^*\|^2 + \tau_n^2\|A^*(I-T)Aq_n\|^2 - (1-\mu)\tau_n\|(I-T)Aq_n\|^2 \\
&= \|q_n - x^*\|^2 + (1-\mu)^2\gamma^2 \frac{\|(I-T)Aq_n\|^4}{(\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2)^2}\|A^*(I-T)Aq_n\|^2 \\
&\quad - (1-\mu)^2\gamma \frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} \\
&\leq \|q_n - x^*\|^2 + (1-\mu)^2\gamma^2 \frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} \\
&\quad - (1-\mu)^2\gamma \frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} \\
&= \|q_n - x^*\|^2 - (1-\mu)^2\gamma(1-\gamma) \frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2}.
\end{aligned} \tag{9}$$

This implies that
$$\|u_n - x^*\| \le \|q_n - x^*\|. \tag{10}$$
We have $v_{n+1} = U_\lambda u_n$, we get
$$\begin{aligned}\|v_{n+1} - x^*\|^2 &= \|(1-\lambda)u_n + \lambda S u_n - x^*\|^2 \\ &= \|u_n - x^* + \lambda(Su_n - u_n)\|^2 \\ &= \|u_n - x^*\|^2 + 2\lambda\langle Su_n - u_n, u_n - x^*\rangle + \lambda^2\|Su_n - u_n\|^2 \\ &\le \|u_n - x^*\|^2 + \lambda(\beta - 1)\|Su_n - u_n\|^2 + \lambda^2\|Su_n - u_n\|^2 \\ &= \|u_n - x^*\|^2 - \lambda(1-\beta-\lambda)\|Su_n - u_n\|^2. \end{aligned} \tag{11}$$
Hence, we get
$$\|v_{n+1} - x^*\| \le \|u_n - x^*\|. \tag{12}$$
Combining (10) and (12) we get
$$\|v_{n+1} - x^*\| \le \|u_n - x^*\| \le \|q_n - x^*\|. \tag{13}$$
On the other hand, from the definition of v_n, we get
$$\begin{aligned}\|q_n - x^*\| &= \|(1-\beta_n)(v_n + \alpha_n(v_n - v_{n-1})) - x^*\| \\ &= \|(1-\beta_n)(v_n - x^*) + (1-\beta_n)\alpha_n(v_n - v_{n-1}) - \beta_n x^*\| \\ &\le (1-\beta_n)\|v_n - x^*\| + (1-\beta_n)\alpha_n\|v_n - v_{n-1}\| + \beta_n\|x^*\| \\ &= (1-\beta_n)\|v_n - x^*\| + \beta_n[(1-\beta_n)\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| + \|x^*\|]. \end{aligned} \tag{14}$$
From the definition of α_n we get $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| = 0$. Therefore, we deduce
$$\lim_{n\to\infty}\left[(1-\beta_n)\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| + \|x^*\|\right] = \|x^*\|.$$
Thus there exists $M > 0$ such that
$$(1-\beta_n)\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| + \|x^*\| \le M. \tag{15}$$
Combining (14) and (15), we obtain
$$\|q_n - x^*\| \le (1-\beta_n)\|v_n - x^*\| + \beta_n M. \tag{16}$$
Using (13) and (16) we get
$$\begin{aligned}\|v_{n+1} - x^*\| &\le (1-\beta_n)\|v_n - x^*\| + \beta_n M \\ &= \max\{\|v_n - x^*\|, M\} \le ... \le \max\{\|v_1 - x^*\|, M\}.\end{aligned}$$
Therefore, the sequence $\{v_n\}$ is indeed bounded, as claimed.

Claim 2.
$$\begin{aligned}(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} &+ \lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 \\ &\le \|v_n - x^*\|^2 - \|v_{n+1} - x^*\|^2 + \beta_n M_1,\end{aligned}$$
for some $M_1 > 0$. Indeed, by (16) we get
$$\begin{aligned}\|q_n - x^*\|^2 &\le [(1-\beta_n)\|v_n - x^*\| + \beta_n M]^2 \\ &\le [\|v_n - x^*\| + \beta_n M]^2 \\ &= \|v_n - x^*\|^2 + 2\beta_n\|v_n - x^*\|M + \beta_n^2 M^2 \\ &= \|v_n - x^*\|^2 + \beta_n[2\|v_n - x^*\|M + \beta_n M^2] \\ &\le \|v_n - x^*\|^2 + \beta_n M_1. \end{aligned} \tag{17}$$

where $M_1 = \max_n\{2\|v_n - x^*\|M + \beta_n M^2\}$. Now, substituting (17) into (9) we get

$$\|u_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \beta_n M_1 - (1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2}. \tag{18}$$

Substituting (18) into (11) we obtain

$$\begin{aligned}\|v_{n+1} - x^*\|^2 \leq \|v_n - x^*\|^2 + \beta_n M_1 - (1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} \\ - \lambda(1-\beta-\lambda)\|Su_n - u_n\|^2.\end{aligned}$$

Hence

$$\begin{aligned}(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} + \lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 \\ \leq \|v_n - x^*\|^2 - \|v_{n+1} - x^*\|^2 + \beta_n M_1.\end{aligned}$$

Claim 3.

$$\begin{aligned}\|v_{n+1} - x^*\|^2 \leq &(1-\beta_n)\|v_n - x^*\|^2 + \beta_n\big[2(1-\beta_n)\|v_n - x^*\|\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| \\ &+ \alpha_n\|v_n - v_{n-1}\|\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| + 2\|z\|\|q_n - v_{n+1}\| + 2\langle -x^*, v_{n+1} - x^*\rangle\big].\end{aligned}$$

Indeed, using the inequalities (13) and (6), we get

$$\begin{aligned}\|v_{n+1} - x^*\|^2 \leq &\|q_n - x^*\|^2 \\ =&\|(1-\beta_n)[v_n + \alpha_n(v_n - v_{n-1})] - x^*\|^2 \\ =&\|(1-\beta_n)(v_n - x^*) + (1-\beta_n)\alpha_n(v_n - v_{n-1}) - \beta_n x^*\|^2 \\ \leq&\|(1-\beta_n)(v_n - x^*) + (1-\beta_n)\alpha_n(v_n - v_{n-1})\|^2 + 2\beta_n\langle -x^*, q_n - x^*\rangle \\ \leq&(1-\beta_n)^2\|v_n - x^*\|^2 + 2(1-\beta_n)\alpha_n\|v_n - x^*\|\|v_n - v_{n-1}\| + \alpha_n^2\|v_n - v_{n-1}\|^2 \\ &+ 2\beta_n\langle -x^*, q_n - v_{n+1}\rangle + 2\beta_n\langle -x^*, v_{n+1} - x^*\rangle \\ \leq&(1-\beta_n)\|v_n - x^*\|^2 + \beta_n\big[2(1-\beta_n)\|v_n - x^*\|\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| \\ &+ \alpha_n\|v_n - v_{n-1}\|\frac{\alpha_n}{\beta_n}\|v_n - v_{n-1}\| + 2\|x^*\|\|q_n - v_{n+1}\| + 2\langle -x^*, v_{n+1} - x^*\rangle\big].\end{aligned}$$

Claim 4. $\{\|v_n - x^*\|^2\}$ converges to zero. Indeed, by Lemma 2.2 it suffices to show that

$$\limsup_{k\to\infty}\langle x^*, v_{n_k+1} - x^*\rangle \leq 0 \quad \text{and} \quad \lim_{k\to\infty}\|q_{n_k} - v_{n_k+1}\| = 0$$

for every subsequence $\{\|v_{n_k} - x^*\|\}$ of $\{\|v_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|) \geq 0.$$

For this, suppose that $\{\|v_{n_k} - x^*\|\}$ is a subsequence of $\{\|v_n - x^*\|\}$ such that $\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|) \geq 0$. Then

$$\begin{aligned}&\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\|^2 - \|v_{n_k} - x^*\|^2) \\ &= \liminf_{k\to\infty}[(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|)(\|v_{n_k+1} - x^*\| + \|v_{n_k} - x^*\|)] \geq 0.\end{aligned}$$

By Claim 2 we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Big((1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_{n_k}\|^4}{\|q_{n_k}-Sq_{n_k}\|^2+\|A^*(I-T)Aq_{n_k}\|^2}+\lambda(1-\beta-\lambda)\|Su_{n_k}-u_{n_k}\|^2\Big)\\
&\le \limsup_{k\to\infty}[\|v_{n_k}-x^*\|^2-\|v_{n_k+1}-x^*\|^2+\beta_{n_k}M_1]\\
&\le \limsup_{k\to\infty}[\|v_{n_k}-x^*\|^2-\|v_{n_k+1}-x^*\|^2]+\limsup_{k\to\infty}\beta_{n_k}M_1\\
&= -\liminf_{k\to\infty}[\|v_{n_k+1}-x^*\|^2-\|v_{n_k}-x^*\|^2]\\
&\le 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\frac{\|(I-T)Aq_{n_k}\|^4}{\|q_{n_k}-Sq_{n_k}\|^2+\|A^*(I-T)Aq_{n_k}\|^2}=0 \text{ and } \lim_{k\to\infty}\|(I-S)u_{n_k}\|=0. \tag{19}$$

Moreover, since $\{v_n\}$ is bounded, using the inequality (16) we also obtain $\{q_n\}$ is bounded. By Lemma 2.1 we get $\{q_{n_k}-Sq_{n_k}\}$ is bounded and $\{A^*(I-T)Aq_{n_k}\}$ is bounded. Therefore it follows from (19) that

$$\lim_{n\to\infty}\|(I-T)Aq_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\|u_{n_k}-Su_{n_k}\|=0. \tag{20}$$

On the other hand, using the definition of $\{u_{n_k}\}$, see that

$$\begin{aligned}
\|u_{n_k}-q_{n_k}\|&=\tau_{n_k}\|A^*(I-T)Aq_{n_k}\|\\
&\le \frac{\|(I-T)Aq_{n_k}\|^2}{\|q_{n_k}-Sq_{n_k}\|^2+\|A^*(I-T)Aq_{n_k}\|^2}\|A^*\|\|(I-T)Aq_{n_k}\|\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{21}$$

Using (20) and the definition of $\{v_n\}$, we get

$$\|v_{n_k+1}-u_{n_k}\|=\lambda\|Su_{n_k}-u_{n_k}\|\to 0 \text{ as } n\to\infty. \tag{22}$$

Now, we show that

$$\|v_{n_k+1}-v_{n_k}\|\to 0 \text{ as } n\to\infty. \tag{23}$$

Indeed, using the definition of $\{v_n\}$ we have

$$\begin{aligned}
\|q_{n_k}-v_{n_k}\|&=\|(1-\beta_{n_k})\alpha_{n_k}(v_{n_k}-v_{n_k-1})-\beta_{n_k}v_{n_k}\|\\
&\le(1-\beta_{n_k})\alpha_{n_k}\|v_{n_k}-v_{n_k-1}\|+\beta_{n_k}\|v_{n_k}\|\\
&=\beta_{n_k}\big[(1-\beta_{n_k})\frac{\alpha_{n_k}}{\beta_{n_k}}\|v_{n_k}-v_{n_k-1}\|+\|v_{n_k}\|\big]\to 0 \text{ as } n\to+\infty.
\end{aligned} \tag{24}$$

Combining (21) and (24) we have

$$\|v_{n_k}-u_{n_k}\|\to 0. \tag{25}$$

Combining (22) and (25), we deduce that

$$\lim_{k\to+\infty}\|v_{n_k+1}-v_{n_k}\|=0. \tag{26}$$

Combining (24) and (26), we deduce that

$$\lim_{k\to+\infty}\|v_{n_k+1}-q_{n_k}\|=0.$$

Since the sequence $\{v_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$, which converges weakly to some $z\in H$, such that

$$\limsup_{k\to\infty}\langle x^*,v_{n_k}-x^*\rangle=\lim_{j\to\infty}\langle x^*,v_{n_{k_j}}-x^*\rangle=\langle x^*,z-x^*\rangle. \tag{27}$$

From (24) and (21) we get

$$q_{n_k} \rightharpoonup z, \text{ and } u_{n_k} \rightharpoonup z,$$

this together with (20) and the demiclosedness of $I-S$ and $I-T$, we have $z \in \Omega$ and, from (27) and the definition of $x^* = P_\Omega(0)$, we have

$$\limsup_{k\to\infty} \langle x^*, v_{n_k} - x^* \rangle = \langle x^*, z - x^* \rangle \le 0. \tag{28}$$

Combining (23) and (28), we have

$$\begin{aligned} \limsup_{k\to\infty} \langle x^*, v_{n_k+1} - x^* \rangle &\le \limsup_{k\to\infty} \langle x^*, v_{n_k} - x^* \rangle \\ &= \langle x^*, z - x^* \rangle \\ &\le 0. \end{aligned} \tag{29}$$

Hence, by (29), $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \|v_n - v_{n-1}\| = 0$, Claim 3 and Lemma 2.2, we have $\lim_{n\to\infty} \|v_n - x^*\| = 0$. This is the desired result.

□

4. **Conclusions**

In this work, we proposed a new iteration algorithm to obtain the strong convergence results for split common fixed point problems. Our main results are an extension of the related results announced in this direction. This paper's research highlights include a novel algorithm and its analysis techniques, which use inertia to improve algorithm convergence rate.

REFERENCES

[1] R. Abaidoo, E.K. Agyapong, Financial development and institutional quality among emerging economies. J. Econ. Dev. **24** (2022), 198-216.

[2] F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Anal. **9** (2001), 3-11.

[3] O.A. Boikanyo, A strongly convergent algorithm for the split common fixed point problem. Appl Math Comput. **265** (2015), 844-853.

[4] R.E. Bruck, S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces. Houston J. Math. **3** (1977), 459-470.

[5] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Prob. **18** (2002), 441-453.

[6] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Prob. **18** (2004), 103-120.

[7] A. Cegielski, S. Reich, R. Zalas, Regular sequences of quasi-nonexpansive operators and their applications. SIAM J. Optim. **28** (2018), 1508–1532.

[8] Y. Censor, A. Segal, The split common fixed point problem for directed operators. J. Convex Anal. **16** (2009), 587-600.

[9] P.L. Combettes, Hilbertian convex feasibility problem: Convergence of projection methods. Applied Mathematics and Optimization. 35(3) (1997), 311-330.

[10] H. Cui, F. Wang, Iterative methods for the split common fixed point problem in Hilbert spaces. Fixed Point Theory Appl. 2014, 1-8 (2014)

[11] M. Eslamian, A. Kamandi, A. Tahmasbi, Inertial methods for split common fixed point problems: application to binary classification in machine learning. Comp. Appl. Math. **43** (2024), 371.

[12] R. Kraikaew, S. Saejung, On split common fixed point problems. J. Math. Anal. Appl. **415** (2014), 513-524.

[13] A. Kumar, B.S., Thakur, M. Postolache, Dynamic stepsize iteration process for solving split common fixed point problems with applications. Math. Comput. Simulat. **218** (2024), 498-511.

[14] A. Moudafi, The split common fixed point problem for demicontractive mappings. Inverse Prob. **26** (2010), 055007.

[15] B.T. Polyak, Some methods of speeding up the convergence of iterative methods. Zh. Vychisl. Mat. Mat. Fiz. **4** (1964), 1-17
[16] R.T. Rockafellar, Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14** (1976), 877-898.
[17] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. **75** (2012), 742–750.
[18] D.V. Thong, Viscosity approximation methods for solving fixed point problems and split common fixed point problems. J. Fixed Point Theory Appl. **19** (2017), 1481-1499.
[19] F. Wang, H.K. Xu, Cyclic algorithms for split feasibility problems in Hilbert spaces. Nonlinear Anal. **74** (2011), 4105-4111.
[20] F. Wang, A new iterative method for the split common fixed point problem in Hilbert spaces. Optimization **66** (2017), 407-415.
[21] F. Wang, The split feasibility problem with multiple output sets for demicontractive mappings. J Optim Theory Appl **195**, 837–853 (2022)
[22] Y. Wang, J. Chen, A. Pitea, The split equality fixed point problem of demicontractive operators with numerical example and application. Symmetry-Basel. **12** (2020), Art. No. 902.
[23] D. Wu, M. Postolache, Two iterative algorithms for solving the split common fixed point problems. Filomat **34** (2020), No. 13, 4375-4386.
[24] H.K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. Inverse Prob. **22** (2006), 2021-2034.
[25] H.K. Xu, Iterative methods for the split feasibility problem in infinite dimensional Hilbert spaces. Inverse Prob. **26** (2010), 105018.
[26] Z.Y. Xu, W.Y. Qian, The split feasibility problem and its application to matrix equations. Applied Mathematics and Computation. **218**(9) (2012), 5012-5021.
[27] Q. Yang, The relaxed CQ algorithm for solving the split feasibility problem. Inverse Prob. **20** (2004), 1261-1266.
[28] Y.H. Yao, Y.C. Liou, M. Postolache, Self-adaptive algorithms for the split problem of the demicontractive operators. Optimization **67** (2017), 1309-1319.
[29] L. Yang, L.J. Zhou, M. Postolache, An accelerated Mann-type method for split common fixed point problem in Hilbert spaces. U. Politeh Buch. Ser. A **86**(2024), No. 1, 3-16.
[30] Y. Yao, J.C., Yao, Y.C. Liou, M. Postolache, Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms. Carpathian J. Math. **34** (2018), No. 3, 459-466.
[31] Y. Yao, L. Leng, M. Postolache, X. Zheng, Mann-type iteration method for solving the split common fixed point problem. J. Nonlinear Convex Anal. **18** (2017), No. 5, 875-882.
[32] T.C. Yin, A. Pitea, Strong convergence of an iterative procedure for pseudomonotone variational inequalities and fixed point problems. Filomat. **36** (2022), 4111-4122.