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CHARACTERIZATIONS OF RIGHT MODULAR GROUPOIDS BY $(\in,\in\vee q_k)$ -FUZZY INTERIOR IDEALS

Madad KHAN[1], Young Bae JUN[2], Shamas-ur-REHMAN[3]

In this paper, we introduce the concept of $(\in,\in\vee q)$*-fuzzy ideals in a right modular groupoid. We discuss several important features of a completely regular right modular groupoid by using the generalized fuzzy ideals.*

Keywords: Right modular groupoid, completely regular, $(\in,\in\vee q)$-fuzzy ideals and $(\in,\in\vee q_k)$-fuzzy ideals.

1. Introduction

The specific models of real world problems in various fields such as computer science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and many other fields are hard to construct with precision because of uncertainty on many occasions. For handling such difficulties we need some natural tools such as probability theory and theory of fuzzy sets [18] which have already been developed. Associative Algebraic structures are mostly used for applications of fuzzy sets. Mordeson, Malik and Kuroki [10] have discovered the vast field of fuzzy semigroups, where theoretical exploration of fuzzy semigroups and their applications are used in fuzzy coding, fuzzy finite-state machines and fuzzy languages. The use of fuzzification in automata and formal language has widely been explored. Moreover complete l-semigroups have a wide range of applications in the theories of automata, formal languages and programming.

The fundamental concept of fuzzy sets was first introduced by Zadeh [18]. Rosenfeld introduced the definition of a fuzzy subgroup of a group [15]. Kuroki initiated the theory of fuzzy bi ideals in semigroups [8]. The thought of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was defined by Murali [11]. The concept of quasi-coincidence of a fuzzy point to a fuzzy set was introduced in [14]. Jun and Song introduced (α,β)-fuzzy interior ideals in semigroups [4].

[1] Dr. Head, Department of Mathematics COMSATS Institute of Information Technology Abbottabad, Pakistan. **madadmath@yahoo.com**
[2] Professor, Department of Mathematics Education Gyeongsang National University Chinju, Korea. skywine@gmail.com
[3] PhD, Department of Mathematics COMSATS Institute of Information Technology Abbottabad, Pakistan. **shamas_200814@yahoo.com**

In this paper we characterize non-associative algebraic structures called right modular groupoids by their $(\in,\in\vee q_k)$-fuzzy ideals. A right modular groupoid M is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup.
The concept of a left almost semigroup (LA-semigroup) [5] or a right modular groupoid was first given by M. A. Kazim and M. Naseeruddin in 1972. A right modular groupoid M is a groupoid having the left invertive law,

$$(ab)c=(cb)a, \text{ for all } a,b,c\in M. \tag{1}$$

In a right modular groupoid M, the following medial law [5] holds,

$$(ab)(cd)=(ac)(bd), \text{ for all } a,b,c,d\in M. \tag{2}$$

The left identity in a right modular groupoid if exists is unique[12]. In a right modular groupoid M with left identity the following paramedial law holds [13],

$$(ab)(cd)=(dc)(ba), \text{ for all } a,b,c,d\in M. \tag{3}$$

If a right modular groupoid M contains a left identity, then,

$$a(bc)=b(ac), \text{ for all } a,b,c\in M. \tag{4}$$

2. Preliminaries

Let M be a right modular groupoid. By a subgroupoid of M, we mean a non-empty subset A of M such that $A^2\subseteq A$. A non-empty subset A of a right modular groupoid M is called left (right) ideal of M if $MA\subseteq A$ $(AM\subseteq A)$. A is called two-sided ideal or simply ideal if it is both a left and a right ideal of M. A non-empty subset A of a right modular groupoid M is called generalized bi-ideal of M if $(AM)A\subseteq A$. A subgroupoid A of M is called bi-ideal of M if $(AM)A\subseteq A$. A subgroupoid A of M is called interior ideal of M if $(MA)M\subseteq A$. A non-empty subset A of a right modular groupoid M is called quasi-ideal of M if $QM\cap MQ\subseteq Q$. Every one sided ideal is quasi ideal and every bi-ideal is generalized bi-ideal. Also every two sided ideal is an interior ideal but converse is not true.

Definition *1. A fuzzy subset F of a right modular groupoid M is called a fuzzy interior ideal of M if it satisfies the following conditions,*

(i) $F(xy)\geq \min\{F(x),F(y)\}$ *for all* $x,y\in M$.

(ii) $F((xa)y)\geq F(a)$ *for all* $x,a,y\in M$.

Definition 2. *A fuzzy subset F of a right modular groupoid M of the form*

$$F(y)=\begin{cases} t\in(0,1] & \text{if } y=x\\ 0 & \text{if } y\neq x\end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

A fuzzy point x_t is said to *belong* to (*resp. quasi-coincident with*) a fuzzy set F, written as $x_t\in F$ (*resp.* x_tqF) if $F(x)\geq t$ (*resp.* $F(x)+t>1$). If $x_t\in F$ or x_tqF, then we write $x_t\in\vee q$. The symbol $\overline{\in\vee q}$ means $\in\vee q$ does not hold. For any two fuzzy subsets f and g of S, $f\leq g$ means that, for all $x\in S$, $f(x)\leq g(x)$.

Lemma 1. *(cf. [7]) A fuzzy subset F of a right modular groupoid M is a fuzzy interior ideal of M if and only if $U(F;t)$ (non-empty) is an interior ideal of M.*

Definition ***3***. *A fuzzy subset F of a right modular groupoid M is called an $(\in,\in\vee q)$-fuzzy interior ideal of M if for all $t,r\in(0,1]$ and $x,a,y\in M$.*

$(A1)$ $x_t\in F$ *and* $y_r\in F$ *implies that* $(xy)_{\min\{t,r\}}\in\vee qF$.

$(A2)$ $a_t\in F$ *implies* $((xa)y)_t\in\vee qF$.

Theorem 1. *(cf. [7]) For a fuzzy subset F of a right modular groupoid M. The conditions $(A1)$ and $(A2)$ of Definition 4, are equivalent to the following,*

$(A3)$ $(\forall x,y\in M)F(xy)\geq\min\{F(x),F(y),0.5\}$

$(A4)$ $(\forall x,a,y\in M)F((xa)y)\geq\min\{F(a),0.5\}$.

Definition 4. *A fuzzy subset F of a right modular groupoid M is called an $(\in,\in\vee q)$-fuzzy bi-ideal of M if for all $t,r\in(0,1]$ and $x,y,z\in M$.*

$(B1)$ $x_t\in F$ *and* $y_r\in F$ *implies that* $(xy)_{\min\{t,r\}}\in\vee qF$.

$(B2)$ $x_t\in F$ *and* $z_r\in F$ *implies* $((xy)z)_{\min\{t,r\}}\in\vee qF$.

Theorem 2. *For a fuzzy subset F of a right modular groupoid M. The conditions $(B1)$ and $(B2)$ of Definition 5 are equivalent to the following,*

$(B3)$ $(\forall x,y\in M)F(xy)\geq\min\{F(x),F(y),0.5\}$

$(B4)$ $(\forall x,y,z\in M)F((xy)z)\geq\min\{F(x),F(y),0.5\}$.

Proof. It is similar to proof of Theorem 1.

Example 1. *Let $M=\{1,2,3\}$ be a right regular modular groupoid and "$\cdot$" be any binary operation defined as follows:*

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Let F be a fuzzy subset of M such that

$$F(1) = 0.6, \quad F(2) = 0.3, \quad F(3) = 0.2.$$

Then we can see easily $F(1\cdot 3) \geq F(3) \wedge 0.5$ that is F is an $(\in, \in \vee q)$-fuzzy left ideal but F is not an $(\in, \in \vee q)$-fuzzy right ideal.

3. Completely Regular Right Modular Groupoids

Definition *5*. *A right modular groupoid M is called regular, if for each $a \in M$ there exists $x \in M$ such that $a = (ax)a$.*

Definition 6. *A right modular groupoid M is called left (right) regular, if for each $a \in M$ there exists $z \in M$ $(y \in M)$ such that $a = za^2$ $(a = a^2 y)$.*

Definition *7*. *A right modular groupoid M is called completely regular if it is regular, left regular and right regular.*

Example 2. *Let $M = \{1, 2, 3, 4\}$ and the binary operation "$\cdot$" defined on M as follows:*

$\cdot$	1	2	3	4
1	4	1	2	3
2	3	4	1	2
3	2	3	4	1
4	1	2	3	4

Then clearly $(M, \cdot)$ is a completely regular right modular groupoid with left identity 4.

Theorem *3.* *If M is a right modular groupoid with left identity e, then it is completely regular if and only if $a \in (a^2 M)a^2$.*

Proof. Let M be a completely regular right modular groupoid with left identity e, then for each $a \in M$ there exist $x, y, z \in M$ such that $a = (ax)a$, $a = a^2 y$ and $a = za^2$, so by using (1), (4) and (3), we get

$$
\begin{aligned}
a &= (ax)a = ((a^2y)x)(za^2) = ((xy)a^2)(za^2) = ((za^2)a^2)(xy) \\
&= ((a^2a^2)z)(xy) = ((xy)z)(a^2a^2) = a^2(((xy)z)a^2) \\
&= (ea^2)(((xy)z)a^2) = (a^2((xy)z))(a^2e) = (a^2((xy)z))((aa)e) \\
&= (a^2((xy)z))((ea)a) = (a^2((xy)z))a^2 \in (a^2M)a^2.
\end{aligned}
$$

Conversely, assume that $a \in (a^2M)a^2$ then clearly $a = a^2y$ and $a = za^2$, now using $(3), (1)$ and (4), we get

$$
\begin{aligned}
a &\in (a^2M)a^2 = (a^2M)(aa) = (aa)(Ma^2) = (aa)(M(aa)) \\
&= (aa)((eM)(aa)) = (aa)((aa)(Me)) \subseteq (aa)((aa)M) \\
&= (aa)(a^2M) = ((a^2M)a)a = (((aa)M)a)a = ((aM)(aa))a \\
&= (a((Ma)a))a \subseteq (aM)a.
\end{aligned}
$$

Therefore M is completely regular.

4. $(\in,\in\vee q_k)$ -fuzzy Ideals in Right Modular Groupoids

It has been given in [3] that $x_t q_k F$ is the generalizations of $x_t qF$, where k is an arbitrary element of $[0,1)$ as $x_t q_k F$ if $F(x)+t+k>1$. If $x_t \in F$ or $x_t q_k F$ implies $x_t \in\vee q_k F$. Here we discuss the behavior of $(\in,\in\vee q_k)$ -fuzzy left ideal, $(\in,\in\vee q_k)$ -fuzzy right ideal, $(\in,\in\vee q_k)$ -fuzzy interior ideal, $(\in,\in\vee q_k)$ -fuzzy bi-ideal, $(\in,\in\vee q_k)$ -fuzzy quasi-ideal in the completely regular right modular groupoid M.

Example 3. *Let $M = \{1, 2, 3\}$ be any right modular groupoid with binary operation "$\cdot$" defined as in Example 1. Let F be a fuzzy subset of M such that* $F(1) = 0.7$, $F(2) = 0.4$, $F(3) = 0.3$. *Then F is an $(\in,\in\vee q_k)$ -fuzzy ideal but clearly F is not an $(\in,\in\vee q)$ -fuzzy ideal.*

Definition 8. *A fuzzy subset F of a right modular groupoid M is called an $(\in,\in\vee q_k)$ -fuzzy subgroupoid of M if for all $x, y \in M$ and $t, r \in (0,1]$ the following condition holds* $x_t \in F, y_r \in F$ implies $(xy)_{\min\{t,r\}} \in\vee q_k F$.

Theorem 4. *Let F be a fuzzy subset of M. Then F is an $(\in,\in\vee q_k)$ -fuzzy subgroupoid of M if and only if* $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$.

Proof. It is similar to the proof of Theorem 1.

Definition ***9***. *A fuzzy subset* F *of a right modular groupoid* M *is called an* $(\in,\in\vee q_k)$*-fuzzy left (right) ideal of* M *if for all* $x,y\in M$ *and* $t,r\in(0,1]$ *the following condition holds*

$$y_t\in F \text{ implies } (xy)_t\in\vee q_k F\ \left(y_t\in F \text{ implies } (yx)_t\in\vee q_k F\right).$$

Theorem ***5***. *Let* F *be a fuzzy subset of* M. *Then* F *is an* $(\in,\in\vee q_k)$*-fuzzy left (right) ideal of* M *if and only if* $F(xy)\geq\min\{F(y),\frac{1-k}{2}\}\left(F(xy)\geq\min\{F(x),\frac{1-k}{2}\}\right)$.

Proof. It is same as in [16].

Definition ***10***. *A fuzzy subset* F *of a right modular groupoid* M *is called an* $(\in,\in\vee q_k)$*-fuzzy generalized bi-ideal of* M *if for all* $x,y,z\in M$ *and* $t,r\in(0,1]$ *the following condition hold*

$$\text{if } x_t\in F \text{ and } z_r\in F \text{ implies } ((xy)z)_{\min\{t,r\}}\in\vee q_k F.$$

An $(\in,\in\vee q_k)$*-fuzzy generalized bi-ideal of* M *is called an* $(\in,\in\vee q_k)$*-fuzzy bi-ideal of* S *if it is also an* $(\in,\in\vee q_k)$*-fuzzy AG-subgroupoid of* M.

Definition ***11.*** *Let* M *be a right modular groupoid and* f *be a fuzzy subset of* M. *Then* f *is an* $(\in,\in\vee q)$*-fuzzy generalized bi-ideal of* M, *if for all* x, y, $z\in M$ *and* t, $r\in(0,1]$, *we have*

$$x_t\in f, z_r\in f\Rightarrow((xy)z)_{t\wedge r}\in\vee qf.$$

An $(\in,\in\vee q)$*-fuzzy generalized bi-ideal of* M *is called an* $(\in,\in\vee q)$*-fuzzy bi-ideal of* S *if it is also an* $(\in,\in\vee q)$*-fuzzy AG-subgroupoid of* M.

Theorem ***6***. *Let* F *be a fuzzy subset of* M. *Then* F *is an* $(\in,\in\vee q_k)$*-fuzzy bi-ideal of* M *if and only if*

$(i)\,F(xy)\geq\min\{F(x),F(y),\frac{1-k}{2}\}$ *for all* $x,y\in M$ *and* $k\in[0,1)$,

$(ii)\,F((xy)z)\geq\min\{F(x),F(z),\frac{1-k}{2}\}$ *for all* $x,y,z\in M$ *and* $k\in[0,1)$.

Proof. It is similar to the proof of Theorem 1.

Corollary ***1.*** *Let* F *be a fuzzy subset of* M. *Then* F *is an* $(\in,\in\vee q_k)$*-fuzzy generalized bi-ideal of* M *if and only if* $F((xy)z)\geq\min\{F(x),F(z),\frac{1-k}{2}\}$ *for all* $x,y,z\in M$ *and* $k\in[0,1)$.

Definition ***12***. *A fuzzy subset* F *of a right modular groupoid* M *is called an* $(\in,\in\vee q_k)$*-fuzzy interior ideal of* M *if for all* $x,a,y\in M$ *and* $t,r\in(0,1]$ *the following conditions hold*

(i) *If* $x_t\in F$ *and* $y_r\in M$ *implies* $(xy)_{\min\{t,r\}}\in\vee q_k F$,

(ii) *If* $a_t \in M$ *implies* $((xa)y)_{\min\{t,r\}} \in \vee q_k F$.

Theorem *7.* *Let* F *be a fuzzy subset of* M . *Then* F *is an* $(\in, \in \vee q_k)$ *-fuzzy interior ideal of* M *if and only if*

(i) $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ *for all* $x, y \in M$ *and* $k \in [0,1)$,

(ii) $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\}$ *for all* $x, a, y \in M$ *and* $k \in [0,1)$.

Proof. It is similar to the proof of Theorem 1.

Definition *13*. *Let* f *and* g *be a fuzzy subsets of the AG-groupoid* S , *then the* k *-product of* f *and* g *is defined by*

$$(f \circ_k g)(a) = \begin{cases} \bigvee_{a=bc} \min\{f(a),\ f(b),\ \frac{1-k}{2}\} & \text{if there exists } b, c \in S \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

where $k \in [0,1)$.

The k *intersection of* f *and* g *is defined by*

$$(f \cap_k g)(a) = \{f(a) \wedge f(b) \wedge \frac{1-k}{2}\} \text{ for all } a \in S.$$

Definition *14*. *A fuzzy subset* F *of a right modular groupoid* M *is called an* $(\in, \in \vee q_k)$ *-fuzzy quasi-ideal of* M *if following condition holds*

$$F(x) \geq \min\left\{(F \circ 1)(x), (1 \circ F)(x), \frac{1-k}{2}\right\}.$$

where 1 *is the fuzzy subset of* M *mapping every element of* M *on* 1.

Lemma *2*. *If* M *is a completely regular right modular groupoid with left identity, then a fuzzy subset* F *is an* $(\in, \in \vee q_k)$ *-fuzzy right ideal of* M *if and only if* F *is an* $(\in, \in \vee q_k)$ *-fuzzy left ideal of* M .

Proof. Let F be an $(\in, \in \vee q_k)$ -fuzzy right ideal of a completely regular right modular groupoid M , then for each $a \in M$ there exists $x \in M$ such that $a = (a^2 x)a^2$, then by using (1) , we have

$$F(ab) = F(((a^2x)a^2)b) = F((ba^2)(a^2x)) \geq F(ba^2) \wedge \frac{1-k}{2} \geq F(b) \wedge \frac{1-k}{2}.$$

Conversely, assume that F is an $(\in, \in \vee q_k)$ -fuzzy right ideal of M , then by using (1), we have

$$F(ab) = F(((a^2x)a^2)b) = F((ba^2)(a^2x)) \geq F(a^2x) \wedge \frac{1-k}{2} = F((aa)x) \wedge \frac{1-k}{2}$$
$$= F((xa)a) \wedge \frac{1-k}{2} \geq F(a) \wedge \frac{1-k}{2}.$$

Theorem *8*. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy interior ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2x)a^2$, then by using (4) and (1), we have

$$F(ab) = F(((a^2x)a^2)b) \geq F(aa) \geq F(a) \wedge F(a) \wedge \frac{1-k}{2}, \text{ and}$$

$$F(ab) = F(a((b^2y)b^2))F((b^2y)(ab^2)) = F(((bb)y)(ab^2))$$
$$= F(((yb)b)(ab^2)) \geq F(b) \wedge \frac{1-k}{2}.$$

The converse is obvious.

Theorem *9*. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2x)a^2$, then by using (4), we have

$$F(ab) = F(((a^2x)a^2)b) = F(((a^2x)(aa))b)$$
$$= F((a((a^2x)a))b) \geq F(a) \wedge F(b) \wedge \frac{1-k}{2}.$$

The converse is obvious.

Theorem *10*. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of M.*

Proof. Let F be an $(\in,\in\vee q_k)$-fuzzy bi-ideal of a completely regular right modular groupoid M, then for each $a\in M$ there exists $x\in M$ such that $a=(a^2x)a^2$, then by using (1) and (4), we have

$$\begin{aligned}F(ab) &= F(((a^2x)a^2)b)=F((((aa)x)a^2)b)=F((((xa)a)a^2)b)\\ &= F((ba^2)((xa)a))=F((b(aa))((aa)x))=F((a(ba))((aa)x))\\ &= F((aa)((a(ba))x))=F((((a(ba))x)a)a)=F((((b(aa))x)a)a)\\ &= F((((x(aa))b)a)a)=F((((a(xa))b)a)a)=F(((ab)(a(xa)))a)\\ &= F((a((ab)(xa)))a)\geq F(a)\wedge F(a)\wedge\frac{1-k}{2}=F(a)\wedge\frac{1-k}{2}.\end{aligned}$$

And, by using (4),(1) and (3), we have

$$\begin{aligned}F(ab) &= F(a((b^2y)b^2))=F((b^2y)(ab^2))=F(((bb)y)(a(bb)))\\ &= F(((a(bb)y)(bb))=F(((a(bb))(ey))(bb))=F(((ye)((bb)a))(bb))\\ &= F((bb)((ye)a))(bb))\geq F(bb)\wedge\frac{1-k}{2}\geq F(b)\wedge F(b)\wedge\frac{1-k}{2}.\\ &= F(b)\wedge\frac{1-k}{2}.\end{aligned}$$

The converse is obvious.

Theorem 11. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of M if and only if F is an $(\in,\in\vee q_k)$-fuzzy two sided ideal of M.*

Proof. Let F be an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of a completely regular right modular groupoid M, then for each $a\in M$ there exists $x\in M$ such that $a=(a^2x)a^2$, then by using (1), (3) and (4), we have

$$\begin{aligned}ab &= ((a^2x)a^2)b=(ba^2)(a^2x)=(xa^2)(a^2b)\\ &= (x(aa))(a^2b)=(a(xa))(a^2b)=((a^2b)(xa))a.\end{aligned}$$

Then

$$F(ab) \geq (F \circ 1)(ab) \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2}$$
$$= \bigvee_{ab=pq} \{F(p) \wedge 1(q)\} \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2}$$
$$\geq F(a) \wedge 1(b) \wedge \bigvee_{ab=lm} \{F(l) \wedge 1(m)\} \wedge \frac{1-k}{2}$$
$$= F(a) \wedge \bigvee_{ab=((a^2b)(xa))a} \{1((a^2b)(xa)) \wedge F(a)\}$$
$$\geq F(a) \wedge 1((a^2b)(xa)) \wedge F(a) \wedge \frac{1-k}{2} = F(a) \wedge \frac{1-k}{2}.$$

Also by using (4) and (1), we have

$$ab = a[(b^2y)b^2] = (b^2y)(ab^2) = ((bb)y)(ab^2) = \{(ab^2)y\}(bb) = b[\{(ab^2)y\}b].$$

Then

$$F(ab) \geq (F \circ 1)(ab) \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2}$$
$$= \bigvee_{ab=pq} \{F(p) \wedge 1(q)\} \wedge \bigvee_{ab=lm} \{1(l) \wedge F(m)\} \wedge \frac{1-k}{2}$$
$$\geq F(b) \wedge 1(((ab^2)y)b) \wedge 1(a) \wedge F(b) \wedge \frac{1-k}{2} = F(b) \wedge \frac{1-k}{2}.$$

The converse is obvious.

Remark *1*. *We note that in a completely regular right modular groupoid* M *with left identity,* $(\in, \in \vee q_k)$*-fuzzy left ideal* $(\in, \in \vee q_k)$*-fuzzy right ideal,* $(\in, \in \vee q_k)$ *fuzzy ideal,* $(\in, \in \vee q_k)$*-fuzzy interior ideal,* $(\in, \in \vee q_k)$*-fuzzy bi-ideal,* $(\in, \in \vee q_k)$*-fuzzy generalized bi-ideal and* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal coincide with each other.*

Theorem *12*. *If* M *is a completely regular right modular groupoid then* $F \wedge_k G = F \circ_k G$ *for every* $(\in, \in \vee q_k)$ *- fuzzy ideal* F *and* G *of* M.

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy right ideal of a completely regular right modular groupoid M and G is an $(\in, \in \vee q_k)$-fuzzy left ideal of M, and M is a completely regular then for each $a \in M$ there exists $x \in M$ such that $a = (a^2x)a^2$, so we have

Therefore $F \wedge_k G \leq F \circ_k G$, again

$$(F \circ_k G)(a) = (F \circ G)(a) \wedge \frac{1-k}{2} = \left(\bigvee_{a=pq} \{F(p) \wedge G(q)\} \right) \wedge \frac{1-k}{2}$$

$$= \bigvee_{a=pq} \left\{ F(p) \wedge G(q) \wedge \frac{1-k}{2} \right\} \leq \bigvee_{a=pq} \left\{ (F(pq \wedge G(pq)) \wedge \frac{1-k}{2} \right\}$$

$$= F(a) \wedge G(a) \wedge \frac{1-k}{2} = (F \wedge_k G)(a).$$

Therefore $F \wedge_k G \geq F \circ_k G$. Thus $F \wedge_k G = F \circ_k G$.

REFERENCES

[1]. *S. K. Bhakat and P. Das*, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Sys., **vol. 80**, 1996, pp. 359-368.

[2]. *S. K. Bhakat and P. Das*, On the definition of fuzzy subgroups, Fuzzy Sets and Sys., **vol. 51**, 1992, pp. 235-241.

[3]. *Y. B. Jun*, Generalizations of $(\in, \in \vee q)$ -fuzzy subalgebras in BCK/BCI-algebra, Comput. Math. Appl., **vol. 58**, 2009, pp. 1383-1390.

[4]. *Y. B. Jun and S. Z. Song*, Generalized fuzzy interior ideals in semigroups, Inform Sci., **vol. 176**, 2006, pp. 3079-3093.

[5]. *M. A. Kazim and M. Naseeruddin*, On almost semigroups, The Alig. Bull. Math., **vol. 2**, 1972, pp. 1-7.

[6]. *A. Khan and M. Shabir*, (α, β)-fuzzy interior ideals in ordered semigroups, Lobachevskii J. Math., **vol. 30**, no. 1, 2009, pp. 30-39.

[7]. *A. Khan, Young Bae Jun, and Tahir Mahmood*, Generalized fuzzy interior ideals in Abel-Grassmann's groupoids, Internat. J. Math. Math. Sci., 2010, Article ID 838392.

[8]. *N. Kuroki*, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli, **vol. 28**, 1979, pp. 17-21.

[9]. *Z. Liao, M. Hu, M. Chen, L. Yi, M. Kui, J. Lu and C. Liu*, Generalized $(\alpha, \beta)-$ fuzzy interior ideals in semigroups, Third International Joint Conference on Computational Science and Optimization, 2010, pp. 197-200.

[10]. *J. N. Mordeson, D. S. Malik and N. Kuroki*, *Fuzzy semigroups*. Springer-Verlag, Berlin, Germany, 2003.

[11]. *V. Murali*, Fuzzy points of equivalent fuzzy subsets, Inform. Sci., **vol. 158**, 2004, pp. 277-288.

[12]. *Q. Mushtaq and S. M. Yusuf*, On LA-semigroups, The Alig. Bull. Math., **vol. 8**, 1978, pp. 65-70.

[13]. *P. V. Protić and N. Stevanović*, AG-test and some general properties of Abel-Grassmann's groupoids, PU. M. A, **vol. 4**, no. 6, 1995, pp. 371-383.

[14]. *P. M. Pu and Y. M. Liu*, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., **vol. 76**, 1980, pp. 571-599.

[15]. *A. Rosenfeld*, Fuzzy groups, J. Math. Anal. Appl., **vol. 35**, 1971, pp. 512-517.

[16]. *M. Shabir, Y. B. Jun and Y. Nawaz*, Semigroups characterized by $\left(\in, \in \vee q_k\right)$-fuzzy ideals, Computer and Mathematics with Applications, vol. 60, 2010, pp. 1473-1493.
[17]. *Y. Yin and H. Li*, Note on generalized fuzzy interior ideals in semigroups, Inform. Sci., **vol. 177**, 2007, pp. 5798-5800.
[18]. *L. A. Zadeh*, Fuzzy sets, Inform. Control, **vol. 8**, 1965, pp. 338-353.
[19]. *J. Zhan and Y. B. Jun*, Generalized fuzzy interior ideals of semigroups, Neural Comput. & Applic., **vol. 19**, 2010, pp. 515-519.
[20]. *J. Zhan and X. Ma*, On fuzzy interior ideals in semigroups, J. Math. Res. Expo. (Chinese), **vol. 28**, 2008, pp. 103-111.