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A NEW VARIABLE-COEFFICIENT BERNOULLI EQUATION-BASED SUB-EQUATION METHOD FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

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In this paper, a new variable-coefficient Bernoulli equation-based sub-equation method is proposed to establish exact solutions for nonlinear differential equations. For illustrating the validity of this method, we apply it to the (2+1)-dimensional breaking soliton equation and the (2+1)-dimensional dispersive long wave equations. As a result, some new exact solutions for them are successfully obtained.

Keywords: Variable-coefficient sub-equation method; Bernoulli equation; Breaking soliton equation; Dispersive long wave equations; Nonlinear differential equation; Exact solution.

MSC2010: 35Q51, 35Q53.

1. Introduction

Nonlinear differential equations (NLDEs) can be used to describe many nonlinear phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. In the research of the theory of NLDEs, searching for more explicit exact solutions to NLDEs is one of the most fundamental and significant study in recent years. With the help of computerized symbolic computation, much work has been focused on the various extensions and applications of the known algebraic methods to construct the solutions to NLDEs. There have been a variety of powerful methods. For example, these methods include the known homogeneous balance method [1,2], the tanh-method [3-5], the inverse scattering transform [6], the Backlund transform [7,8], the Hirotas bilinear method [9,10], the generalized Riccati sub-equation method[11,12], the Jacobi elliptic function expansion [13,14], the F-expansion method [15], the exp-function expansion method [16,17], the (G'/G)-expansion method [18,19] and so on. However, we notice that most of the existing methods are companied with constant coefficients, while very few methods are concerned of variable-coefficients.

In this paper, by introducing a new ansatz, we develop a new variable-coefficient Bernoulli equation-based sub-equation method for solving NLDEs. First we give the description of the variable-coefficient Bernoulli equation-based sub-equation method. Then we apply the method to solve the (2+1)-dimensional breaking soliton equation and the (2+1)-dimensional dispersive long wave equations. Some conclusions are presented at the end of the paper.

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2. **Description of the variable-coefficient Bernoulli equation-based sub-equation method**

We consider the following Bernoulli equation:

$$G' + \lambda G = G^2, \tag{1}$$

where $\lambda \neq 0$ is a complex number, $G = G(\xi)$. The solutions of Eq. (1) is denoted by

$$G(\xi) = \frac{\lambda}{1 + \lambda d e^{\lambda \xi}}, \tag{2}$$

where d is an arbitrary constant. Especially, when λ is a real number and $d = \frac{1}{\lambda}$, we obtain

$$G(\xi) = \frac{\lambda}{2}(1 - tanh(\frac{\lambda \xi}{2})). \tag{3}$$

When $d = \frac{1}{\lambda}$, $\lambda = i\widetilde{\lambda}$, where $\widetilde{\lambda}$ is a real number, i is the unit of imaginary number, we obtain

$$G(\xi) = \frac{\lambda}{2} - \frac{\lambda i}{2} tan(\frac{\widetilde{\lambda}\xi}{2}). \tag{4}$$

Suppose that a nonlinear equation, say in two or three independent variables $x,\ y,\ t$, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{xy} ...) = 0, \tag{5}$$

where $u = u(x, y, t)$ is an unknown function, P is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. Suppose that

$$u(x, y, t) = u(\xi), \quad \xi = \xi(x, y, t), \tag{6}$$

and then Eq. (5) can be turned into the following form $\widetilde{P}(u,\ u', u'',\ ...) = 0$.

Step 2. Suppose that the solution of (7) can be expressed by a polynomial in G as follows:

$$u(\xi) = a_m(x, y, t)G^m + a_{m-1}(x, y, t)G^{m-1} + \ ... + a_0(x, y, t), \tag{7}$$

where $G = G(\xi)$ satisfies Eq. (1), and $a_m(x, y, t),\ a_{m-1}(x, y, t),\ ...,\ a_0(x, y, t)$ are all unknown functions to be determined later with $a_m(x, y, t) \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (7).

Step 3. Substituting (8) into (7) and using (1), collecting all terms with the same order of G together, the left-hand side of (7) is converted to another polynomial in G. Equating each coefficient of this polynomial to zero, yields a set of partial differential equations for $a_m(x, y, t),\ a_{m-1}(x, y, t) ...,\ a_0(x, y, t)$, $\xi(x, y, t),\ \lambda$.

Step 4. Solving the equations system in Step 3, and using the solutions of Eq. (1), we can construct exact coefficient function solutions of Eq. (7).

Remark 2.1. *As the partial differential equations in Step 3 are usually over-determined, we may choose some special forms of* $a_m,\ a_{m-1},\ ...,\ a_0$ **as will be done** *in the following.*

3. Applications of the proposed method

In this section, we will present some **applications of the method described** in Section 2.

3.1. (2+1)-dimensional breaking soliton equation

We consider the (2+1)-dimensional breaking soliton equation [20-22]:

$$u_{xxxy} - 2u_y - 4u_x u_{xy} + u_{xt} = 0. \tag{8}$$

Eq. (9) is used to describe the (2+1)-dimensional interaction of Riemann wave propagated along the y-axis with long wave propagated along the x-axis [23]. Some types of exact solutions for Eq. (9) have been obtained by the Riccati sub-equation method [20, 21] and the symbolic computation method [22].

To apply the method described above, we assume that $u(x,y,t) = U(\xi)$, where $\xi = \xi(x,y,t)$. Then Eq. (9) can be turned into

$$\xi_x^3\xi_y U'''' + (3\xi_x^2\xi_{xy} + 3\xi_x\xi_y\xi_{xx})U''' + (3\xi_{xy}\xi_{xx} + 3\xi_x\xi_{xxy} + \xi_{xxx}\xi_y + \xi_x\xi_t)U'' \\ +(\xi_{xxxy} - 2\xi_y + \xi_{xt})U' - 4\xi_x\xi_{xy}U'^2 - 4\xi_x^2\xi_y U'U'' = 0. \tag{9}$$

We will proceed to solve Eq. (10) in two cases.

Case 1: Assume that $U(\xi) = \sum_{i=0}^{m} a_i(y,t)G^i$. By balancing the order of U'''' and $U'U''$ in Eq. (10), we can obtain $m = 1$. So

$$U(\xi) = a_1(y,t)G + a_0(y,t). \tag{10}$$

Substituting (12) into (10) and collecting all the terms with the same power of G together, equating each coefficient to zero yields a set of under-determined **partial differential equations** for $a_0(y,t)$, $a_1(y,t)$ and $\xi(x,y,t)$. Solving these equations with the aid of Maple software yields

Family 1:

$$\xi(x,y,t) = C_1x + f(t),\ a_1(y,t) = \frac{3C_1}{4},\ a_0(y,t) = \frac{2f'(t)y}{C_1} + g(t),$$

where C_1 is an arbitrary constant, and $f(t)$, $g(t)$ are two arbitrary functions. Combining with Eq. (2) we can obtain the following exact solutions for (2+1)-dimensional breaking soliton equation:

$$u_1(x,y,t) = \frac{3C_1}{4}[\frac{\lambda}{1+\lambda de^{\lambda(C_1x+f(t))}}] + \frac{2f'(t)y}{C_1} + g(t). \tag{11}$$

In the special case of Eq. (3) we obtain the following solitary wave solutions:

$$u_2(x,y,t) = \frac{3C_1}{4}[\frac{\lambda}{2}(1 - tanh(\frac{\lambda(C_1x+f(t))}{2}))] + \frac{2f'(t)y}{C_1} + g(t). \tag{12}$$

Using Eq. (4) we obtain the following trigonometric function solutions:

$$u_3(x,y,t) = \frac{3C_1}{4}[\frac{\lambda}{2} - \frac{\lambda i}{2}tan(\frac{\widetilde{\lambda}(C_1x+f(t))}{2}))] + \frac{2f'(t)y}{C_1} + g(t). \tag{13}$$

Family 2:

$$\xi(x,y,t)=\frac{C_1C_2}{2}(C_3e^{-\frac{x+C_4}{C_1}}-e^{\frac{x+C_4}{C_1}}),\ a_1(y,t)=C_5,\ a_0(y,t)=f(t),$$

where $C_i,\ i=1,2,3,4,5$ are an arbitrary constant, and $f(t)$ is an arbitrary function. Combining with Eq. (2) we can obtain the following exact solutions:

$$u_4(x,y,t)=\frac{C_5\lambda}{1+\lambda de^{\lambda[\frac{C_1C_2}{2}(C_3e^{-\frac{x+C_4}{C_1}}-e^{\frac{x+C_4}{C_1}})]}}+f(t). \tag{14}$$

Case 2: Assume that

$$U(\xi)=\sum_{i=0}^{m}a_i(x)G^i. \tag{15}$$

Similarly, by balancing the order of U'''' and $U'U''$ in Eq. (10), we can obtain $m=1$. So

$$U(\xi)=a_1(x)G+a_0(x). \tag{16}$$

Substituting (18) into (10) and collecting all the terms with the same power of G together, equating each coefficient to zero yields a set of under-determined partial differential equations for $a_0(x)$, $a_1(x)$ and $\xi(x,y,t)$. **Solving these equations yields**

Family 3:

$$\xi(x,y,t)=F_1(y)+F_2(x),\ a_1(x)=2F_2'(x),$$

$$a_0(x)=\int\frac{2F_2'(x)F_2'''(x)-4\lambda F_2'^2(x)F_2''(x)-F_2''^2(x)+\lambda^2F_2'^4(x)}{4F_2'^2(x)}+C_1,$$

where C_1 is an arbitrary constant, and $F_1(y)$, $F_2(x)$ are two arbitrary functions with respect to the variable y and x respectively. Then we have

$$u_5(x,y,t)=\frac{2F_2'(x)\lambda}{1+\lambda de^{\lambda(F_1(y)+F_2(x))}}$$
$$+\int\frac{2F_2'(x)F_2'''(x)-4\lambda F_2'^2(x)F_2''(x)-F_2''^2(x)+\lambda^2F_2'^4(x)}{4F_2'^2(x)}+C_1. \tag{17}$$

Family 4:

$$\xi(x,y,t)=F_1(y),\ a_1(x)=-\frac{1}{2}(3C_1x+3C_2)^{\frac{1}{3}}\pm i\frac{\sqrt{3}}{2}(3C_1x+3C_2)^{\frac{1}{3}},$$

$$a_0(x)=\frac{3C_3(C_1x+C_2)^{\frac{1}{3}}}{C_1}+\frac{5C_1}{36(C_1x+C_2)}+C_4,$$

where $C_i,\ i=1,2,3,4$ are arbitrary constants, and $F_1(y)$ is an arbitrary function. Then we have

$$u_6(x,y,t)=\frac{\lambda[-\frac{1}{2}(3C_1x+3C_2)^{\frac{1}{3}}\pm i\frac{\sqrt{3}}{2}(3C_1x+3C_2)^{\frac{1}{3}}]}{1+\lambda de^{\lambda F_1(y)}}$$
$$+\frac{3C_3(C_1x+C_2)^{\frac{1}{3}}}{C_1}+\frac{5C_1}{36(C_1x+C_2)}+C_4. \tag{18}$$

Family 5:

$$\xi(x,y,t)=F_1(y),\ a_1(x)=(3C_1x+3C_2)^{\frac{1}{3}},$$

$$a_0(x)=\frac{3C_3(C_1x+C_2)^{\frac{1}{3}}}{C_1}+\frac{5C_1}{36(C_1x+C_2)}+C_4,$$

where $C_i,\ i=1,2,3,4$ are arbitrary constants, and $F_1(y)$ is an arbitrary function. Then we have

$$u_7(x,y,t)=\frac{\lambda(3C_1x+3C_2)^{\frac{1}{3}}}{1+\lambda de^{\lambda F_1(y)}}+\frac{3C_3(C_1x+C_2)^{\frac{1}{3}}}{C_1}+\frac{5C_1}{36(C_1x+C_2)}+C_4. \quad (19)$$

Family 6:

$$\xi(x,y,t)=C_1t+F_1(y),\ a_1(x)=C_2,\ a_0(x)=C_3x+C_4,$$

where $C_i,\ i=1,2,3,4$ are arbitrary constants, and $F_1(y)$ is an arbitrary function. Then we have

$$u_8(x,y,t)=\frac{\lambda C_2}{1+\lambda de^{\lambda(C_1t+F_1(y))}}+C_3x+C_4. \quad (20)$$

Family 7:

$$\xi(x,y,t)=C_1t+C_2+C_3y,\ a_1(x)=-\frac{1}{2}(3C_4x+3C_5)^{\frac{1}{3}}\pm i\frac{\sqrt{3}}{2}(3C_4x+3C_5)^{\frac{1}{3}},$$

$$a_0(x)=\int\frac{(36C_3C_4^2C_6x^2+72C_3C_4C_5C_6x+36C_3C_5^2C_6)}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx,$$

$$+\int\frac{(9C_1C_4^2x^2+18C_1C_4C_5x+9C_1C_5^2-5C_3C_4^2)(C_4x+C_5)^{\frac{2}{3}}}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx+C_7,$$

where $C_i,\ i=1,2,...,7$ are arbitrary constants. Then we have

$$u_9(x,y,t)=\frac{\lambda[-\frac{1}{2}(3C_4x+3C_5)^{\frac{1}{3}}\pm i\frac{\sqrt{3}}{2}(3C_4x+3C_5)^{\frac{1}{3}}]}{1+\lambda de^{\lambda(C_1t+C_2+C_3y)}}$$

$$+\int\frac{(36C_3C_4^2C_6x^2+72C_3C_4C_5C_6x+36C_3C_5^2C_6)}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx,$$

$$+\int\frac{(9C_1C_4^2x^2+18C_1C_4C_5x+9C_1C_5^2-5C_3C_4^2)(C_4x+C_5)^{\frac{2}{3}}}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx+C_7. \quad (21)$$

Family 8:

$$\xi(x,y,t)=C_1t+C_2+C_3y,\ a_1(x)=(3C_4x+3C_5)^{\frac{1}{3}},$$

$$a_0(x)=\int\frac{(36C_3C_4^2C_6x^2+72C_3C_4C_5C_6x+36C_3C_5^2C_6)}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx,$$

$$+\int\frac{(9C_1C_4^2x^2+18C_1C_4C_5x+9C_1C_5^2-5C_3C_4^2)(C_4x+C_5)^{\frac{2}{3}}}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx+C_7,$$

where $C_i,\ i=1,2,...,7$ are arbitrary constants. Then we have

$$u_{10}(x,y,t)=\frac{\lambda(3C_4x+3C_5)^{\frac{1}{3}}}{1+\lambda de^{\lambda(C_1t+C_2+C_3y)}}$$

$$+\int\frac{(36C_3C_4^2C_6x^2+72C_3C_4C_5C_6x+36C_3C_5^2C_6)}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx,$$

$$+\int\frac{(9C_1C_4^2x^2+18C_1C_4C_5x+9C_1C_5^2-5C_3C_4^2)(C_4x+C_5)^{\frac{2}{3}}}{36C_3(C_4x+C_5)^{\frac{8}{3}}}dx+C_7. \tag{22}$$

Family 9:

$$\xi(x,y,t)=[\frac{C_3}{(C_1y+C_2)(C_3-1)^2C_4}]^{\frac{C_3}{1-C_3}},$$

$$a_1(x)=-\frac{1}{2}(3C_5x+3C_6)^{\frac{1}{3}}\pm i\frac{\sqrt{3}}{2}(3C_5x+3C_6)^{\frac{1}{3}},$$

$$a_0(x)=\frac{3C_7(C_5x+C_6)^{\frac{1}{3}}}{C_5}+\frac{5C_5}{36(C_5x+C_6)}+C_8,$$

where $C_i,\ i=1,2,...,8$ are arbitrary constants. Then we have

$$u_{11}(x,y,t)=\frac{\lambda[-\frac{1}{2}(3C_5x+3C_6)^{\frac{1}{3}}\pm i\frac{\sqrt{3}}{2}(3C_5x+3C_6)^{\frac{1}{3}}]}{1+\lambda de^{\lambda[\frac{C_3}{(C_1y+C_2)(C_3-1)^2C_4}]^{\frac{C_3}{1-C_3}}}}$$

$$+\frac{3C_7(C_5x+C_6)^{\frac{1}{3}}}{C_5}+\frac{5C_5}{36(C_5x+C_6)}+C_8. \tag{23}$$

Family 10:

$$\xi(x,y,t)=F_1(x)-C_2ln[\frac{C_1(C_3y+C_4)}{C_2^2}],\ a_1(x)=2F_1'(x),$$

$$a_0(x)=\int\frac{2F_1'(x)F_1'''(x)-4\lambda F_1'^2(x)F_1''(x)-F_1''^2(x)+\lambda^2F_1'^4(x)}{4F_1'^2(x)}+C_1,$$

where $C_i,\ i=1,2,3,4$ are arbitrary constants, and $F_1(x)$ is an arbitrary function. Then we have

$$u_{12}(x,y,t)=\frac{2\lambda F_1'(x)}{1+\lambda de^{\lambda(F_1(x)-C_2ln[\frac{C_1(C_3y+C_4)}{C_2^2}])}}$$

$$+\int\frac{2F_1'(x)F_1'''(x)-4\lambda F_1'^2(x)F_1''(x)-F_1''^2(x)+\lambda^2F_1'^4(x)}{4F_1'^2(x)}+C_1. \tag{24}$$

Remark 3.1. *In Families 2-10, if we* **use** *Eqs. (3)-(4), then we also obtain corresponding hyperbolic function solutions and trigonometric function solutions as the analysis in Family 1, which are omitted here.*

Remark 3.2. *We note that the established exact solutions for the (2+1)-dimensional breaking soliton equation above are different from those in [20-22] essentially as we have used a new variable-coefficient method based on a different Bernoulli sub-equation here, and are new exact solutions which have not been reported by other authors in the literature.*

3.2. (2+1)-dimensional dispersive long wave equation

We consider the known (2+1)-dimensional dispersive long wave equations [24-37]:

$$u_{yt} + v_{xx} + (uu_x)_y = 0, \tag{25}$$

$$v_t + u_x + (uv)_x + u_{xxy} = 0, \tag{26}$$

Some types of exact solutions for Eqs. (27)-(28) have been obtained in [24-37] by use of various methods including the Riccati sub-equation method [24, 25, 30], the nonlinear transformation method [26], Jacobi function method [28, 29, 37], (G'/G)-expansion method [27], modified CK's direct method [31], EXP-function method [32], Hopf-Cole transformation method [33], modified extended Fan's sub-equation method [34, 35], generalized algebraic method [36].

To apply the proposed method, **similar to the process** above, we assume that $u(x,y,t) = U(\xi),\ \xi = \xi(x,y,t)$, and suppose

$$U(\xi) = \sum_{i=0}^{m} a_i(y,t)G^i,\ V(\xi) = \sum_{i=0}^{n} b_i(y,t)G^i, \tag{27}$$

where $G = G(\xi)$ satisfies Eq. (1). By balancing the highest order derivatives and nonlinear terms in Eqs. (27)-(28) we have $m = 1,\ n = 2$. So

$$U(\xi) = a_1(y,t)G + a_0(y,t),\ V(\xi) = b_2(y,t)G^2 + b_1(y,t)G + b_0(y,t), \tag{28}$$

Combining (27), (28), (30), collecting all the terms with the same power of G together, equating each coefficient to zero yields a set of under-determined partial differential equations for $a_0(y,t),\ a_1(y,t),\ b_0(y,t),\ b_1(y,t),\ b_2(y,t)$ and $\xi(x,y,t)$. Solving these equations with the aid of Maple software yields

$Family$ 1:

$$\xi(x,y,t) = \frac{C_1 x}{2} - \frac{1}{2}\int C_1 F_1(t)dt - \frac{1}{4}C_1^2\lambda t - \frac{1}{2}C_1 t F_2(y) + F_3(y),$$

$$a_1(y,t) = C_1,\ a_0(y,t) = F_1(t) + F_2(y),$$

$$b_2(y,t) = \frac{C_1}{2}(C_1 t F_2'(y) - 2F_3'(y)),\ b_1(y,t) = C_1\lambda(F_3'(y) - \frac{C_1 t}{2}F_2'(y)),$$

$$b_0(y,t) = -F_2'(y) - 1,$$

where C_1 is an arbitrary constant, and $F_1(t),\ F_2(y),\ F_3(y)$ are arbitrary functions. Combining with Eq. (2) we can obtain the following exact solutions for the (2+1)-dimensional dispersive long wave equation:

$$u_1(x,y,t) = [\frac{C_1\lambda}{1 + \lambda d e^{\lambda[\frac{C_1 x}{2} - \frac{1}{2}\int C_1 F_1(t)dt - \frac{1}{4}C_1^2\lambda t - \frac{1}{2}C_1 t F_2(y) + F_3(y)]}}] + F_1(t) + F_2(y),$$

$$v_1(x,y,t) = \frac{C_1}{2}(C_1 t F_2'(y) - 2F_3'(y)) \tag{29}$$

$$[\frac{\lambda}{1+\lambda d e^{\lambda[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}}]^2$$

$$+C_1\lambda(F_3'(y)-\frac{C_1t}{2}F_2'(y))[\frac{\lambda}{1+\lambda d e^{\lambda[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}}]$$

$$-F_2'(y)-1. \quad (30)$$

In the special case of Eq. (3) we obtain the following solitary wave solutions:

$$u_2(x,y,t)=\frac{C_1\lambda}{2}$$

$$\{1-tanh(\frac{\lambda[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}{2})\}+F_1(t)+F_2(y), \quad (31)$$

$$v_2(x,y,t)=\frac{C_1\lambda^2(C_1tF_2'(y)-2F_3'(y))}{8}$$

$$\{1-tanh(\frac{\lambda[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}{2})\}^2$$

$$+\frac{C_1\lambda^2(F_3'(y)-\frac{C_1t}{2}F_2'(y))}{2}$$

$$\{1-tanh(\frac{\lambda[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}{2})\}-F_2'(y)-1. \quad (32)$$

Using Eq. (4) we obtain the following trigonometric function solutions:

$$u_3(x,y,t)=\frac{C_1\lambda}{2}$$

$$\{1-itan(\frac{\widetilde{\lambda}[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}{2})\}+F_1(t)+F_2(y), \quad (33)$$

$$v_3(x,y,t)=\frac{C_1\lambda^2(C_1tF_2'(y)-2F_3'(y))}{8}$$

$$\{1-itan(\frac{\widetilde{\lambda}[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}{2})\}^2$$

$$+\frac{C_1\lambda^2(F_3'(y)-\frac{C_1t}{2}F_2'(y))}{2}$$

$$\{1-itan(\frac{\widetilde{\lambda}[\frac{C_1x}{2}-\frac{1}{2}\int C_1F_1(t)dt-\frac{1}{4}C_1^2\lambda t-\frac{1}{2}C_1tF_2(y)+F_3(y)]}{2})\}-F_2'(y)-1. \quad (34)$$

Family 2:

$$\xi(x,y,t)=-\frac{C_1x}{2}+\frac{1}{2}\int C_1F_1(t)dt+\frac{1}{4}C_1^2\lambda t+\frac{1}{2}C_1tF_2(y)+F_3(y),$$

$$a_1(y,t)=C_1,\ a_0(y,t)=F_1(t)+F_2(y),$$

$$b_2(y,t)=\frac{C_1}{2}(C_1tF_2'(y)+2F_3'(y)),\ b_1(y,t)=-C_1\lambda(F_3'(y)+\frac{C_1t}{2}F_2'(y)),$$
$$b_0(y,t)=F_2'(y)-1,$$
where C_1 is an arbitrary constant, and $F_1(t),\ F_2(y),\ F_3(y)$ are arbitrary functions. Then combining with Eq. (2) we obtain the following exact solutions:

$$u_4(x,y,t)=\frac{C_1\lambda}{1+\lambda de^{\lambda[-\frac{C_1x}{2}+\frac{1}{2}\int C_1F_1(t)dt+\frac{1}{4}C_1^2\lambda t+\frac{1}{2}C_1tF_2(y)+F_3(y)]}}+F_1(t)+F_2(y),\tag{35}$$

$$\begin{aligned}v_4(x,y,t)=&\frac{C_1(C_1tF_2'(y)+2F_3'(y))}{2}\\&\{\frac{\lambda}{1+\lambda de^{\lambda[-\frac{C_1x}{2}+\frac{1}{2}\int C_1F_1(t)dt+\frac{1}{4}C_1^2\lambda t+\frac{1}{2}C_1tF_2(y)+F_3(y)]}}\}^2\\&-C_1\lambda(F_3'(y)+\frac{C_1t}{2}F_2'(y))\\&\{\frac{\lambda}{1+\lambda de^{\lambda[-\frac{C_1x}{2}+\frac{1}{2}\int C_1F_1(t)dt+\frac{1}{4}C_1^2\lambda t+\frac{1}{2}C_1tF_2(y)+F_3(y)]}}\}+F_2'(y)-1.\end{aligned}\tag{36}$$

Family 3:

$$\xi(x,y,t)=C_1x-C_1\int F_1(t)dt\mp C_1^2\lambda t+F_2(y),\ a_1(y,t)=\pm2C_1,\ a_0(y,t)=F_1(t),$$

$$b_2(y,t)=-2C_1F_2'(y),\ b_1(y,t)=2C_1\lambda F_2'(y),\ b_0(y,t)=-1,$$
where C_1 is an arbitrary constant, and $F_1(t),\ F_2(y)$ are arbitrary functions. Then we have

$$u_5(x,y,t)=\frac{\pm2C_1\lambda}{1+\lambda de^{\lambda[C_1x-C_1\int F_1(t)dt\mp C_1^2\lambda t+F_2(y)]}}+F_1(t),\tag{37}$$

$$\begin{aligned}v_5(x,y,t)=&-2C_1F_2'(y)\{\frac{\lambda}{1+\lambda de^{\lambda[C_1x-C_1\int F_1(t)dt\mp C_1^2\lambda t+F_2(y)]}}\}^2\\&+2C_1\lambda F_2'(y)\{\frac{\lambda}{1+\lambda de^{\lambda[C_1x-C_1\int F_1(t)dt\mp C_1^2\lambda t+F_2(y)]}}\}-1.\end{aligned}\tag{38}$$

Family 4:

$$\xi(x,y,t)=C_1x+F_1(y),\ a_1(y,t)=\pm2C_1,\ a_0(y,t)=\mp C_1\lambda,$$

$$b_2(y,t)=-2C_1F_1'(y),\ b_1(y,t)=2C_1\lambda F_1'(y),\ b_0(y,t)=-1,$$
where C_1 is an arbitrary constant, and $F_1(y)$ is an arbitrary function. Then we have

$$u_6(x,y,t)=\frac{\pm2C_1\lambda}{1+\lambda de^{\lambda[C_1x+F_1(y)]}}\mp C_1\lambda,\tag{39}$$

$$\begin{aligned}v_6(x,y,t)=&-2C_1F_1'(y)\{\frac{\lambda}{1+\lambda de^{\lambda[C_1x+F_1(y)]}}\}^2\\&+2C_1\lambda F_1'(y)\{\frac{\lambda}{1+\lambda de^{\lambda[C_1x+F_1(y)]}}\}-1.\end{aligned}\tag{40}$$

Family 5:

$$\xi(x,y,t) = C_1x + C_2t + F_1(y),\ a_1(y,t) = \pm 2C_1,\ a_0(y,t) = \frac{\mp C_1^2\lambda - C_2}{C_1},$$

$$b_2(y,t) = -2C_1F_1'(y),\ b_1(y,t) = 2C_1\lambda F_1'(y),\ b_0(y,t) = -1,$$

where $C_1,\ C_2$ are arbitrary constants, and $F_1(y)$ is an arbitrary function. Then we have

$$u_7(x,y,t) = \frac{\pm 2C_1\lambda}{1+\lambda d e^{\lambda[C_1x+C_2t+F_1(y)]}} + \frac{\mp C_1^2\lambda - C_2}{C_1}, \tag{41}$$

$$v_7(x,y,t) = -2C_1F_1'(y)\{\frac{\lambda}{1+\lambda d e^{\lambda[C_1x+C_2t+F_1(y)]}}\}^2$$
$$+2C_1\lambda F_1'(y)\{\frac{\lambda}{1+\lambda d e^{\lambda[C_1x+C_2t+F_1(y)]}}\} - 1. \tag{42}$$

Family 6:

$$\xi(x,y,t) = C_1x + C_2y + F_1(t),\ a_1(y,t) = \pm 2C_1,\ a_0(y,t) = \frac{\mp C_1^2\lambda - C_2}{C_1},$$

$$b_2(y,t) = -2C_1C_2,\ b_1(y,t) = 2C_1C_2\lambda,\ b_0(y,t) = -1,$$

where $C_1,\ C_2$ are arbitrary constants, and $F_1(t)$ is an arbitrary function. Then we have

$$u_8(x,y,t) = \frac{\pm 2C_1\lambda}{1+\lambda d e^{\lambda[C_1x+C_2y+F_1(t)]}} + \frac{\mp C_1^2\lambda - C_2}{C_1}, \tag{43}$$

$$v_8(x,y,t) = -2C_1C_2\{\frac{\lambda}{1+\lambda d e^{\lambda[C_1x+C_2y+F_1(t)]}}\}^2$$
$$+2C_1C_2\lambda\{\frac{\lambda}{1+\lambda d e^{\lambda[C_1x+C_2y+F_1(t)]}}\} - 1. \tag{44}$$

Remark 3.3. *By the combination of Families 2-6 with Eqs. (3)-(4), we also obtain some hyperbolic function solutions and trigonometric function solutions, which are omitted here.*

Remark 3.4. *The established solutions Eqs. (31)-(46) for the (2+1)-dimensional dispersive long wave equations can not be obtained by the methods in [24-37], and are new exact solutions to our best knowledge.*

4. **Conclusions**

We have proposed **a variable**-coefficient Bernoulli equation-based sub-equation method method for solving nonlinear differential equations. Based on this method, abundant exact solutions have been obtained with the aid of Maple software for the (2+1)-dimensional breaking soliton equation and the (2+1)-dimensional dispersive long wave equations. As one can see, this method is concise, powerful, and can be applied to solve other nonlinear differential equations.

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