GLOBAL DYNAMIC OF A HEROIN EPIDEMIC MODEL

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In this paper, we propose and study an epidemic model consider-ing the effect of educational/prevention programs on the control of illicit drug uses. We compute the threshold quantity R_0 and show the occurrence of backward bifurcation leading to bistability, both directly and by Castillo-Chavez and Song theorem. Furthermore, the global stability of the equi-librium points of the model is investigated using Lyapunov functions and compound matrices, i.e., geometric stability method.

Keywords: Backward bifurcation; Global stability; Illicit drugs; Epidemic models.

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1. Introduction

As far as usage of illicit drugs damages the physical, mental and social well being of individuals, their families and societies, the widespread, pervasive concerns about illicit drug usage and its controlling strategies, significantly reflected on health, educational and political programs.
In general, there exist three main strategies to restrict and delimitate illicit drug consumption in all countries: legal strategies, educational-training strategies, and treatment strategies. The most important educational-training activities are the increasing of awareness among peoples about the physical, mental and social dangers of drug use. There is evidence that education/prevention programs can mitigate chronic drug addiction, [6], demonstrates the significant, long-term benefits to programmes that reduce or delay first use or prevent the transition from experimental use to addiction. Furthermore, as indicated in [13], school-age and teenage years are critical in terms of experimentation with drugs and the development of behaviors that can lead to dependence and abuse in adulthood. The earlier young people start to use psychoactive substances, the more likely they are to develop drug abuse disorders in later life, [14].
Among various drug users, heroin users are at high risk of addiction and criminal actions. The heroin was first considered as an epidemic problem in 1981-1983 in Ireland. White and Comiskey, have introduced the first model for the

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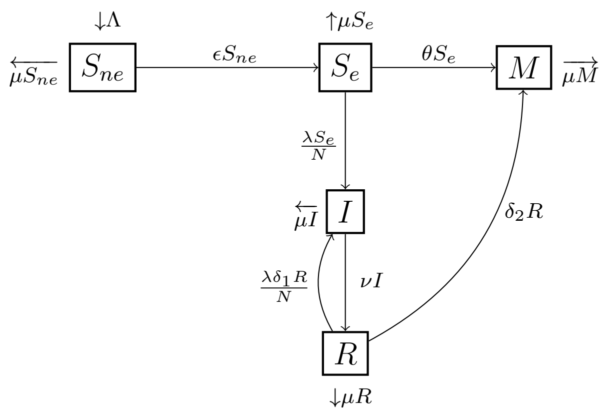
dynamics of heroin users, see [17]. Their model was revisited by Mulone and Straughan, [10]. After White and Comiskey's work, the epidemiology of drugs has been studied by several authors. We propose and study a modified form of White-Comiskey's model by considering the effect of education/prevention activities on the drug users. For this aim, we split the susceptible populations into three compartments, noneducated susceptibles, educated susceptibles and individuals who are completely aware of drug harms, so that they will not use drugs forever.
We will investigate the dynamical behaviors of the model such as steady states, backward bifurcation, and local and global stability. The paper is organized as follows. In section 2, we present the model and some preliminaries such as boundedness and the basic reproduction number of the system. In section 3, we study the existence of endemic equilibrium points and show that backward bifurcation leading to bistability occurs. We prove the occurrence of backward bifurcation, both directly and by the theorem of Castillo-Chavez and Song. In section 4, we obtain sufficient conditions for the local and global stability of both drug-free and endemic equilibrium points by Lyapunov functions and compound matrices, i.e., geometric stability method.

2. Model formulation and basic properties



Our proposed model is based on subdividing the given community into five compartments: S_{ne}, noneducated susceptibles, S_e, educated susceptibles, M, the susceptible individuals who are completely aware from drug harms and will not use drugs forever, I, infective individuals, i.e., drug users and R drug users in treatment and rehabilitation. We denote the number of this compartments by $S_{ne}(t), S_e(t), M(t), I(t)$ and $R(t)$, respectively. We assume that, recruits (including travelers, newborns,...) enter the susceptible population at a constant rate Λ, noneducated susceptibles become under educational programs with constant rate ϵ, and educated susceptible individuals to become infected, i.e., drug use at rate $\frac{\beta I}{N}$. We also assume that infected individuals, i.e., drug users, become under treatment/rehabilitation at rate ν, and drug users under treatment/rehabilitation relapse to the class of untreated drug users at rate $\frac{\delta_1 \beta I}{N}$. On the other hand, educated susceptibles and drug users in treatment/rehabilitation, become completely aware of drugs at rate θ and δ_2, respectively. Furthermore all individuals suffer from natural death rate μ, while α is drug-related death rate. Because of the importance of early ages, i.e. school and teenage years, indicated in [13] and mentioned above, we suppose that almost all of the population goes under educational programs, mostly at school age and teenage years. Furthermore all recruited populations take the educational programs. Therefore we can neglect the flow from S_{ne} to I. The evolution of the life of an individual in various stages can be represented by the above diagram, and the parameters are defined as in Table 1. Based on the flow diagram of the model depicted in the above figure, we obtain the following ODE system:

$$\begin{cases} \dfrac{dS_{ne}}{dt} = \Lambda - \epsilon S_{ne} - \mu S_{ne} \\ \dfrac{dS_e}{dt} = \epsilon S_{ne} - \dfrac{\lambda S_e}{N} - \theta S_e - \mu S_e \\ \dfrac{dM}{dt} = \theta S_e + \delta_2 R - \mu M \\ \dfrac{dI}{dt} = \dfrac{\lambda S_e}{N} + \dfrac{\lambda \delta_1 R}{N} - \nu I - \mu I \\ \dfrac{dR}{dt} = \nu I - \dfrac{\lambda \delta_1 R}{N} - \delta_2 R - \mu R \end{cases} \tag{1}$$

In which $\lambda = \beta I$ is the force of infection. On the positivity of solutions of (1), we have the following result.

Theorem 2.1. *If initial data $S_{ne}(0) > 0$, $S_e(0) > 0$, $M(0) > 0$, $I(0) > 0$ and $R(0) > 0$, then the solution $(S_{ne}(t), S_e(t), M(t), I(t), R(t))$ of (1) is positive for all $t \geq 0$.*

Proof: Let $(S_{ne}(t), S_e(t), M(t), I(t), R(t))$ be the solution of the system (1) with initial data $S_{ne}(0) > 0$, $S_e(0) > 0$, $M(0) > 0$, $I(0) > 0$ and $R(0) > 0$. Suppose that the conclusion is not true, then there is a $t^* > 0$ such that,

$$min\{S_{ne}(t^*), S_e(t^*), M(t^*), I(t^*), R(t^*)\} = 0$$

and

$$min\{S_{ne}(t), S_e(t), M(t), I(t), R(t)\} > 0$$

for all $t \in [0, t^*)$. If $min\{S_{ne}(t^*), S_e(t^*), M(t^*), I(t^*), R(t^*)\} = S_{ne}(t^*)$, then we have, $\dfrac{dS_{ne}}{dt} \geq -\epsilon S_{ne} - \mu S_{ne}$, for all $t \in [0, t^*)$. Hence, $0 = S_{ne}(t^*) \geq S_{ne}(0)exp(-\int_0^{t^*} (\epsilon+\mu)dt) > 0$, which leads to a contradiction. Similarly, we can obtain contradictions when, $min\{S_{ne}(t^*), S_e(t^*), M(t^*), I(t^*), R(t^*)\}$, is equal to other variables of the system. This completes the proof. □

We consider the total population of the community, i.e., N, to be constant. Hence $\Lambda = \mu S_{ne} + \mu S_e + \mu M + \mu I + \mu R$. Now we replace Λ in (1), and then use the substitutions $s_{ne} = \dfrac{S_{ne}}{N}$, $s_e = \dfrac{S_e}{N}$, $m = \dfrac{M}{N}$, $i = \dfrac{I}{N}$ and $r = \dfrac{R}{N} = 1 - s_{ne} - s_e - m - i$, which yields the following final form of our system:

$$\begin{cases} \dfrac{ds_{ne}}{dt} = \mu - \epsilon s_{ne} - \mu s_{ne} \\ \dfrac{ds_e}{dt} = \epsilon s_{ne} - \beta i s_e - \theta s_e - \mu s_e \\ \dfrac{dm}{dt} = \theta s_e + \delta_2 - \delta_2 s_{ne} - \delta_2 s_e - \delta_2 m - \delta_2 i - \mu m \\ \dfrac{di}{dt} = \beta i s_e + \beta\delta_1 i - \beta\delta_1 i s_{ne} - \beta\delta_1 i s_e - \beta\delta_1 i m - \beta\delta_1 i^2 - \nu i - \mu i \end{cases} \tag{2}$$

We study (2) in the following feasible region:

$$\Omega = \{(s_{ne}, s_e, m, i) \in \mathbb{R}^4_+ : s_{ne} \geq 0, s_e \geq 0, m \geq 0, i \geq 0, s_{ne} + s_e + m + i \leq 1\}$$

Which is positively invariant with respect to (2). This system has a unique drug-free equilibrium,

$$P_0 = (s^*_{ne}, s^*_e, m^*, i^*) = (\frac{\mu}{\epsilon + \mu}, \frac{\epsilon\mu}{(\epsilon + \mu)(\theta + \mu)}, \frac{\theta\epsilon}{(\epsilon + \mu)(\theta + \mu)}, 0),$$

and the Jacobian matrix of P_0 has the following form:

$$J(P_0) = \begin{bmatrix} -\epsilon - \mu & 0 & 0 & 0 \\ \epsilon & -\theta - \mu & 0 & -\beta s^*_e \\ -\delta_2 & \theta - \delta_2 & -\delta_2 - \mu & -\delta_2 \\ 0 & 0 & 0 & \beta s^*_e - \nu - \mu \end{bmatrix}.$$

Which has the eigenvalues $-\epsilon - \mu, -\theta - \mu, -\delta_2 - \mu, \beta s^*_e - \nu - \mu$. Now we define the basic reproduction number by $R_0 = \frac{\beta s^*_e}{\mu + \nu} = \frac{\beta\epsilon\mu}{(\epsilon+\mu)(\theta+\mu)(\nu+\mu)}$. See [9], for the definition and properties of the basic reproduction number.
It is clear that $\beta s^*_e - \nu - \mu < 0$ if and only if $R_0 < 1$, and we obtain the following result on the local stability of the drug-free equilibrium.

Theorem 2.2. *The drug-free equilibrium point P_0 of 2 has asymptotic stability when $R_0 < 1$ and instability when $R_0 > 1$.*

3. **Endemic equilibrium points and backward bifurcation**

The endemic equilibrium points of 2 satisfy the following system,

$$\begin{cases} \mu - \epsilon s_{ne}^* - \mu s_{ne}^* = 0 \\ \epsilon s_{ne}^* - \beta i^* s_e^* - \theta s_e^* - \mu s_e^* = 0 \\ \theta s_e^* + \delta_2 - \delta_2 s_{ne}^* - \delta_2 s_e^* - \delta_2 m^* - \delta_2 i^* - \mu m^* = 0 \\ \beta i^* s_e^* + \beta\delta_1 i^* - \beta\delta_1 i^* s_{ne}^* - \beta\delta_1 i^* s_e^* - \beta\delta_1 i^* m^* - \beta\delta_1 (i^2)^* - \nu i^* - \mu i^* = 0 \end{cases}$$

which yields that, i^* is the positive root of

$$A(i^*)^3 + B(i^*)^2 + Ci^* + D = 0 \tag{3}$$

where $A = -\beta^2\delta_1\mu$, $B = \beta(\delta_1\mu(\beta(1-q_1) - (\theta+\mu)) - (\delta_2+\mu)(\nu+\mu))$, $C = \beta(\mu(1-q_1)(\mu+\delta_2+\delta_1(\theta+\mu)) - \delta_1\epsilon q_1(\mu+\theta)) - (\delta_2+\mu)(\theta+\mu)(\nu+\mu) = (\mu+\delta_2)(\theta+\mu)(\nu+\mu)(R_0-1)$, and
$D = \mu\theta\delta_2 + \mu^2\theta + \mu^2\delta_2 + \mu^3 - \mu q_1\theta\delta_2 - \mu^2 q_1\theta - \mu^2 q_1\delta_2 - \mu^3 q_1 - \mu\epsilon q_1\delta_2 - \mu^2\epsilon q_1 + \delta_2^2\theta + \delta_2\theta\mu + \delta_2^2\mu + \delta_2\mu^2 - \delta_2^2 q_1\theta - \delta_2 q_1\theta\mu - \delta_2^2 q_1\mu - \delta_2 q_1\mu^2 - \delta_2^2\varepsilon q_1 - \delta_2\epsilon q_1\mu - \delta_2\theta\epsilon q_1 - \delta_2^2\theta - \delta_2^2\mu + \delta_2^2 q_1\theta + \delta_2^2 q_1\mu + \delta_2^2\epsilon q_1 - \mu\theta\epsilon q_1 - \mu\delta_2\theta - \mu^2\delta_2 + \mu\delta_2 q_1\theta + \delta_2 q_1\mu^2 + \mu\delta_2\epsilon q_1 = 0$, in which $q_1 = \dfrac{\mu}{\epsilon+\mu}$. Since $D = 0$, i^* is the root of the following quadratic equation, $F(i^*) = A(i^*)^2 + Bi^* + C = 0$.
Now $F''(i^*) = 2A < 0$, hence the quadratic polynomial $F(i^*)$ is a concave parabola and has a maximum point $i^*_{\max} = -\frac{B}{2A}$ with $F(i^*_{\max}) = \frac{4AC-B^2}{4A}$. If $R_0 > 1$, since $F(0) = C > 0$, $\Delta = B^2 - 4AC > 0$ and $A < 0$, the equation $F(i^*) = 0$ has exactly one positive solution (an endemic equilibrium).
In most epidemic models, at the critical value of the reproduction number $R_0 = 1$, an endemic equilibrium bifurcates and exists when $R_0 > 1$. However, there are epidemic models in which the bifurcating endemic equilibrium exists for $R_0 < 1$. In such cases, it is said that backward bifurcation occurs, and there is a range of the reproduction number $R_0^c < R_0 < 1$, where there are at least two endemic equilibria. See [9] for more details.
For the occurrence of backward bifurcation, we must have $i^*_{\max} > 0$ and $F(i^*_{\max}) \geq 0$, which are equivalent to $B > 0$ and $\Delta \geq 0$. For the computation of R_0^c we solve $\Delta = 0$, which yields, $R_0^c = 1 - \dfrac{B^2}{4\beta^2\delta_1\mu(\mu+\delta_2)(\theta+\mu)(\nu+\mu)}$.
The above arguments imply the following theorem.

Theorem 3.1. *If $R_0 > 1$, system (2) has a unique endemic equilibrium point, and when $R_0^c < R_0 < 1$ and $B > 0$, it has two endemic equilibrium points.*

Now we study the bifurcation of drug-free equilibrium point P_0 when $R_0 = 1$, by using the Castillo-Chavez and Song theorem (theorem 4.1. in [4]), which is proved by center manifold theory. See [2, 3, 11, 12, 18] for applications of this theorem. The relation $R_0 = 1$ can be interpreted in term of parameter

β as $\beta = \beta^* = \dfrac{\nu+\mu}{s_e^*}$. The eigenvalues of the Jacobian matrix, $J(P_0, \beta^*)$ are $\lambda_1 = -\theta - \mu$, $\lambda_2 = -\epsilon - \mu$, $\lambda_3 = -\delta_2 - \mu$ and $\lambda_4 = 0$. Now since 0 is simple and nonzero eigenvalues are nonnegative real numbers, when $\beta = \beta^*$ (or $R_0 = 1$) the assumption (A1) of the Castillo-Chavez and Song theorem, is verified. Let $w = (w_1, w_2, w_3, w_4, w_5)^T$, be the right eigenvector of $J(P_0, \beta^*)$ associated with eigenvalue $\lambda_4 = 0$, and founded by:

$$\begin{cases} (-\epsilon - \mu)w_1 = 0 \\ \epsilon w_1 - (\theta+\mu)w_2 - \beta^* s_e^* w_4 = 0 \\ -\delta_2 w_1 + (\theta - \delta_2)w_2 - (\delta_2+\mu)w_3 - \delta_2 w_4 = 0 \end{cases}$$

A simple computation implies, $w_1 = 0$, $w_3 = -\dfrac{\theta(\nu+\mu)+\delta_2(\nu+\theta)}{\delta_2+\mu}$, $w_4 = \theta + \mu$ and $w_2 = -(\nu+\mu)$. On the other hand, $v = (v_1, v_2, v_3, v_4, v_5)$, the left eigenvector associated with zero eigenvalue which is founded by $vA = 0$ and has the following form, $v = (0, 0, 0, 1)$. Now we compute the quantities **a** and **b** of theorem 3.2., that is,

$$\mathbf{a} = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(P_0, \beta^*) = 2\beta v_4 w_4(-\delta_1(w_2 + w_3 + w_4) + w_2)$$

and $\mathbf{b} = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(P_0, \beta^*) = v_4 w_4 s_e^*$. We observe that **b** is positive, so that, it is the sign of **a** which determines the behavior of the system around $\beta = \beta^*$. We consider $A_1 = \nu + \mu$ and $A_2 = \dfrac{\delta_1 \nu(2\delta_2 + \mu + \theta)}{\delta_2 + \mu}$. Hence if $A_2 > A_1$, $\mathbf{a} > 0$ and $\mathbf{a} < 0$ if $A_2 < A_1$. Now part (4) in the theorem of Castillo-Chavez and Song implies the following result.

Theorem 3.2. *If $A_2 < A_1$, in the ODE system (2), backward bifurcation occurs when $R_0 = 1$. Furthermore, endemic equilibrium has asymptotic stability when $R_0 > 1$ and close to one.*

4. Global stability of equilibrium points

In this section, we discuss the global stability of steady states. At first, we consider the drug-free equilibrium point.

Proposition 4.1. *The drug-free equilibrium point, P_0, is globally asymptotically stable in Ω, if, $R_0 \le R_0^* = \dfrac{\epsilon\mu}{(\epsilon+\mu)(\theta+\mu)(1+\delta_1)}$.*

Proof. Define $V : \{(s_{ne}, s_e, m, i) \in \Omega : s_{ne} > 0, s_e > 0, m > 0\} \to \mathbb{R}$ by

$$V(s_{ne}, s_e, m, i) = i$$

The time derivative of V along the solution curves of (2) is,

$$\frac{dV}{dt} = \frac{di}{dt} \leq (\beta(1+\delta_1) - (\nu+\mu))i = (\nu+\mu)(\frac{R_0(1+\delta_1)}{s_e^*} - 1)i$$

Therefore, $\frac{dV}{dt} \leq 0$ when $R_0 \leq R_0^* = \frac{s_e^*}{1+\delta_1} = \frac{\epsilon\mu}{(\epsilon+\mu)(\theta+\mu)(1+\delta_1)}$. Furthermore, $\frac{dV}{dt} = 0$ if and only if $i = 0$. Hence Lasalle invariance principle shows the global asymptotic stability of P_0 with respect to the invariant set Ω. See [7, 16] for the proofs and applications of the notion of asymptotic stability with respect to invariant sets. □

Now we present the geometric method for the global stability problem, proved in [8], see [1, 3, 8] for applications of the method. Let us denote unit ball of $\mathbb{R}^2$ and its boundary and closure by, $\mathcal{B}$, $\partial\mathcal{B}$, and $\bar{\mathcal{B}}$ respectively. We also denote the collection of all Lipschitzian functions from X to Y, by $Lip(X \to Y)$. We consider a function $\phi \in Lip(\bar{\mathcal{B}} \to \Omega)$ as a simply connected and rectifiable surface in $\Omega \subseteq \mathbb{R}^n$. A closed and rectifiable curve in Ω, can be described as a function $\phi \in Lip(\partial\mathcal{B} \to \Omega)$ and called simple if it is one to one. Suppose $\Sigma(\psi, \Omega) = \{\psi \in Lip(\bar{\mathcal{B}} \to \Omega) : \phi|_{\partial\mathcal{B}} = \psi\}$. Let Ω be an open domain which is simply connected, then $\Sigma(\psi, \Omega)$ is a nonvoid set, for any simple, closed and rectifiable curve in Ω. Consider a norm $\|.\|$ on $\mathbb{R}^{\binom{n}{2}}$. We define a functional $\mathcal{S}$ on the surfaces in Ω by the following relation: $\mathcal{S}\phi = \int_{\bar{\mathcal{B}}} \|P(\phi)(\frac{\partial\phi}{\partial u_1} \wedge \frac{\partial\phi}{\partial u_2})\|du$, in which the mapping $u \mapsto \phi(u)$ is Lipschitzian on $\bar{\mathcal{B}}$, and $\frac{\partial\phi}{\partial u_1} \wedge \frac{\partial\phi}{\partial u_2}$ is the wedge product in $\mathbb{R}^{\binom{n}{2}}$. Furthermore, the $\binom{n}{2} \times \binom{n}{2}$ matrix function P, is invertible and $\|P^{-1}\|$ is a bounded function on $\phi(\bar{\mathcal{B}})$. Consider the vector field $x \mapsto f(x) \in \mathbb{R}^n$, which is a C^1 function on the set $\Omega \subset \mathbb{R}^n$, and the following ODE system, $\frac{dx}{dt} = f(x)$. We consider the function $\phi_t(u) = x(t, \phi(u))$ as the solution of the system passing through $(0, \phi(u))$, for any ϕ. We define the right-hand derivative of $\mathcal{S}\phi_t$, by the following relation, $D_+\mathcal{S}\phi_t = \int_{\bar{\mathcal{B}}} \lim_{h\to 0^+} \frac{1}{h}(\|z + hQ(\phi_t(u))z\| - \|z\|)du$, in which $Q = P_f P^{-1} + P\frac{\partial f^{[2]}}{\partial x}P^{-1}$, where P_f is the matrix obtained by replacing each entry p_{ij} of P by its directional derivative in the direction of f, i.e., $\frac{\partial p_{ij}}{\partial x} \cdot f$, and $\frac{\partial f^{[2]}}{\partial x}$ denotes the second additive compound matrix of $\frac{\partial f}{\partial x}$, see [8]. Furthermore, we consider the following differential equation, $\frac{dz}{dt} = Q(\phi_t(u))z$ for which the solution is of the form $z = P(\phi)(\frac{\partial\phi}{\partial u_1} \wedge \frac{\partial\phi}{\partial u_2})$. The formula $D_+\mathcal{S}\phi_t$ can be expressed as, $D_+\mathcal{S}\phi_t = \int_{\bar{\mathcal{B}}} D_+\|z\|du$. The Jacobian matrix of (2) is given by, $\frac{\partial f}{\partial x} = [a_{ij}]$, in which,

$a_{11} = -\epsilon - \mu, a_{12} = 0, a_{13} = 0, a_{14} = 0, a_{21} = \epsilon, a_{22} = -\beta i - \theta - \mu$
$a_{23} = 0, a_{24} = -\beta s_e, a_{31} = -\delta_2, a_{32} = \theta - \delta_2, a_{33} = -\delta_2 - \mu, a_{34} = -\delta_2$
$a_{41} = -\beta\delta_1 i, a_{42} = \beta i - \beta\delta_1 i, a_{43} = -\beta\delta_1 i$
$a_{44} = \beta s_e + \beta\delta_1 - \beta\delta_1 s_{ne} - \beta\delta_1 s_e - \beta\delta_1 m - 2\beta\delta_1 i - \nu - \mu$

And the second additive compound matrix of $\frac{\partial f}{\partial x}$ has the following form: $M = \dfrac{\partial f^{[2]}}{\partial x} = [M_{ij}]$, with the following components:
$M_{11} = -\epsilon - 2\mu - \beta i - \theta, M_{12} = 0, M_{13} = -\beta s_e, M_{14} = 0, M_{15} = 0$
$M_{16} = 0, M_{21} = \theta - \delta_2, M_{22} = -\epsilon - 2\mu - \delta_2, M_{23} = -\delta_2, M_{24} = 0, M_{25} = 0$
$M_{26} = 0, M_{31} = \beta i - \beta\delta_1 i, M_{32} = -\beta\delta_1 i$
$M_{33} = -\epsilon - 2\mu + \beta s_e + \beta\delta_1 - \beta\delta_1 s_{ne} - \beta\delta_1 s_e - \beta\delta_1 m - 2\beta\delta_1 i - \nu, M_{34} = 0$
$M_{35} = 0, M_{36} = 0, M_{41} = \delta_2, M_{42} = \epsilon, M_{43} = 0$
$M_{44} = -\beta i - \theta - 2\mu - \delta_2, M_{45} = -\delta_2, M_{46} = \beta s_e, M_{51} = \beta\delta_1 i, M_{52} = 0$
$M_{53} = \epsilon, M_{54} = -\beta\delta_1 i$
$M_{55} = -\beta i - \theta - 2\mu + \beta s_e + \beta\delta_1 - \beta\delta_1 s_{ne} - \beta\delta_1 s_e - \beta\delta_1 m - 2\beta\delta_1 i - \nu$
$M_{56} = 0, M_{61} = 0, M_{62} = \beta\delta_1 i, M_{63} = -\delta_2, M_{64} = -\beta i + \beta\delta_1 i, M_{65} = \theta - \delta_2$
$M_{66} = -\delta_2 - 2\mu + \beta s_e + \beta\delta_1 - \beta\delta_1 s_{ne} - \beta\delta_1 s_e - \beta\delta_1 m - 2\beta\delta_1 i - \nu$

Let $P = [p_{ij}]$ be the matrix with, $p_{11} = p_{22} = p_{34} = p_{43} = p_{55} = p_{66} = \frac{1}{i}$, in which other arrays are zero. The inverse of P is $P^{-1} = [q_{ij}]$ with the $q_{11} = q_{22} = q_{34} = q_{43} = q_{55} = q_{66} = i$, in which other arrays are zero. Furthermore, $P_f = [l_{ij}]$ with $l_{11} = l_{22} = l_{34} = l_{43} = l_{55} = l_{66} = -\frac{i'}{i^2}$ and other arrays are zero. Hence we have the relation, $P_f P^{-1} = -diag(\frac{i'}{i}, \frac{i'}{i}, \frac{i'}{i}, \frac{i'}{i}, \frac{i'}{i}, \frac{i'}{i})$, therefore the matrix Q has the following form, $Q = P_f P^{-1} + PMP^{-1} = [A_{ij}]$ in which,
$A_{11} = -\epsilon - \mu - \beta i - \theta - \beta s_e - \beta\delta_1 + \beta\delta_1 s_{ne} + \beta\delta_1 s_e + \beta\delta_1 m + \beta\delta_1 i + \nu$
$A_{14} = -\beta s_e, A_{21} = \theta - \delta_2$
$A_{22} = -\epsilon - \mu - \delta_2 - \beta s_e - \beta\delta_1 + \beta\delta_1 s_{ne} + \beta\delta_1 s_e + \beta\delta_1 m + \beta\delta_1 i + \nu, A_{24} = -\delta_2$
$A_{31} = \delta_2, A_{32} = \epsilon$
$A_{33} = -\beta i - \theta - \mu - \delta_2 - \beta s_e - \beta\delta_1 + \beta\delta_1 s_{ne} + \beta\delta_1 s_e + \beta\delta_1 m + \beta\delta_1 i + \nu$
$A_{35} = -\delta_2, A_{36} = \beta s_e, A_{41} = \beta i - \beta\delta_1 i, A_{42} = -\beta\delta_1 i$
$A_{44} = -\epsilon - \mu - \beta\delta_1 i, A_{51} = \beta\delta_1 i, A_{53} = -\beta\delta_1 i, A_{54} = \epsilon$
$A_{55} = -\beta i - \theta - \mu - \beta\delta_1 i, A_{62} = \beta\delta_1 i, A_{63} = -\beta i + \beta\delta_1 i, A_{64} = -\delta_2, A_{65} = \theta - \delta_2$
$A_{66} = -\delta_2 - \mu - \beta\delta_1 i$

Now we use the norm introduced in [5], for $\mathbb{R}^6$.

Lemma 4.1. *There is a constant $\tau > 0$, for which $D_+\|z\| \leq -\tau\|z\|$ for all $z \in \mathbb{R}^6$ and $s_{ne}, s_e, m, i > 0$, where z is the solution of $\frac{dz}{dt} = Q(\phi_t(u))z$, provided that,*

$$\mu + 2\beta > \nu + \theta + \delta_2, \mu + \epsilon > 2\beta + 2\beta\delta_1 + \delta_2 + \theta, 2\beta < \mu + min\{\delta_2, \theta\}. \quad (4)$$

Proof. We prove the existence of a $\tau > 0$ for which $D_+\|z\| \leq -\tau\|z\|$. The full calculation to demonstrate this relation contains sixteen cases related to the different orthants and the above norm, see [2]. We present the argumentation of one case; all the others are treated in the same way.

Case 1: $U_1 < U_2$ and $z_4, z_5, z_6 > 0$. In this case $||z|| = |z_4| + |z_5| + |z_6|$ and

$$\begin{aligned} D_+||z|| =& z_4' + z_5' + z_6' = A_{41}z_1 + A_{42}z_2 + A_{44}z_4 + A_{51}z_1 + A_{53}z_3 + A_{54}z_4 \\ &+ A_{55}z_5 + A_{62}z_2 + A_{63}z_3 + A_{64}z_4 + A_{65}z_5 + A_{66}z_6 \le (\beta i + \beta\delta_1 i)|z_1| \\ &+ (\beta\delta_1 i)|z_2| + (-\epsilon - \mu - \beta\delta_1 i)|z_4| + (\beta\delta_1 i)|z_1| + (\beta\delta_1 i)|z_3| + (\epsilon)|z_4| \\ &+ (-\beta i - \theta - \mu - \beta\delta_1 i)|z_5| + (\beta\delta_1 i)|z_2| + (\beta i + \beta\delta_1 i)|z_3| + (-\delta_2)|z_4| \\ &+ (\theta - \delta_2)|z_5| + (-\delta_2 - \mu - \beta\delta_1 i)|z_6| \end{aligned}$$

thus $D_+||z|| < (-3\mu - 3\delta_2 + 3\beta\delta_1 i + \beta i)||z||$, and $\beta i + 3\beta\delta_1 i \le \beta + 3\beta\delta_1 < \frac{5}{2}(\mu + \delta_2)$, hence $-3\mu - 3\delta_2 + 3\beta\delta_1 i + \beta i < 0$. □

In [8], the geometric method is applied to investigate the global stability of a unique steady state. In such cases, there exists a compact absorbing set. Hence surfaces remain in Ω for all time. But in models with backward bifurcation, such as model 1, such a set will not exist. Hence as in [1], we prove the existence of the following sequence φ^k of surfaces in the next lemma.

Lemma 4.2. *For an arbitrary simple and closed curve ψ in Ω, there is $\epsilon > 0$ and surfaces φ^k which minimizes $\mathcal{S}$ with respect to $\Sigma(\psi, \Omega)$, in such a way that, for all $k = 2, 3, ...$ and $t \in [0, \epsilon]$, $\varphi_t^k \subseteq \Omega$.*

Proof. Let $\xi = \dfrac{1}{2} min\{i : (s_{ne}, s_e, m, i) \in \psi\}$, in which i is the infective variable of system (2). It is easy to see that $\psi > 0$. From (2) and positivity of solutions, we have,

$$\frac{di}{dt} \ge -\beta\delta_1 i(s_{ne}+s_e+m+i) - (\nu+\mu)i = -\beta\delta_1 i(1-r) - (\nu+\mu)i \ge -(\beta\delta_1+\nu+\mu)i$$

which holds in Ω. Hence there exists $\epsilon > 0$ such that, the solutions with $i(0) \ge \xi$, remains in Ω, for $t \in [0, \epsilon]$. Hence we must show the existence of a sequence $\{\varphi^k\}$ which minimizes $\mathcal{S}$ with respect to $\Sigma(\psi, \widetilde{\Omega})$, in which $\widetilde{\Omega} = \{(s_{ne}, s_e, m, i) \in \Omega : i \ge \xi\}$. Now for $\varphi(u) = (s_{ne}(u), s_e(u), m(u), i(u)) \in \Sigma(\psi, \Omega)$, we define another surface $\widetilde{\varphi}(u) = (\widetilde{s}_{ne}(u), \widetilde{s}_e(u), \widetilde{m}(u), \widetilde{i}(u))$ by,

$$\begin{cases} \varphi(u) & if \;\; i(u) \ge \xi \\ (s_{ne}, s_e, m, \xi) & if \;\; i(u) < \xi, s_{ne} + s_e + m + \xi \le 1 \\ A & if \;\; i(u) < \xi, s_{ne} + s_e + m + \xi > 1 \end{cases}$$

in which,

$$A = (\frac{s_{ne}}{\sqrt{3}(s_{ne} + s_e + m)}(1-\xi), \frac{s_e}{\sqrt{3}(s_{ne} + s_e + m)}(1-\xi), \frac{m}{\sqrt{3}(s_{ne} + s_e + m)}(1-\xi), \xi)$$

It is easy to see that $\widetilde{\varphi}(u) \in \Sigma(\psi, \widetilde{\Omega})$. We will prove $\mathcal{S}\widetilde{\varphi} \le \mathcal{S}\phi$. We denote $\dfrac{\partial\widetilde{\varphi}}{\partial u_1} \wedge \dfrac{\partial\widetilde{\varphi}}{\partial u_2} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{x}_6)^T$ and $\dfrac{\partial\varphi}{\partial u_1} \wedge \dfrac{\partial\varphi}{\partial u_2} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$, and prove $\|\frac{\partial\widetilde{\phi}}{\partial u_1} \wedge \frac{\partial\widetilde{\phi}}{\partial u_2}\| \le \|\frac{\partial\phi}{\partial u_1} \wedge \frac{\partial\phi}{\partial u_2}\|$.

Case 1. If $i(u) \geq \xi$ then $\widetilde{\varphi} = \varphi$ and therefore, $|\widetilde{x}_j| = |x_j|, (j = 1, 2, \cdots, 6)$.
Case 2. If $i(u) < \xi$ and $s_{ne}+s_e+m+\xi \leq 1$, then $\widetilde{\varphi}(u) = (s_{ne}(u), s_e(u), m(u), \xi)$. Therefore it follows $\widetilde{x}_j = x_j (j = 1, 2, 4)$ and $\widetilde{x}_j = 0 (j = 3, 5, 6)$. Thus $|\widetilde{x}_j| \leq |x_j|$.
Case 3. If $i(u) < \xi$ and $s_{ne}+s_e+m+\xi > 1$, then $\widetilde{\varphi}(u) = (\frac{s_{ne}}{\sqrt{3}(s_{ne}+s_e+m)}(1-\xi), \frac{s_e}{\sqrt{3}(s_{ne}+s_e+m)}(1-\xi), \frac{m}{\sqrt{3}(s_{ne}+s_e+m)}(1-\xi), \xi)$. In this case, using $\frac{\partial \widetilde{s_{ne}}}{\partial u_j} + \frac{\partial \widetilde{s}_e}{\partial u_j} + \frac{\partial \widetilde{m}}{\partial u_j} = 0$, we can obtain, $\frac{\partial \widetilde{\varphi}}{\partial u_1} = z_1(u_1) f_1 + z_2(u_1) f_2$ and $\frac{\partial \widetilde{\varphi}}{\partial u_2} = z_1(u_2) f_1 + z_2(u_2) f_2$ in which, $f_1 = [-1, 1, 0, 0]^T$ and $f_2 = [-1, 0, 1, 0]^T$ and

$$z_1(u_j) = \frac{1-\xi}{\sqrt{3}} \frac{(s_{ne}+m)\frac{\partial s_e}{\partial u_j} - s_e(\frac{\partial s_{ne}}{\partial u_j} + \frac{\partial m}{\partial u_j})}{(s_{ne}+s_e+m)^2}$$

$$z_2(u_j) = \frac{1-\xi}{\sqrt{3}} \frac{(s_{ne}+s_e)\frac{\partial m}{\partial u_j} - m(\frac{\partial s_{ne}}{\partial u_j} + \frac{\partial s_e}{\partial u_j})}{(s_{ne}+s_e+m)^2}$$

for $j = 1, 2$. Therefore,

$$\frac{\partial \widetilde{\varphi}}{\partial u_1} \wedge \frac{\partial \widetilde{\varphi}}{\partial u_2} = (z_1(u_1) z_2(u_2) - z_2(u_1) z_1(u_2)) f_1 \wedge f_2 =$$

$$\frac{(1-\xi)^2}{3(s_{ne}+s_e+m)^4} K \, [1, -1, 0, 1, 0, 0]^T$$

in which, $K = s_{ne}(s_{ne}+s_e+m)x_1 - m(s_{ne}+s_e+m)x_2 + s_e(s_{ne}+s_e+m)x_4$, which yields, $||\frac{\partial \widetilde{\phi}}{\partial u_1} \wedge \frac{\partial \widetilde{\phi}}{\partial u_2}|| \leq |x_1| + |x_2| + |x_4| \leq ||\frac{\partial \phi}{\partial u_1} \wedge \frac{\partial \phi}{\partial u_2}||$. Furthermore $\widetilde{i}(u) = \max\{i(u), \xi\}$, hence $\frac{1}{\widetilde{i}} \leq \frac{1}{i}$. Now let $\widetilde{P} = [\widetilde{p}_{ij}]$, with $\widetilde{p}_{11} = \widetilde{p}_{22} = \widetilde{p}_{34} = \widetilde{p}_{43} = \widetilde{p}_{55} = \widetilde{p}_{66} = \frac{1}{\widetilde{i}}$, in which other arrays are zero. Since $|\frac{1}{\widetilde{i}}\widetilde{x}_j| \leq |\frac{1}{i}x_j|$ $(j = 1, 2, 4)$ and $|\frac{1}{\widetilde{i}}\widetilde{x}_j| \leq |\frac{1}{i}x_j|$ $(j = 3, 5, 6)$, by an easy computation $\mathcal{S}\widetilde{\phi} = \int_{\widetilde{\mathcal{B}}} \|\widetilde{P}(\widetilde{\phi})(\frac{\partial \widetilde{\phi}}{\partial u_1} \wedge \frac{\partial \widetilde{\phi}}{\partial u_2})\| du \leq \int_{\widetilde{\mathcal{B}}} \|P(\phi)(\frac{\partial \phi}{\partial u_1} \wedge \frac{\partial \phi}{\partial u_2})\| du = \mathcal{S}\phi$. Using lemma, we can choose $\delta = \inf\{\mathcal{S}\phi : \phi \in \Sigma(\psi, \Omega)\}$. Suppose that $\{\phi^k\}$, minimizes $\mathcal{S}$ with respect to $\Sigma(\psi, \Omega)$, then $\lim_{k\to\infty} \mathcal{S}\phi^k = \delta$. Now consider the sequence $\{\widetilde{\phi}^k\} \subset \Sigma(\psi, \widetilde{\Omega})$ as in the above definition, from the boundedness of $\{\mathcal{S}\widetilde{\phi}^k\}$ and $\mathcal{S}\widetilde{\phi}^k \leq \mathcal{S}\phi^k$, we have $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}_k \leq \delta$. Furthermore $\widetilde{\phi}^k \in \Sigma(\psi, \Omega)$, hence $\mathcal{S}\widetilde{\phi}^k \geq \delta$, and $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}_k \geq \delta$, which implies $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}_k = \delta$. Now

$$\inf\{\mathcal{S}\widetilde{\phi} : \widetilde{\phi} \in \Sigma(\psi, \widetilde{\Omega})\} \leq \inf\{\mathcal{S}\phi : \phi \in \Sigma(\psi, \Omega)\} = \delta$$

and from $\widetilde{\phi} \in \Sigma(\psi, \Omega)$ we have $\inf\{\mathcal{S}\widetilde{\phi} : \widetilde{\phi} \in \Sigma(\psi, \widetilde{\Omega})\} \geq \delta$, which implies $\inf\{\mathcal{S}\widetilde{\phi} : \widetilde{\phi} \in \Sigma(\psi, \widetilde{\Omega})\} = \delta$. At the final we can show that $\lim_{k\to\infty} \mathcal{S}\widetilde{\phi}_k = \delta =$

$\inf\{\mathcal{S}\widetilde{\phi} : \widetilde{\phi} \in \Sigma(\psi, \widetilde{\Omega})\}$, i.e., $\{\widetilde{\phi}^k\}$ minimizes $\mathcal{S}$ relative to $\Sigma(\psi, \widetilde{\Omega})$.

From lemmas 4.1 and 4.2, we have the following result.

Theorem 4.1. *Any ω-limit point of 1 in Ω° is an equilibrium point, and therefore each positive semi-trajectory of system converges to an equilibrium point.*

Finally, from the above theorem, we obtain the following result.

Theorem 4.2. *Assume that inequality (4) is satisfied, then:*
(1) If there is no endemic equilibrium, then all solutions of (1), tend to the drug-free equilibrium P_0;
(2) If $R_0 > 1$, then all solutions of (1), converge to the unique endemic equilibrium;
(3) If there are two endemic equilibrium points, which occurs when $R_0^c < R_0 < 1$, solutions of the system either go to the drug-free equilibrium P_0 or tend to the upper equilibrium point.

5. **Conclusions**

The White and Comiskey's model of heroin epidemics is extended in this paper. This extension includes the split of the susceptible populations into three compartments, noneducated susceptibles, educated susceptibles and individuals who are completely aware of drug harms so that they will not use drugs forever. A complete qualitative study of the model including the existence and local and global stability of the equilibrium points are carried out. The drug-free equilibrium P_0, is shown to be locally and globally stable under suitable conditions. Using compound matrices the sufficient conditions for the local and global stability of the endemic equilibrium points is obtained. The occurrence of backward bifurcation is also proved for the model which shows under some conditions, it is not enough to reduce R_0 to the region $R_0 < 1$, to control the drug epidemic. In fact when $R_0 < 1$, the drug problem may be persistent. Hence we compute another threshold, $R_0^c < 1$, and show that for the control of drug epidemic, R_0 should be reduced to below R_0^c.

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