A SEQUENTIAL TWO-STAGE METHOD FOR SOLVING GENERALIZED SADDLE POINT PROBLEMS

Davod Khojasteh Salkuyeh[1], Fatemeh Panjeh Ali Beik[2], Davod Hezari[3]

In [Appl. Math. Comput. 217 (2011) 5596-5602], Li et al. suggested an effective iterative method for solving large sparse saddle point problems with symmetric positive definite $(1,1)$*-block. Recently, Zhu et al. [Appl. Math. Comput. 242 (2014) 907-916] developed the method for the saddle point problems with* $(1,1)$*-block being non-symmetric positive definite. This paper deals with extending their idea to obtain a new iterative scheme for solving the generalized saddle point problems with non-symmetric positive definite (1,1)-block and symmetric positive semidefinite (2,2)-block. To this end, the original linear system is split into two smaller subsystems. One of them is solved directly by the Cholesky factorization and the other by an iterative method. The convergence analysis of the method is investigated and some numerical experiments are reported to demonstrate the effectiveness of the proposed approach.*

Keywords: Iterative method, generalized saddle point, positive definite, convergence.
AMS Subject Classification: 65F10, 65F50, 65N22.

1. Introduction

Let us first introduce some symbols exploited throughout this paper. The real and imaginary parts of a complex number z are respectively represented by $\Re(z)$ and $\Im(z)$. The notation $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) stands for the set of all $m \times n$ real (complex) matrices. The transpose and conjugate transpose of a given matrix A are respectively denoted by A^T and A^H. The identity matrix of order m is denoted by I_m. For a given square matrix A, the spectral radius of A is signified by $\rho(A)$. For two given arbitrary symmetric square real matrices A and B, $A \succ B$ means $A - B$ is symmetric positive definite.

[1]Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran, e-mail: khojasteh@guilan.ac.ir

[2]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran, e-mail: f.beik@vru.ac.ir

[3]Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran, e-mail: d.hezari@gmail.com

In this paper, we are concerned with the solution of the following large and sparse generalized saddle point problem

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is non-symmetric positive definite, i.e., $A + A^T$ is symmetric positive definite, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, $B \in \mathbb{R}^{m\times n}$ has full row rank, $x, f \in \mathbb{R}^n$, $y, g \in \mathbb{R}^m$ and $m \leq n$. These assumptions guarantee the existence and uniqueness of the solution of the linear systems (1)(see [8, Lemma 1.1]).

In practice, systems of the form (1) appear in a variety of computational sciences and engineering applications such as constrained optimization, computational fluid dynamics, mixed finite element discretization of the Navier-Stokes equations, the linear elasticity problem, elliptic and parabolic interface problems, constrained least-squares problem and so on; for more details see [9, 13, 17, 18, 21] and references therein.

In the literature there has been a considerable expansion in the field of solution techniques for solving systems of the form (1), recently. More precisely, several researches were devoted to studying the performance of different kinds of iterative methods including, for instance, the Uzawa-type methods [6, 7, 12], HSS-based methods [2, 3, 4, 8], as well as preconditioned Krylov subspace methods such as MINRES and GMRES together with suitable preconditioners [1, 5, 10, 11, 14, 20, 23, 24]. For a comprehensive review on iterative methods for large and sparse linear systems in saddle point form, one may refer to [9].

In the case that the $(2, 2)$-block C in (1) is zero and the matrix A is symmetric positive definite, Li et al. [25] have proposed an efficient splitting iterative method. In fact the original system is transformed into two sub-systems with smaller sizes by premultiplying both sides of (1) with the following matrix

$$\mathcal{P}_\alpha = \begin{pmatrix} I_n & -B^T(BB^T)^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ B & -\alpha BB^T \end{pmatrix}. \tag{2}$$

Afterward, the conjugate gradient (CG) method and a splitting iteration method are respectively used to solve the linear system with a SPD coefficient matrix and the other sub-system. By making use of the same technique and a new matrix splitting based on the Hermitian and skew-Hermitian splitting (HSS) of the $(1, 1)$-block, Zhu et al. [27] have presented an efficient sequential two-stage method for solving saddle point problems when the $(1, 1)$-block and $(2, 2)$-block are non-symmetric positive definite and zero matrices, respectively. In this paper, we develop the idea of the technique presented in ([25, 27]) for the generalized saddle point problem (1) with $C \neq 0$ and split the original system into two smaller subsystems. One of the obtained subsystems is directly solved by the Cholesky factorization and the other one by an iterative method. Convergence properties of the proposed approach are investigated and numerical experiments are reported to confirm the validity of the established results and reveal the efficiency of the proposed method.

The remainder of the paper is organized as follows. In Section 2, we present a generalization of the methods presented in [25, 27] and investigate its properties. Section 3 is devoted to examining some numerical experiments to illustrate the effectiveness of the proposed method. Finally the paper is ended with a brief conclusion in Section 4.

2. Main results

This section deals with applying a new iterative scheme to solve the generalized saddle point problem (1) whose $(2,2)$-block may be a nonzero matrix. To this end, let us first consider the following generalized preconditioner which incorporates (2) as its special case,

$$\mathcal{P}_{\alpha,\gamma} = \begin{pmatrix} I_n & -B^T(\gamma I_m + \alpha C)^{-1}(BB^T)^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ \gamma B & -\alpha BB^T \end{pmatrix}, \qquad (3)$$

where α and γ are two given positive constants.

Premultiplying both sides of Eq. (1) by $\mathcal{P}_{\alpha,\gamma}$ yields

$$\mathcal{P}_{\alpha,\gamma}\mathcal{A}u \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{P}_{\alpha,\gamma}b. \qquad (4)$$

By some straightforward calculations, the system (4) can be rewritten as follows:

$$\begin{pmatrix} \tilde{A} & 0 \\ \tilde{B} & \tilde{C} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix}, \qquad (5)$$

where

$$\begin{aligned} \tilde{A} &= A + \alpha B^T(\gamma I_m + \alpha C)^{-1}B - \gamma B^T(\gamma I_m + \alpha C)^{-1}(BB^T)^{-1}BA, \\ \tilde{B} &= B(\gamma A - \alpha B^T B), \quad \tilde{C} = BB^T(\gamma I_m + \alpha C), \\ \tilde{f} &= f - B^T(\gamma I_m + \alpha C)^{-1}\left[\gamma(BB^T)^{-1}Bf - \alpha g\right], \quad \tilde{g} = B(\gamma f - \alpha B^T g). \end{aligned}$$

Both of the matrices $\mathcal{P}_{\alpha,\gamma}$ and $\mathcal{A}$ are nonsingular. Hence, we conclude that $\mathcal{P}_{\alpha,\gamma}\mathcal{A}$, as well as $\tilde{A}$ and $\tilde{C}$ are nonsingular. System (5) is equivalent to

$$\begin{cases} \tilde{A}x = \tilde{f}, \\ \tilde{B}x + \tilde{C}y = \tilde{g}. \end{cases} \qquad (6)$$

Then, the vector x can be obtained from the first equation of (6) by solving a linear system with coefficient matrix $\tilde{A}$. Upon substituting this into the second equation, the vector y is obtained by solving the linear system $\tilde{C}y = \tilde{g} - \tilde{B}x$. Note that both of the matrices $BB^T \in \mathbb{C}^{m\times m}$ and $(\gamma I_m + \alpha C) \in \mathbb{C}^{m\times m}$ are symmetric positive definite and of small size. Hence, we can use the Cholesky factorization of these matrices to compute y. However, the coefficient matrix $\tilde{A}$ of the first equation in (6) is generally large and dense, so for this system, direct computations are very costly and impractical in actual implementations. Therefore, we consider the splitting $\tilde{A} = M_{\alpha,\gamma} - N_{\alpha,\gamma}$ for the matrix $\tilde{A}$, where

$$M_{\alpha,\gamma} = A + \alpha B^T(\gamma I_m + \alpha C)^{-1}B = A + rB^T(I_m + rC)^{-1}B =: M_r,$$
$$N_{\alpha,\gamma} = \gamma B^T(\gamma I_m + \alpha C)^{-1}(BB^T)^{-1}BA = B^T(I_m + rC)^{-1}(BB^T)^{-1}BA =: N_r,$$

in which $r = \alpha/\gamma$. Using $\tilde{A} = M_r - N_r$, we then propose the following stationary iterative method

$$M_r x^{(k+1)} = N_r x^{(k)} + \tilde{f}, \tag{7}$$

where $x^{(0)}$ is an initial guess. Since A is positive definite and the matrix $rB^T(I_m+rC)^{-1}B$ is symmetric positive semidefinite, it follows that the matrix M_r is positive definite. Hence, one may use a direct method such as the LU factorization or an iterative method such as the restarted GMRES(m) to solve the system with the coefficient matrix M_r. Here, we mention that for real positive definite matrices the GMRES(m) converges for any $m \geq 1$ (see [26, Theorem 6.30]).

As known, the necessary and sufficient condition for the convergence of the iterative method (7) for all initial vectors $x^{(0)}$ and right-hand side $\tilde{f}$ is that $\rho(M_r^{-1}N_r) < 1$. In order to demonstrate the convergence behaviour of the proposed iterative manner, we present the following two theorems. The first theorem reveals that the spectral radius of the iteration matrix corresponding to (7) tends to zero as $r \to \infty$. The second theorem gives a sufficient condition for the constant r (α and γ) which guarantees the convergence of the iterative method (7) for any initial guess.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be positive definite, $C \in \mathbb{R}^{m\times m}$ be symmetric positive semidefinite and $B \in \mathbb{R}^{m\times n}$ be of full row rank. Then, $\rho(M_r^{-1}N_r) \to 0$ as $r \to 0$.*

Proof. Let (λ, x) be an eigenpair of $G_r = M_r^{-1}N_r$ with $\|x\|_2 = 1$. Hence,

$$G_r x = \lambda x \Rightarrow N_r x = \lambda M_r x \Rightarrow x^H N_r x = \lambda x^H M_r x.$$

If $Bx = 0$, then $x^H N_r x = 0$, which implies that $\lambda = 0$, and there is nothing to prove. Therefore, without loss of generality, it is assumed that $Bx \neq 0$. Since the matrix $(I_m + rC)^{-1}$ is symmetric positive definite, it follows that

$$x^H B^T(I_m + rC)^{-1}Bx = (Bx)^H(I_m + rC)^{-1}(Bx) > 0.$$

Therefore, we have

$$\begin{aligned} \mid x^H M_r x \mid^2 &= \left(\Re(x^H Ax) + rx^H B^T(I_m + rC)^{-1}Bx\right)^2 + \Im(x^H Ax)^2 \\ &\geq \left(\Re(x^H Ax) + rx^H B^T(I_m + rC)^{-1}Bx\right)^2. \end{aligned}$$

Hence,

$$\mid \lambda \mid = \frac{\mid x^H N_r x \mid}{\mid x^H M_r x \mid} \leq \frac{\mid x^H B^T(I_m + rC)^{-1}(BB^T)^{-1}BAx \mid}{\Re(x^H Ax) + rx^H B^T(I_m + rC)^{-1}Bx}.$$

Let $\mu_1 = \cdots = \mu_k = 0 < \mu_{k+1} \leq \cdots \leq \mu_m$ be the eigenvalues of C. Since C is symmetric, there is an orthogonal matrix U (i.e., $U^TU = I$) such that $C = U^TDU$, where $D = \mathrm{diag}(\mu_1, \ldots, \mu_m)$. Now, we have

$$\begin{aligned}
&\mid x^H B^T(I_m + rC)^{-1}(BB^T)^{-1}BAx \mid^2 \\
&\qquad \leq \|Ax\|_2^2 \|B^T(BB^T)^{-1}(I_m + rC)^{-1}Bx\|_2^2 \\
&\qquad \leq \|A\|_2^2 \left(x^H B^T(I_m + rC)^{-1}(BB^T)^{-1}(I_m + rC)^{-1}Bx\right) \\
&\qquad \leq \frac{\|A\|_2^2}{\sigma_{\min}^2(B)} \left(x^H B^T(I_m + rC)^{-2}Bx\right) \\
&\qquad = \frac{\|A\|_2^2}{\sigma_{\min}^2(B)} \left(w^H(I_m + rD)^{-2}w\right),
\end{aligned}$$

where $w = UBx \neq 0$ and $\sigma_{\min}(B)$ is the smallest singular value of B. On the other hand, we see that

$$rx^H B^T(I_m + rC)^{-1}Bx = rw^H(I_m + rD)^{-1}w.$$

Consequently,

$$\lim_{r\to\infty} w^H(I_m + rD)^{-1}w = \lim_{r\to\infty} w^H(I_m + rD)^{-2}w = \|\tilde{w}\|_2^2, \tag{8}$$

where $\tilde{w} = (w_1, \ldots, w_k, 0, \ldots, 0)^T \in \mathbb{C}^m$. Therefore,

$$\mid \lambda \mid \leq \frac{\|A\|_2}{\sigma_{\min}(B)} \frac{\sqrt{w^H(I_m + rD)^{-2}w}}{\Re(x^H Ax) + rw^H(I_m + rD)^{-1}w}.$$

From Eq. (8) we deduce that the right-hand side of the latter inequality tends to zero as $r \to \infty$ and this completes the proof. □

Remark 2.1. *From Theorem 2.1 and that $r = \alpha/\gamma$, it follows that for fixed value of α, we have $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}) \to 0$ as $\gamma \to 0$. In addition, we deduce that for fixed α and a sufficiently small value of γ, we have $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}) < 1$ which guarantees the convergence of the method.*

Proposition 2.1. *Suppose that C is a nonzero symmetric semipositive definite matrix and r is a positive constant. Then, $(I + rC)^{-1} \succ (I + rC)^{-2}$.*

Proof. Note that if C is a diagonal matrix the validity of the assertion can be easily seen. In general situation, since C is symmetric positive semidefinite, there exists an orthogonal matrix U such that $C = U^TDU$ where D is a diagonal matrix with nonnegative diagonal entries. Consequently, we may conclude the result immediately from the fact that

$$x^H \left(I + rC\right)^{-k} x = y^H \left(I + rD\right)^{-k} y, \qquad \forall x \in \mathbb{C}^m,$$

where $y = Ux$ and $k = 1, 2$. □

Theorem 2.2. *Assume that $A \in \mathbb{R}^{n\times n}$ is non-symmetric positive definite, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite and $B \in \mathbb{R}^{m\times n}$ has full row rank. Furthermore, suppose that $A = H + S$ where H and S are, respectively,* *the symmetric and skew-symmetric part of A, i.e., $H = (A + A^T)/2$ and $S = (A - A^T)/2$. Let α and γ be two given positive parameters so that*

$$r = \frac{\alpha}{\gamma} \geq \left(\frac{\|A\|_2^2}{\lambda_{\min}\sigma_{\min}} \right),$$

where $\lambda_{\min}$ and $\sigma_{\min}$ stand for the smallest eigenvalues of H and BB^T, respectively. Then $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}) < 1$.

Proof. Let (λ, x) be an eigenpair of $G_{\alpha,\gamma} = M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma} = M_r^{-1}N_r$ with $\|x\|_2 = 1$. In what follows, without loss of generality, we may assume that $Bx \neq 0$. Notice that positive definiteness of the matrix A implies that H is a symmetric positive definite matrix, and hence $\lambda_{\min} > 0$. Now, by the Cauchy-Schwarz inequality we conclude that

$$\begin{aligned}
|x^H N_\alpha x|^2 &= |\, x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} BAx \,|^2 \\
&\leq \|Ax\|_2^2 \|B^T (BB^T)^{-1} (I_m + rC)^{-1} Bx\|_2^2 \\
&\leq \|A\|_2^2 \left(x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} (I_m + rC)^{-1} Bx \right) \\
&< \lambda_{\min}^2 + \|A\|_2^2 \left(x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} (I_m + rC)^{-1} Bx \right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
|x^H N_\alpha x| &< \sqrt{\lambda_{\min}^2 + \|A\|_2^2 x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} (I_m + rC)^{-1} Bx} \\
&< \frac{1}{\lambda_{\min}} \left(\lambda_{\min}^2 + \|A\|_2^2 x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} (I_m + rC)^{-1} Bx \right) \\
&= \lambda_{\min} + \frac{\|A\|_2^2}{\lambda_{\min}} \left(x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} (I_m + rC)^{-1} Bx \right). \quad (9)
\end{aligned}$$

As a consequence of the well-known Courant-Fischer theorem [19, Theorem 4.2.11], it is seen that $x^H Hx \geq \lambda_{\min}$. Using (9), the fact that $\Re(x^H Ax) = x^H Hx$ and Proposition 2.1, we derive

$$\begin{aligned}
|\, x^H M_r x \,| - |\, x^H N_r x \,| &\geq x^H Hx + r x^H B^T (I_m + rC)^{-1} Bx - |\, x^H N_r x \,| \\
&> r x^H B^T (I_m + rC)^{-1} Bx - \frac{\|A\|_2^2}{\lambda_{\min}} \left(z^H (BB^T)^{-1} z \right) \\
&> r x^H B^T (I_m + rC)^{-2} Bx - \frac{\|A\|_2^2}{\lambda_{\min}} \left(z^H (BB^T)^{-1} z \right) \\
&\geq r z^H z - \frac{\|A\|_2^2}{\lambda_{\min}} \left(z^H (BB^T)^{-1} z \right) \\
&= z^H \left(rI_m - \frac{\|A\|_2^2}{\lambda_{\min}} (BB^T)^{-1} \right) z,
\end{aligned}$$

where $z = (I_m + rC)^{-1} Bx$. By the assumption $r > \frac{\|A\|_2^2}{\lambda_{\min}\sigma_{\min}}$, so we conclude that

$$z^H \left(rI_m - \frac{\|A\|_2^2}{\lambda_{\min}} (BB^T)^{-1} \right) z > 0,$$

which completes the proof. □

We now propose the following algorithm to solve the saddle point problem.

Algorithm 1.

(1) Choose an initial guess $x^{(0)}$, $\epsilon > 0$ and $\alpha, \gamma > 0$.
(2) Set $r = \alpha/\gamma$ and $r^{(0)} = \tilde{f} - \tilde{A}x^{(0)}$.
(3) Set

$$M = A + rB^T(I_m + rC)^{-1}B \quad \text{and} \quad N = B^T(I_m + rC)^{-1}(BB^T)^{-1}BA.$$

(4) For $k = 0, 1, 2 \ldots,$ Do
(5) Solve $Mx^{(k+1)} = Nx^{(k)} + \tilde{f}$ for $x^{(k+1)}$
(6) If $\|\tilde{f} - \tilde{A}x^{(k+1)}\|_2/\|r^{(0)}\|_2 < \epsilon$, then $x_a = x^{(k+1)}$ and stop.
(7) EndDo
(8) Solve $\tilde{C}y_a = \tilde{g} - \tilde{B}x_a$ for y_a.

Algorithm 1 computes the approximate solution $(x_a; y_a)$ for the problem (1). It is noted that, in practice, we use the GMRES(m) to solve the system involving Step 5 of the algorithm. Moreover, when A is symmetric positive definite, the matrix M is symmetric positive definite, too. In this case we can use the conjugate gradient method to solve the system of Step 5. Similar to [25, Theorem 2.2] the convergence of the method can be verified in both of the cases.

3. Numerical experiments

In this section, we present two examples, which are of the block linear system of the form (1) with (symmetric and nonsymmetric) real positive definite matrix A, to assess the feasibility and effectiveness of the proposed method. We compare the numerical behaviors of the proposed method with the Inexact Hermitian and skew-Hermitian Splitting (IHSS) method, presented by Benzi and Golub in [8], in terms of number of iteration steps (denoted by Iters) and the CPU time in seconds (denoted by CPU). For Examples 3.1 and 3.2, the matix A of the corresponding system is symmetric and nonsymmetric, respectively, so, the corresponding matrix $M_{\alpha,\gamma}$ is symmetric and nonsymmetric positive definite, respectively. Therefore, in the implementation of the proposed method, we apply the CG and GMRES(50) algorithms to solve the linear system with the coefficient matrix $M_{\alpha,\gamma}$ for Examples 3.1 and 3.2, respectively, and the Cholesky factorization for the remaining systems. The parameter α, adopted in HSS method, is the experimentally found optimal one that minimizes the total number of iteration steps. In addition, the parameters α and γ for the proposed method are set to be 1 and 10^{-5}, respectively.

All iteration processes are started from zero and terminate once the Euclidean norms of the current residuals are reduced by a factor of 10^6 from those of the initial residuals. Further, all codes are computed in double precision in

TABLE 1. Numerical results Example 3.1.

	Proposed method $(\alpha,\gamma)=(1,10^{-5})$		HSS method		
n	Iters	CPU	α	Iters	CPU
5000	2	0.16	0.04	173	1.51
10000	2	0.28	0.04	173	2.97
15000	2	0.40	0.04	173	4.33
20000	2	0.50	0.04	173	5.87

TABLE 2. Numerical results Example 3.2 with $\nu = 1/50$.

		Proposed method $(\alpha,\gamma)=(1,10^{-5})$		HSS method		
Grid	(n,m)	Iters	CPU	α	Iters	CPU
8×8	(162,62)	2	0.02	0.07	187	0.02
16×16	(578,256)	2	0.04	0.04	222	0.10
32×32	(2178,1024)	3	0.75	0.02	349	1.67
64×64	(8450,4096)	8	24.65	0.006	1152	71.68

MATLAB, and all experiments are performed on a personal Laptop with Intel Core i7 CPU 1.8 GHz, 6GB RAM.

Example 3.1. [22] Consider the saddle point problem (1) with the coefficient matrices $A=(a_{ij})_{n\times n}$, $B=[T,O]$, and $C=I$, where

$$a_{ij}=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-|i-j|^2}{2\sigma^2}},\quad T=\frac{1}{1000}\text{tridiag}(1,4,1)\in\mathbb{R}^{m\times m},\quad O\in\mathbb{R}^{m\times(n-m)}.$$

We set $\sigma = 1.5$ and $m=\frac{n}{2}$. The right-hand side vector $(f;g)$ is taken such that the vector $(x;y)$ of all ones is the exact solution of (1).

Example 3.2. We consider the steady-state Navier-Stokes equation

$$\begin{cases} -\nu\triangle\mathbf{u}+(\mathbf{u}.\nabla)\mathbf{u}+\nabla p &= \mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \end{cases}\quad \text{in}\quad \Omega=[0,1]\times[0,1],$$

where $\nu > 0$. By the IFISS package [15], this problem is linearized by the Picard iteration and then discretized by using the stabilized Q1-P0 finite elements (see [16]). This yields a generalized saddle point problem of the form (1). The right-hand side vectors f and g are taken such that x and y are two vectors of all ones.

The number of iteration steps and the CPU times for the IHSS and proposed methods for the Examples 3.1 and 3.2, with respect to different values of the problem sizes n and m, are listed in Tables 1 and 2. From these tables we see that the proposed method can successfully compute satisfactory

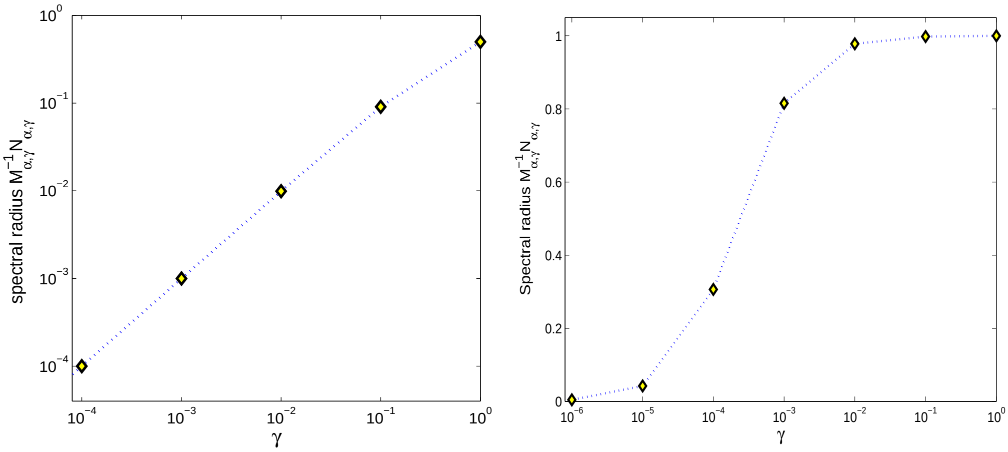

FIGURE 1. Spectrum radius of the $M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}$ for Example 3.1: $n = 2178$ (*left*) and for Example 3.2: $n = 5000$ (*right*).

approximations to the exact solutions of the above two examples in a few iteration steps and when comparing the IHSS and the proposed method, the latter method performs better than the former.

In Fig. 3, we plot $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma})$ with respect to the fixed $\alpha = 1$ and different values of γ for the above two examples, respectively. As expected, it can be observed that $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma})$ converges to zero as γ tends to zero.

4. **Conclusion**

We have presented an efficient method to solve generalized saddle point problems. Theoretical analysis of the method have been presented. The performance of the proposed method have been numerically compared with the inexact Hermitian and skew-Hermitian splitting method. The obtained numerical results have shown that the proposed method outperforms the IHSS method for the presented examples.

REFERENCES

[1] *Z. Z. Bai*, Structured preconditioners for nonsingular matrices of block two-by-two structures, Math. Comp., **75** (2006), 791-815.

[2] *Z. Z. Bai*, Optimal parameters in the HSS-like methods for saddle-point problems, Numer. Linear Algebra Appl., **16** (2009), 447-479.

[3] *Z. Z. Bai and G. H. Golub*, Accelerated Hermitian and skew-Hermitian splitting methods for saddle-point problems, IMA J. Numer. Anal., **27** (2007), 1-23.

[4] *Z. Z. Bai, G. H. Golub and J. Y. Pan*, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefnite linear systems, Numer. Math., **98** (2004), 1-32.

[5] *Z. Z. Bai and M. K. Ng*, On inexact preconditioners for nonsymmetric matrices, SIAM J. Sci. Comput., **26** (2005), 1710-1724.

[6] *Z. Z. Bai, B. Parlett and Z. Q. Wang*, On generalized successive overrelaxation methods for augmented linear systems, Numer. Math., **102** (2005), 1-38.

[7] *Z. Z. Bai and Z. Q. Wang*, On parameterized inexact Uzawa methods for generalized saddle point problems, Linear Algebra Appl., **428** (2008), 2900-2932.

[8] *M. Benzi and G.H. Golub*, A preconditioner for generalized saddle point problems, SIAM J. Matrix Anal. Appl., **26** (2004), 20-41.

[9] *M. Benzi, G.H. Golub and J. Liesen*, Numerical solution of saddle point problems, Acta Numer., **14** (2005), 1-137.

[10] *M. Benzi and X.P. Guo*, A dimensional split preconditioner for Stokes and linearized Navier-Stokes equations, Appl. Numer. Math., **61** (2011), 66-76.

[11] *M. Benzi, M. K. Ng, Q. Niu and Z. Wang*, A relaxed dimensional fractorization preconditioner for the incompressible Navier-Stokes equations, J. Comput. Phys., **230** (2011), 6185-6202.

[12] *J. H. Bramble, J. E. Pasciak, and A. T. Vassilev*, Analysis of the inexact Uzawa algorithm for saddle point problems, SIAM J. Numer. Anal., **34** (1997), 1072-1092.

[13] *F. Brezzi and M. Fortin*, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[14] *Y. Cao, L. Q. Yao and M. Q. Jiang*, A modified dimensional split preconditioner for generalized saddle point problems, J. Comput. Appl. Math., **250** (2013), 70-82.

[15] *H. C. Elman, A. Ramage and D. J. Silvester*, IFISS: A Matlab toolbox for modelling incompressible flow, ACM Trans. Math. Software., **33** (2007), Article 14.

[16] *H.C. Elman, D.J. Silvester and A.J. Wathen*, Finite elements and fast iterative solvers: with Applications in incompressible fluid dynamics, Oxford University Press, 2005.

[17] *V. Girault and P. A. Raviart*, Finite Element Methods for NavierStokes Equations, SpringerVerlag, Berlin, 1986.

[18] *R. Glowinski and P. Le Tallec*, Augmented Lagrangian and Operator-splitting Methods in Nonlinear Mechanics. SIAM, Philadelphia, 1989.

[19] *R. A. Horn and C. R. Johnson*, Matrix analysis, Cambridge University Press, Cambridge, 1985.

[20] *C. Keller, N. I. M. Gould and A. J. Wathen*, Constraint preconditioning for indefinite linear systems, SIAM J. Matrix Anal. Appl., **21** (2000), 1300-1317.

[21] *Y. Keung and J. Zou*, An efficient linear solver for nonlinear parameter identification problems, SIAM J. Sci. Comput., **22** (2000), 1511-1526.

[22] *J. Lu and Z. Zhang*, A modified nonlinear inexact Uzawa algorithm with a variable relaxation parameter for the stabilized saddle point problem, SIAM J. Matrix Anal. Appl., **31** (2010), 1934-1957.

[23] *M. F. Murphy, G. H. Golub and A. J. Wathen*, A note on preconditioning for indefinite linear systems, SIAM J. Sci. Comput., **21** (2000), 1969-1972.

[24] *J. Y. Pan, M. K. Ng and Z.Z. Bai*, New preconditioners for saddle point problems, Appl. Math. Comput., 172 (2006), 762-771.

[25] *D. Li, J. Zhao and G. Zhang*, An efficient numerical method for preconditioned saddle point problems, Appl. Math. Comput., **217**, 5596-5602 (2011).

[26] *Y. Saad*, Iterative Methods for Sparse Linear Systems, Second Edition, SIAM, Philadelphia, 2003.

[27] *M. Z. Zhu, G. F. Zhang, Z. Zheng and Z. Z. Liang,* On HSS-based sequential two-stage method for non-Hermitian saddle point problems, Appl. Math. Comput., **242** (2014), 907-916.