CONVERGENCE THEOREMS FOR OPERATORS WITH CONDITION (E) IN HYPERBOLIC SPACES

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We introduce convergence results of iteration process S_n, by Sintunavarat and Pitea [J. Nonlinear Sci. Appl., 2016, 9, 2553-2562] in the framework of hyperbolic spaces, introduced by Kohlenbach [Trans. Am. Math. Soc., 2005, 357, 89-128]. This study is carried out on a remarkable class of operators, that of mappings with condition (E), introduced by Garcia-Falset *et al.* [J. Math. Anal. Appl., 2011, 375, 185-195]. This work presents both results on Δ-convergence and also some strong convergence results, under additional conditions. We conclude the paper with at least two possible ways for further study.

Keywords: hyperbolic spaces, condition (E), Δ-convergence, S_n iteration.
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1. Introduction

Several mathematical problems arising in nonlinear analysis can be solved by means of fixed point theory, currently an active direction of study. This approach supposes finding suitable classes of operators and adequate study framework, which ensure necessary topological and geometric properties for the existence of fixed points. Once the existence is proved, numerical modeling and simulation are necessary for finding them. So, versatile numerical algorithms are desired to make complete such a study.

During the last period, several remarkable classes of operators for fixed point problems are introduced. We mention here: Suzuki in 2008 [25], Garcia-Falset *et al.* in 2011 [10], Aydi *et al.* in 2012 [5], Kamran *et al.* in 2016 [13], Donghan *et al* in 2018 [9], Bejenaru and Postolache in 2020 [7], Ali *et al* in 2021 and 2017, respectively [3, 4]. As concerns recent designed algorithms for fixed point problems, we mention a few references: Sahu *et al.* [21], Thakur *et al.* [27, 28], Usurelu *et al.* [29, 30], Yao *et al.* [31, 32].

Given the theme of the paper, we will focus on hyperbolic spaces, as generalization of metric spaces, in the sense introduced by Kohlenbach [16]. For other works in this regard, please, see: Goebel and Kirk [11], Reich and Shafrir [20]. The axiomatic system of these spaces shows that the convex structure of Takahashi is fructified herein (see [26], [19]). On the other hand, it should also be noted that hyperbolic spaces represent a natural generalization of CAT(0) spaces, which have the additional assumption of satisfying a condition named (CN) (see [8], [12], [14]), in which various convergence properties are studied, as it happens in [1] or in [6].

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Based on the four properties of hyperbolic spaces, certain results can be studied in this direction and, also, the classic iterations can be stated in a new context, as suggested by Ahmad *et al.* [2], in a recent paper. Inspired by all these aspects, we consider that dealing with problems in this context is an important starting point of development, which deserves to be followed closely, given all the results that can be produced.

2. **Preliminaries**

First, we will present some results regarding hyperbolic spaces.

Definition 2.1 ([16])**.** A hyperbolic space is a metric space (X, d) together with a map $W\colon X^2 \times [0,1] \to X$, with the following properties:

(i) $d\left(u, W(x,y,\alpha)\right) \le (1-\alpha)d(u,x) + \alpha d(u,y)$;

(ii) $d\left(W(x,y,\alpha), W(x,y,\beta)\right) = |\alpha-\beta|\, d(x,y)$;

(iii) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(iv) $d\left(W(x,z,\alpha), W(y,w,\alpha)\right) \le (1-\alpha)d(x,y) + \alpha d(z,w)$, for all $x,y,z,w \in X$ and for all $\alpha, \beta \in [0,1]$.

In this case, we will denote the hyperbolic space with (X, d, W).

It is useful to notice that relation (i) is the well known convex structure introduced by Takahashi [26].

Definition 2.2 ([26], [24])**.** We will consider the hyperbolic space (X, d, W).

(i) A subset S of this hyperbolic space is called convex if

$$W(x,y,\alpha) \in S,$$

for all $x, y \in S$ and for all $\alpha \in [0,1]$.

(ii) This hyperbolic space is said to be strictly convex if for any x, $y \in X$ and $\lambda \in [0,1]$, there exists a unique element $z \in X$ such that

$$d(z,x) = \lambda d(x,y)$$

and

$$d(z,y) = (1-\lambda)d(x,y).$$

(iii) This hyperbolic space is said to be uniformly convex if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ so if $d(x,u) \le r$, $d(y,u) \le r$ and $d(x,y) \ge \varepsilon r$, then

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \le (1-\delta)r.$$

(iv) A map $\eta\colon\ (0,\infty) \times (0,2] \to (0,1]$ with the property that $\delta = \eta(r,\varepsilon)$, for a given $r > 0$ and $\varepsilon \in (0,2]$ is said to be a modulus of uniform convexity. Furthermore, the map η is called monotone if it decreases with respect to r, for a fixed ε.

Remark 2.1 ([17])**.** We can see that a uniformly convex hyperbolic space is strictly convex.

Considering the fact that one of the main results is referred to as Δ-convergence, further we will detail some aspects regarding this concept.

Definition 2.3. We will consider the hyperbolic space (X, d, W) and $\{x_n\}$ a bounded sequence in this space.

For $x \in X$, define a continuous functional

$$r\left(\cdot, \{x_n\}\right) : X \to [0,\infty)\,,\ r\left(x, \{x_n\}\right) = \limsup_{n\to\infty} d(x, x_n).$$

(i) The asymptotic radius is defined as follows

$$r\left(\{x_n\}\right) = \inf\left\{r\left(x, \{x_n\}\right) \,:\, x \in X\right\}.$$

(ii) The asymptotic center of a bounded sequence $\{x_n\}$ from a subset S of X is defined as follows

$$A_S\left(\{x_n\}\right) = \left\{x \in X \,:\, r\left(x, \{x_n\}\right) \leq r\left(y, \{x_n\}\right) \text{for any } y \in S\right\}.$$

In the more general case, where the asymptotic center is considered for the space X, it will be denoted by $A\left(\{x_n\}\right)$.

Lemma 2.1 ([18])**.** *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniformly convexity.*

Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset S of X.

Definition 2.4. We will consider the hyperbolic space (X, d, W). A sequence $\{x_n\}$ in X is said to Δ-converge to an element $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$.

So, we will note $\Delta\text{-}\lim\limits_{n\to\infty} x_n = x$ and x is called the Δ-limit of the sequence $\{x_n\}$.

Lemma 2.2 ([15])**.** *Let (X, d, W) be a uniformly convex hyperbolic space with a monotone modulus of uniformly convexity. We will consider an element $x \in X$ and a sequence $\{\alpha_n\}$ in $[a, b]$ provided that $0 < a \leq b < 1$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq r,$$

$$\limsup_{n\to\infty} d(y_n, x) \leq r$$

and

$$\lim_{n\to\infty} d\left(W(x_n, y_n, \alpha_n), x\right) = r,$$

for some $r \geq 0$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

In 2008, Suzuki [25] introduced a new nonexpansivity condition (the class of operators with property (C)), which was later extended in 2011 by Garcia-Falset *et al.* in [10], by introducing the so-called operator class with condition (E).

Definition 2.5 ([10])**.** Let S be a nonempty subset of a Banach space X and $\mu \geq 1$. Any mapping $T\colon S \to X$ which satisfies the inequality

$$\|x - Ty\| \leq \mu \|x - Tx\| + \|x - y\|,$$

for all $x, y \in S$, is said to be endowed with (E_μ) property. Furthermore, T satisfies condition (E) on S if T satisfies condition (E_μ), for some $\mu \geq 1$.

In the following definition, we will adapt Definition 2.5 in the context of hyperbolic spaces, because the study is based on this class of operators with condition (E).

Definition 2.6. Let (X, d, W) be a complete uniformly convex hyperbolic space. A mapping $T\colon S \to X$ satisfies condition (E_μ) provided that

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y),$$

for all $x, y \in S$, where S is a nonempty subset of a space X.

We will present a lemma with important results regarding this class of operators, which will also be used in the context of uniformly convex hyperbolic spaces.

Lemma 2.3 ([10]). *Let (X, d) be a metric space, S a nonempty subset of X and $T\colon S \to S$ a operator with condition* (E).

(i) *If T satisfies condition* (C), *then T satisfies condition* (E).

(ii) *If T has conditions* (E) *and p is a fixed point of the operator T, then*

$$d(Tx, p) \leq d(x, p),$$

for all $x \in S$.

(iii) *If T satisfies condition* (E), *then its fixed point set is always closed.*

In the following, we will review a few things about the iterative process used in the next section. For the beginning, we mention that in the following we will understand by S a subset of a hyperbolic space and, on the other hand, the operator T satisfies condition (E) and F represents the set of its fixed points.

In 2016, Sintunavarat and Pitea [23] introduced an iterative scheme in connection with Berinde-type operators. For an arbitrary $x_1 \in S$, a sequence $\{x_n\}$ results as output of the following three-step procedure:

$$\begin{cases} y_n = (1-\beta_n)\, x_n + \beta_n T x_n \\ z_n = (1-\gamma_n)\, x_n + \gamma_n y_n \\ x_{n+1} = (1-\alpha_n)\, T z_n + \alpha_n T y_n, \end{cases} \tag{2.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$.

This iteration (2.1) can be extended in the context of hyperbolic spaces as follows. For an arbitrary $x_1 \in S$, a sequence $\{x_n\}$ is generated by the iterative scheme:

$$\begin{cases} y_n = W(Tx_n, x_n, \beta_n) \\ z_n = W(y_n, x_n, \gamma_n) \\ x_{n+1} = W(Ty_n, Tz_n, \alpha_n), \end{cases} \tag{2.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$.

Last but not least, we will recall another recently introduced condition, which will be the basis for the generalization of convergence results. It should be noted, however, that this definition is an adaptation of the classic one, introduced in 1974 by Senter and Dotson [22].

Definition 2.7 ([2]). Let (X, d, W) be a complete uniformly convex hyperbolic space, S a nonempty subset of X and $f\colon [0, \infty) \to [0, \infty)$ a nondecreasing function. A mapping $T\colon S \to S$ satisfies the condition (A) if

(i) $f(a) = 0$ if and only if $a = 0$;

(ii) $f(a) > 0$, for each $a > 0$;

(iii) $d(x, Tx) \geq f\left(d\left(x, F\right)\right)$, for each $x \in S$.

3. Main results

To begin with, we will present several preparatory results, useful in developing our convergence results. In the following, by F we will denote the set of fixed points of the operator T.

Lemma 3.1. *Let* (X, d, W) *be a complete uniformly convex hyperbolic space,* S *a closed nonempty convex subset of* X *and* $T\colon S \to S$ *an operator with the property that* $F \neq \varnothing$*, and* $d(Tx, p) \leq d(x, p)$ *for any* $x \in X$*, and* $p \in F$*. If* $p \in F$ *and the sequence* $\{x_n\}$ *is generated by* (2.2)*, then* $\lim_{n\to\infty} d(x_n, p)$ *exists.*

Proof. Using Lemma 2.3 (ii), we can say that

$$\begin{aligned} d(y_n, p) &= d(W(Tx_n, x_n, \beta_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.1}$$

Based on (3.1), we have

$$\begin{aligned} \rho(z_n, p) &= d(W(y_n, x_n, \gamma_n), p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(y_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned} d(x_{n+1}, p) &= d(W(Ty_n, Tz_n, \alpha_n), p) \\ &\leq (1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.3}$$

This last inequality implies that the sequence $\{d(x_n, p)\}_n$ is bounded and nonincreasing, for any $p \in F$. So, we obtain that $\lim_{n\to\infty} d(x_n, p)$ exists, for any $p \in F$. □

Lemma 3.2. *Let* (X, d, W) *be a complete uniformly convex hyperbolic space,* S *a closed nonempty convex subset of* X *and* $T\colon S \to S$ *an operator with condition* (E) *and with the property that* $F \neq \varnothing$*. If the sequence* $\{x_n\}$ *is generated by* (2.2) *with* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{gamma_n\}$ *bounded away from zero, then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. First of all, take $p \in F$. According to Lemma 3.1, the limit $\lim_{n\to\infty} d(x_n, p)$ exists, for a fixed point p of the operator T. Therefore, we can consider that

$$\lim_{n\to\infty} d(x_n, p) = r. \tag{3.4}$$

Using again Lemma 2.3 (ii), we can use that

$$d(Tx_n, p) \leq d(x_n, p)$$

and taking lim sup in (3.4), we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leq r. \tag{3.5}$$

But from the relation (3.1), we know that

$$d(y_n, p) \leq d(x_n, p)$$

and using a procedure similar to the one above mentioned and the relationship (3.4), we obtain

$$\limsup_{n\to\infty} d(y_n, p) \leq r. \tag{3.6}$$

From relation (3.3) and Lemma 3.1, we obtain

$$\begin{aligned} d(x_{n+1},p) &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(y_n,p) \\ &= (1-\alpha_n)d(W(y_n,x_n,\gamma_n),p)+\alpha_n d(y_n,p) \\ &\leq (1-\alpha_n)(1-\gamma_n)d(x_n,p)+[(1-\alpha_n)\gamma_n+\alpha_n]\, d(y_n,p) \\ &= d(x_n,p)+[1-(1-\gamma_n)(1-\alpha_n)]\,(d(y_n,p)-d(x_n,p))\,, \end{aligned}$$

which proves that

$$\frac{d(x_{n+1},p)-d(x_n,p)}{1-(1-\gamma_n)\,(1-\alpha_n)} \leq d(y_n,p)-d(x_n,p).$$

Given these things, there exists $A>0$ so that

$$\frac{1}{A}(d(x_{n+1},p)-d(x_n,p)) \leq \frac{d(x_{n+1},p)-d(x_n,p)}{1-(1-\gamma_n)\,(1-\alpha_n)} \leq d(y_n,p)-d(x_n,p).$$

Using this last inequality and (3.4), we get

$$r \leq \liminf_{n\to\infty} d(y_n,p). \tag{3.7}$$

Obviously, from (3.6) and (3.7), we have

$$\lim_{n\to\infty} d(y_n,p)=r,$$

and from this it follows that

$$\limsup_{n\to\infty} d(W(Tx_n,x_n,\beta_n),p)=r. \tag{3.8}$$

Finally, from (3.4), (3.5), (3.8) and using Lemma 2.2, we deduce that

$$\lim_{n\to\infty} d(x_n,Tx_n)=0,$$

which concludes the proof. □

Considering the previous two results, we will now be able to give a Δ-convergence result.

Theorem 3.1. *Let (X,d,W) be a complete uniformly convex hyperbolic space, S a closed nonempty convex subset of X and $T\colon S\to S$ an operator with condition* (E) *and with the property that $F\neq\varnothing$. Then, the sequence $\{x_n\}$ generated by* (2.2) *with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ bounded away from zero, Δ-converges to a point $p\in F$.*

Proof. Considering the fact that the sequence $\{x_n\}$ is bounded, it has essentially a unique asymptotic center

$$A_S(\{x_n\})=\{x_0\}\,.$$

On the other hand, we will consider a subsequence $\{u_n\}$ of $\{x_n\}$ such that

$$A_S(\{u_n\})=\{u_0\}\,.$$

Using Lemma 3.2, we get

$$\lim_{n\to\infty} d(u_n,Tu_n)=0. \tag{3.9}$$

Next, we will prove that $u_0\in F$.

Using the fact that the operator T satisfies the condition (E), we have

$$d(u_n,Tu_0)\leq \mu d(u_n,Tu_n)+d(u_n,u_0),$$

for some $\mu\geq 1$.

From here and using (3.9), follows that

$$\begin{aligned} r\left(Tu_0, \{u_n\}\right) &= \limsup_{n\to\infty} d(Tu_0, u_n) \\ &\le \limsup_{n\to\infty} d(u_0, u_n) \\ &= r\left(u_0, \{u_n\}\right). \end{aligned}$$

Given the uniqueness of the asymptotic center, we can say that $Tu_0 = u_0$ so $u_0 \in F$. All that remains now is to prove that u_0 is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$ and therefore, we will assume by reductio ad absurdum that there is another asymptotic center, which we will note v_0.

From Lemma 3.1, we deduce that $\lim_{n\to\infty} d(x_n, u_0)$ exists. But, on the other hand, it can be seen that

$$\begin{aligned} \limsup_{n\to\infty} d(u_0, u_n) &= \limsup_{n\to\infty} d(v_0, u_n) \\ &\le \limsup_{n\to\infty} d(v_0, x_n) \\ &\le \limsup_{n\to\infty} d(u_0, x_n) \\ &= \limsup_{n\to\infty} d(u_0, u_n), \end{aligned}$$

which is obviously a contradiction.

So, $u_0 \in F$ is the unique asymptotic center for each subsequence $\{u_n\}$ of $\{x_n\}$ and finally, we conclude that $\{x_n\}$ Δ - converges in the set F. □

In the following, we will detail three results regarding the strong convergence.

Theorem 3.2. *Let (X, d, W) be a complete uniformly convex hyperbolic space, S a compact nonempty convex subset of X and $T\colon S \to S$ an operator with condition* (E) *and with the property that $F \neq \varnothing$. Then, the sequence $\{x_n\}$ generated by* (2.2) *with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ bounded away from zero, converges strongly to a point $p \in F$.*

Proof. In the first place, we will consider an element $x \in S$. Because S is a compact set, we can say that there exists a subsequence, which we will note $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\lim_{k\to\infty} d\left(\{x_{n_k}\}, x\right) = 0.$$

Using now the fact that the operator T is part of the class of mappings with the condition (E), we have that

$$d\left(x_{n_k}, Tx\right) \le \mu d\left(x_{n_k}, Tx_{n_k}\right) + d\left(x_{n_k}, x\right), \tag{3.10}$$

for some $\mu \geq 1$.

From Lemma 3.2, we get

$$\lim_{k\to\infty} d\left(\{x_{n_k}\}, T\{x_{n_k}\}\right) = 0.$$

Taking the limit when k goes to ∞ in relation (3.10), it is immediately inferred that

$$\lim_{k\to\infty} d\left(\{x_{n_k}\}, Tx\right) = 0.$$

Given the uniqueness of the limit for the convergent sequences, we have $Tx = x$, so $x \in F$ and using Lemma 3.1, one has that $\lim_{n\to\infty} d(x_n, x)$ exists, hence the conclusion is obtained. □

Theorem 3.3. *Let (X, d, W) be a complete uniformly convex hyperbolic space, S a closed nonempty convex subset of X and $T\colon S \to S$ an operator with condition* (E) *and with the property that $F \neq \varnothing$. Then, the sequence $\{x_n\}$ generated by* (2.2) *converges strongly to a point $p \in F$ if and only if*

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Proof. We will first assume that the sequence $\{x_n\}$ converges strongly to a point $p \in F$.

From Lemma 3.1, we have

$$\lim_{n\to\infty} d(x_n, p) = 0,$$

so

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

The inverse implication uses the Lemma 2.3 (iii), which guarantees that the set F is closed in S and uses arguments quite similar to Theorem 2 from [22]. □

At the end of this section, using the property (A) of the operators, we will provide another strong convergence result.

Theorem 3.4. *Let (X, d, W) be a complete uniformly convex hyperbolic space, S a closed nonempty convex subset of X and $T\colon S \to S$ an operator with condition* (E) *and with the property that $F \neq \varnothing$. If, in addition, the operator T also satisfies condition* (A)*, then the sequence $\{x_n\}$ generated by* (2.2) *converges strongly to a point of F.*

Proof. First of all, using Lemma 3.2, we get

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.11}$$

As T satisfies condition (A), we can write

$$d(x_n, Tx_n) \geq f\left(d(x_n, F)\right). \tag{3.12}$$

From (3.11) and (3.12), we deduce

$$\liminf_{n\to\infty} f\left(d(x_n, F)\right) = 0$$

and from here it is immediately obtained that

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Finally, using Theorem 3.3, we can conclude that the sequence $\{x_n\}$ admits a strong limit in the set F, which is what we wanted to prove. □

4. **Conclusions**

The objective of the paper is to introduce certain convergence results for an iterative process in three steps, in the context of hyperbolic spaces. These results indicate two aspects: on the one hand, the fact that the chosen iterative process is extremely versatile, can be adapted in various situations. On the other hand, the idea that important results can be obtained, regarding strong convergence, without too many additional conditions, fact which represents an important point of study in the field of nonlinear analysis and fixed point theory. Furthermore, it should be noted that this study is only a first step in the framework given by these spaces, and can be continued in the context of the same iteration or other iterative processes, to obtain common fixed point results. Finally, it is worth to be noted that this can be done using other classes of operators, newly introduced.

REFERENCES

[1] M. Abbas, S. H. Khan, M. Postolache, *Existence and approximation results for SKC mappings in CAT(0) spaces*, J. Inequal. Appl., 2014, Art. No. 212.

[2] J. Ahmad, K. Ullah, M. Arshad, *Convergence, weak w^2 stability, and data dependence results for the F iterative scheme in hyperbolic spaces*, Numer. Algorithms, 2022, https://doi.org/10.1007/s11075-022-01321-y.

[3] M. U. Ali, A. Pitea, *Existence theorem for integral inclusions by a fixed point theorem for multivalued implicit-type contractive mappings*, Nonlinear Anal. Modelling Control, 2021, 26(2), 334-348.

[4] M. U. Ali, T. Kamran, M. Postolache, *Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem*, Nonlinear Anal. Modelling Control, 2017, 22(1), 17-30.

[5] H. Aydi, M. Postolache, W. Shatanawi, *Coupled fixed point results for (psi,phi)-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl., 2012, 63(1), 298-309.

[6] A. Bejenaru, C. Ciobanescu, *Common fixed points of operators with property (E) in CAT(0) spaces*, Mathematics, 2022, 10, 433.

[7] A. Bejenaru, M. Postolache, *A unifying approach for some nonexpansiveness conditions on modular vector spaces*, Nonlinear Anal. Modelling Control, 2020, 25, 827-845.

[8] S. Dhompongsa, B. Panyanak, *On Δ-convergence theorems in CAT(0) space*, Comput. Math. Appl., 2008, 56, 2572-2579.

[9] C. Donghan, S. Jie, L. Weiyi, *Fixed point theorems and convergence theorems for a new generalized nonexpansive mapping*, Numer. Funct. Anal. Optim., 2018, 39, 1742-1754.

[10] J. Garcia Falset, E. Llorens Fuster, T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl., 2011, 375, 185-195.

[11] K. Goebel, W. A. Kirk, *Iteration processes for nonexpansive mappings*, In: S. P. Singh, S. Thomeier, B. Watson (eds.), Topological Methods in Nonlinear Functional Analysis, Contemp. Math., American Mathematical Society, 1983, 21, 115-123.

[12] M. Gromov, *Metric Structure for Riemannian and Non-Riemannian Spaces*; Birkhauser, Boston, MA, USA, 2007.

[13] T. Kamran, M. Postolache, M. U. Ali, Q. Kiran, *Feng and Liu type F-contraction in b-metric spaces with application to integral equations*, J. Math. Anal., 2016, 7(5), 18-27.

[14] W. A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., 2008, 68, 3689-3696.

[15] A. R. Khan, H. Fukhar-ud-din, M. A. A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., 2012, 54, 1-12.

[16] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Am. Math. Soc., 2005, 357, 89-128.

[17] L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0)-spaces*, J. Math. Anal. Appl., 2007, 325, 386-399.

[18] L. Leustean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, In: A. Leizarowitz, B. S. Mordukhovich, I. Shafrir, A. Zaslavski (eds.), Nonlinear Analysis and Optimization I: Nonlinear Analysis, Contemp. Math., American Mathematical Society, 2010, 513, 193-209.

[19] M. O. Olatinwo, M. Postolache *Stability results for Jungck-type iterative processes in conex metric spaces*, Appl. Math. Comput., 2012, 218, 6727-6732.

[20] S. Reich, I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal., 1990, 15, 537-558.

[21] D. R. Sahu, A. Pitea, M. Verma, *A new iteration technique for nonlinear ope-rators as concerns convex programming and feasibility problems*, Numer. Algorithms 2020, 83(2), 421-449.

[22] H. F. Senter, W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., 1974, 44, 375-380.

[23] W. Sintunavarat, A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Sci. Appl., 2016, 9, 2553-2562.

[24] T. Shimizu, W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal., 1996, 8, 197-203.

[25] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., 2008, 340 (2), 1088-1095.

[26] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep., 1970, 22, 142-149.

[27] B. S. Thakur, M. Postolache, *Existence and approximation of solutions for generalized extended nonlinear variational inequalities*, J. Inequal. Appl., 2013, Art. No. 590.

[28] B. S. Thakur, D. Thakur, M. Postolache, *A new iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat 2016, 30(10), 2711-2720.

[29] G. I. Usurelu, M. Postolache, *Algorithm for generalized hybrid operators with numerical analysis and applications*, J. Nonlinear Var. Anal., 2022, 6(3), 255-277.

[30] G. I. Usurelu, T. Turcanu, M. Postolache, *Algorithm for two generalized nonexpansive mappings in uniformly convex spaces*, Mathematics 2022, 10(3), Art. No. 318.

[31] Y. Yao, M. Postolache, S. M, Kang, *Strong convergence of approximated iterations for asymptotically pseudocontractive mappings*, Fixed Point Theory Appl., 2014, Art. No. 100.

[32] Y. Yao, R. P. Agarwal, M. Postolache, Y.-C. Liu, *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem* Fixed Point Theory Appl., 2014, Art. No. 183.