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STABILITY ANALYSIS OF LINEAR DISTRIBUTED ORDER SYSTEM WITH MULTIPLE TIME DELAYS

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In this paper, we study the stability of n-dimensional linear distributed order differential system with timedelays by respect to the nonnegative density function, where the delay matrix is defined in $(\mathbb{R}^{+})^{n\times n}$. We produce necessary and sufficient conditions for asymptotic stability of equations of this type. As an application, one example of distributed order Lotka-Volterra predator-prey system is given to demonstrate our main result.

Keywords: asymptotic stability; distributed order; fractional; delay.

1. Introduction

The history of fractional calculus is more than three centuries old, yet it only receives much attention and interest in the past 20 years; the reader may refer to [1, 2] for the theory and applications of fractional calculus. The generalization of dynamical equations using fractional derivatives proved to be useful and more accurate in mathematical modeling related to many interdisciplinary areas. Applications of fractional calculus and fractional-order differential equations include: dielectric relaxation phenomena in polymeric materials [3], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [4], transport dynamics in systems governed by anomalous diffusion [5, 6], long-time memory in financial time series [7] and so on [8, 9]. Stability analysis and control systems are one of the most important problems that in 1996, Matignon [10] studied stability of n-dimensional linear fractional systems from a point of view of control. However these issues for systems time-delay have been studied in recent years. Delays are encountered in many phenomena, such as pneumatic, hydraulic networks, chemical processes, long transmission lines [11]. Recently, time delays and multiple time delays are introduced to complex dynamical networks, e.g., see [12, 13]. More novelty, Chen and Moore [14] studied stability of 1-dimensional fractional systems with retard time and Deng et al [15] introduce multiple time delays to the fractional differential equations. The idea of fractional derivative of distributed order is stated by Caputo [16] and later

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developed by Caputo himself [17, 18], Bagley and Torvik [19, 20]. Other researchers used this idea, and interesting reviews appeared to describe the related mathematical models of partial fractional differential equation of distributed order. For example, Diethelm and Ford [21] used a numerical technique along with its error analysis to solve the distributed order differential equation and analyze the physical phenomena and engineering problems, see [21] and references therein. Recently H. Saberi Najafi et al [22, 23] studied stability analysis of distributed order differential equations with respect to the nonnegative density function. Furthermore, H. Aminikhah et al [24] investigated sufficient and necessary conditions of stability of nonlinear distributed order fractional system. Now we consider the stability of n-dimensional linear distributed order differential system with time delays by respect to the nonnegative density function.

This paper is organized as follows. In Section 2, we recall some basic definitions of the Caputo fractional derivative operator, systems with fractional derivatives of distributed order. Section 3 contains the main definitions and theorems for checking the stability analysis of linear distributed order system with multiple time delays. Finally, in section 4 we present an example of distributed order Lotka-Volterra predator-prey system to illustrate our main result.

2. Elementary Definitions

In this Section, we consider the main definitions and properties of fractional derivative operators of single and distribute order.

2.1. Fractional Derivative

There are several definitions of a fractional derivative of order $\alpha > 0$ [1, 2], such as Grunwald-Letnikov's definition, Riemann-Liouville's definition, Caputo's fractional derivative. The former two definitions are often used by pure mathematicians, while the last one is adopted by applied scientists, since it is more convenient in engineering applications. The Caputo fractional derivative of $f(t)$ is defined as:

$$ {}_{so}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \quad m-1 \le \alpha \le m,\ m \in \mathbb{N},\ t > 0. \tag{1}$$

Fortunately, the Laplace transform of the Caputo fractional derivative satisfies:

$$\mathcal{L}\left\{ {}_{so}^{C}D_t^{\alpha} f(t) \right\} = s^{\alpha} \ell\{f(t)\} - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \tag{2}$$

where $m-1 < \alpha \le m$ and s is the Laplace variable. The Laplace transform of Caputo fractional derivative requires the knowledge of the initial values of the function and its integer derivatives of order $k = 1, 2, \ldots, m-1$. When $\alpha \in (0,1]$ is given by:

$$\mathcal{L}\left\{ {}_{so}^{C}D_t^{\alpha} f(t) \right\} = s^{\alpha} \ell\left\{ f(t) \right\} - f(0^+)\, s^{\alpha-1}. \tag{3}$$

2.2. Systems with fractional derivatives of distributed order fractional

Fractional derivative operator of distributed order a generalization of the single order ${}_{so}D^{\alpha} = d^{\alpha}/dt^{\alpha}$ with respect to nonnegative density function of $b(\alpha)$ is defined as:

$$ {}^{C}D_t^{b(\alpha)} f(t) = \int_{m-1}^{m} b(\alpha)\, {}_{so}^{C}D_t^{\alpha} f(t)\, d\alpha, \qquad m-1 < \alpha \le m,\ m \in \mathbb{N}. \tag{4}$$

The idea of distributed order is stated by Caputo [16, 17]. Further the Laplace transform of the Caputo distributed order satisfies:

$$\begin{aligned}\mathcal{L}\left\{ {}^{C}D_t^{b(\alpha)} f(t) \right\} &= \int_{m-1}^{m} b(\alpha) \left[s^{\alpha} F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k} \right] d\alpha \\ &= B(s)F(s) - \sum_{k=0}^{m-1} \frac{1}{s^{k+1}} B(s) f^{(k)}(0^+),\end{aligned} \tag{5}$$

where $F(s)$ is the Laplace transform of $f(t)$ and $B(s) = \int_{m-1}^{m} b(\alpha)\, s^{\alpha}\, d\alpha.$ (6)

3. Stability analysis of linear distributed order fractional differential system with multiple time delays

In 2012, H. Saberi Najafi et al [22] studied stability analysis of distributed order fractional differential equations, which is expressed as follows:

$$ {}^{C}D_t^{b(\alpha)} x(t) = Ax(t), \qquad x(0) = x_0,\quad A \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^{n},\ 0 < \alpha \le 1, \tag{7}$$

that we easily generalize this result for linear fractional differential equations with time delay [14]:

$$ {}_{so}^{C}D_t^{\alpha} x(t) = Ax(t-\tau), \tag{8}$$

where α is real and lies in (0,1], $A \in \mathbb{R}^{n\times n}$ and $\tau > 0$ represents the time delay. The initial condition associated to this equation is $x(t) = \varphi(t),\ \forall t \in [-\tau, 0]$ where $\varphi(t) \in \mathbb{C}^0[-\tau, 0]$.

Now, we generalize the main stability properties for the linear distributed order fractional differential system with multiple time delay in the following form:

$$\begin{cases} {}^{C}D_t^{b(\alpha_1)} x_1(t) = a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1n}x_n(t-\tau_{1n}), \\ {}^{C}D_t^{b(\alpha_2)} x_2(t) = a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2n}x_n(t-\tau_{2n}), \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ {}^{C}D_t^{b(\alpha_n)} x_n(t) = a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \cdots + a_{nn}x_n(t-\tau_{nn}), \end{cases} \tag{9}$$

where $b_i(\alpha)$ denotes the nonnegative density function of order $\alpha \in (0,1]$, the initial values $x_i(t)=\varphi_i(t)$, are given for $-\max_{i,j} \tau_{ij} = -\tau_{\max} \le t \le 0$ and $i, j = 1,\dots,n$. In this system, time delay matrix $T = (\tau_{ij})_{n\times n} \in (\mathbb{R}^+)^{n\times n}$, coefficient matrix $A = (a_{ij})_{n\times n}$, state variables $x_i(t), x_i(t-\tau_{ij}) \in \mathbb{R}$ and initial values $\varphi_i(t) \in \mathbb{C}^0\left[-\tau_{\max}, 0\right]$.

We study the stability of system (9) by applying the Laplace transforms on both sides of this system, we have

$$\begin{aligned} B_i(s)X_i(s) - \frac{1}{s}B_i(s)\varphi_i(0) &= \sum_{j=1}^{n} a_{ij}e^{-s\tau_{ij}} X_i(s) \\ &+ \sum_{j=1}^{n} a_{ij}e^{-s\tau_{ij}} X_i(s)\left(\int_{-\tau_{ij}}^{0} e^{-st}\varphi_j(t)dt\right), \end{aligned} \tag{10}$$

for $i = 1,\dots,n$. Where $X_i(s)$ is the Laplace transform of $x_i(t)$.
We can rewrite (10) as follows:

$$\Delta(s).\begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix} = \begin{pmatrix} h_1(s) \\ h_2(s) \\ \vdots \\ h_n(s) \end{pmatrix}, \tag{11}$$

in which

$$h_i(s) = \sum_{j=1}^{n} a_{ij}e^{-s\tau_{ij}}\left(\int_{-\tau_{ij}}^{0} e^{-st}\varphi_j(t)dt\right) + \frac{1}{s}B_i(s)\varphi_i(0), \qquad i = 1,2,\dots,n \tag{12}$$

$$\Delta(s) = \begin{pmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \dots & \Delta_{1n}(s) \\ \Delta_{21}(s) & \Delta_{22}(s) & \dots & \Delta_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1}(s) & \Delta_{n2}(s) & \cdots & \Delta_{nn}(s) \end{pmatrix}, \tag{13}$$

where
$$\Delta_{ij}(s) = \begin{cases} B_i(s) - a_{ii}e^{-s\tau_{ii}} & if\ i = j, \\ -a_{ij}e^{-s\tau_{ij}} & otherwise. \end{cases} \tag{14}$$

For simplicity, we call $\Delta(s)$ a characteristic matrix of (9) with respect to the distributed function $B(s) = (B_1(s), B_2(s),\dots,B_n(s))^T$ where $B_i(s) = \int_0^1 b_i(\alpha)s^{\alpha}d\alpha$. Moreover $\det(\Delta(s)) = 0$ is the characteristic equation of system (9), with respect to the distributed function $B(s)$.

Remark 3.1. If a linear distributed order fractional differential system has a non-zero equilibrium, we can move this equilibrium to the origin by the translation transform. Throughout the paper, we always suppose that system of (9) has a zero solution and all complex computations are done in the branch of the principle value of argument.

Now, we express the main theorem for checking the stability of system (9), but, we first recall the following theorem.

Theorem 3.2. (*Final Value Theorem* [25]) Let $F(s)$ be the Laplace transform of the function $f(t)$. If all poles of $sF(s)$ are in the open left-half plane, then,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{15}$$

Theorem 3.3. The zero solution system of (9) is asymptotically stable if and only if all roots of $\det(\Delta(s)) = 0$ have negative real parts.

proof. Multiplying s on both sides of (11) gives, we have

$$\Delta(s).\begin{pmatrix} sX_1(s) \\ sX_2(s) \\ \vdots \\ sX_n(s) \end{pmatrix} = \begin{pmatrix} sh_1(s) \\ sh_2(s) \\ \vdots \\ sh_n(s) \end{pmatrix}, \tag{16}$$

if all roots of the $\det(\Delta(s)) = 0$ lie in open left half complex plane (i.e, $\Re(s) < 0$), then, we consider (16) in $\Re(s) \geq 0$. In this restricted area, the relation (16) has a unique solution $sX(s) = (sX_1(s), sX_2(s), \dots, sX_n(s))$. Since $\lim_{s\to 0} B_i(s) = 0$, for $i = 1,\dots,n$ so we have

$$\lim_{s\to 0, \Re(s)\geq 0} sX_i(s) = 0, \qquad i = 1,2,\dots,n, \tag{17}$$

which from the final value Theorem 3.2, we get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(x_1(t), x_2(t), \dots, x_n(t)) = \lim_{s\to 0}(sX_1(s), sX_2(s), \dots, sX_n(s)) = 0. \tag{18}$$

The above result shows that the system (9) is asymptotically stable. ■

Definition 3.4. The eigenvalues of A with respect to the distributed function $B(s)$ are the roots of the characteristic equation of system (9) where $B(s) = (B_1(s), B_2(s), \dots, B_n(s))^T$ is the distributed function with respect to the density function $b(\alpha) = (b_1(\alpha), b_2(\alpha), \dots, b_n(\alpha))^T$.

The inertia of a matrix is the triplet of the numbers of eigenvalues of A with positive, negative, and zero real parts. Now, we generalize the inertia concept for analyzing the stability of linear distributed order fractional system with multiple time delays.

Definition 3.5. The inertia of the system (9) is the triple

$$I_{n_{b(\alpha)}}(A - T) = (\pi_{n_{b(\alpha)}}(A - T), \nu_{n_{b(\alpha)}}(A - T), \delta_{n_{b(\alpha)}}(A - T)), \tag{19}$$

where $\pi_{n_{b(\alpha)}}(A-T)$, $\nu_{n_{b(\alpha)}}(A-T)$ and $\delta_{n_{b(\alpha)}}(A-T)$ are, respectively, the number of roots of $\det(\Delta(s))=0$ with positive, negative, and zero real parts.

Definition 3.6. The matrix A is called a stable matrix with respect to the distributed function $B(s)$, if all of the eigenvalue of A with respect to the distributed function $B(s)$ have negative real parts.

Theorem 3.7. The linear distributed order fractional differential system with time delays by respect to the nonnegative density function (9) is asymptotically stable if and only if any of the following equivalent conditions holds.

(1) The matrix A is stable with respect to the distribute function $B(s)$.

(2) $\pi_{n_{b(\alpha)}}(A-T)=\delta_{n_{b(\alpha)}}(A-T)=0.$

(3) All roots s of the characteristic equation of system (9) satisfy $|\arg(s)|>\pi/2$.

***proof*.** According to Theorem 3.3 and the above definitions, proof can be easily obtained. ■

Based on the theorem above, we can obtain the following remarks:

Remark 3.8. If $\tau_{ij}=0$ and $b_i(\alpha)=b(\alpha)$ for $i,j=1,\dots,n$, then the characteristic matrix and characteristic equation of (9) are reduced to $B(s)I-A$ and $\det(B(s)I-A)=0$, respectively. Moreover, $I_{n_{b(\alpha)}}(A-T)=I_{n_{b(\alpha)}}(A)$, which agrees with the results and definitions for distributed order differential equations [22]. Also, the result obtained in Theorem 3.6 of [22] are special case of Theorem 3.7 of the present paper.

Remark 3.9. If $b_i(\alpha)=\delta(\alpha-q_i)$ where $0<q_i\le 1$ for $i=1,\dots,n$ and $\delta(\alpha)$ is the Dirac delta function, then we have the following stability analysis of linear fractional differential system with multiple time delays [13]:

$$\begin{cases} {}_{so}^{C}D_t^{q_1}x_1(t)=a_{11}x_1(t-\tau_{11})+a_{12}x_2(t-\tau_{12})+\cdots+a_{1n}x_n(t-\tau_{1n}),\\ {}_{so}^{C}D_t^{q_2}x_2(t)=a_{21}x_1(t-\tau_{21})+a_{22}x_2(t-\tau_{22})+\cdots+a_{2n}x_n(t-\tau_{2n}),\\ \quad\vdots \qquad\qquad\qquad\qquad \vdots\\ {}_{so}^{C}D_t^{q_n}x_n(t)=a_{n1}x_1(t-\tau_{n1})+a_{n2}x_2(t-\tau_{n2})+\cdots+a_{nn}x_n(t-\tau_{nn}). \end{cases} \tag{20}$$

Also, the characteristic matrix and characteristic equation of (9) are reduced to

$$\Delta(s)=\begin{pmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \dots & \Delta_{1n}(s)\\ \Delta_{21}(s) & \Delta_{22}(s) & \dots & \Delta_{2n}(s)\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_{n1}(s) & \Delta_{n2}(s) & \cdots & \Delta_{nn}(s) \end{pmatrix}, \tag{21}$$

where

$$\Delta_{ij}(s)=\begin{cases} s^{q_i}-a_{ii}e^{-s\tau_{ii}} & if i=j, \\ -a_{ij}e^{-s\tau_{ij}} & otherwise. \end{cases} \tag{22}$$

and $\det(\Delta(s))=0$, respectively. Furthermore, if $\tau_{ij}=\tau>0$ for $i,j=1,\ldots,n$ and $q_1=q_2=\cdots=q_n=1$, then the characteristic matrix and characteristic equation of (20) are reduced to $sI-Ae^{-s\tau}$ and $\det(sI-Ae^{-s\tau})=0$, respectively. They coincide with the usual definitions of the characteristic matrix and characteristic equation of delayed equations [12]. Especially, if $\tau=0$, then the characteristic matrix and characteristic equation of (20) are respectively, reduced to $sI-A$ and $\det(sI-A)=0$, which agree with the typical definitions for typical differential equations.

4. Example

In this Section, we give example to confirm our results. The integer-order Lotka-Volterra predator-prey system with time delay can be modeled as follows:

$$\begin{cases} \dot{x}_1(t)=x_1(t)[r_1-a_{11}x_1(t-\tau)-a_{12}x_2(t-\tau)], \\ \dot{x}_2(t)=x_2(t)[-r_2+a_{21}x_1(t-\tau)-a_{22}x_2(t-\tau)], \end{cases} \tag{23}$$

where $x_1(t)$ and $x_2(t)$ can be interpreted as the population densities of prey and predator at time t, respectively, $\tau>0$ is the feedback time delay of the prey to the growth of the species itself, $r_1>0$ denotes the intrinsic growth rate of the prey and $r_2>0$ denotes the death rate of the predator; the parameters a_{ij} $(i,j=1,2)$ are all positive constants. In 2008, Yan and Zhang [26] investigated the stability and Hopf bifurcation of system (23). The corresponding fractional-order Lotka-Volterra predator-prey system with time delay can be written in the form as below:

$$\begin{cases} {}_{so}^{C}D_t^{\alpha_1}x_1(t)=x_1(t)[r_1-a_{11}x_1(t-\tau)-a_{12}x_2(t-\tau)], \\ {}_{so}^{C}D_t^{\alpha_2}x_2(t)=x_2(t)[-r_2+a_{21}x_1(t-\tau)-a_{22}x_2(t-\tau)], \end{cases} \tag{24}$$

where α_i are real and lies in $(0,1]$.

Now, we consider distributed order fractional Lotka-Volterra predator-prey system with time delays by respect to the nonnegative density function given by:

$$\begin{cases} {}^{C}D_t^{b(\alpha_1)} x_1(t) = x_1(t)[r_1 - a_{11} x_1(t-\tau) - a_{12} x_2(t-\tau)], \\ \\ {}^{C}D_t^{b(\alpha_2)} x_2(t) = x_2(t)[-r_2 + a_{21} x_1(t-\tau) - a_{22} x_2(t-\tau)], \end{cases} \tag{25}$$

with initial values $x_1(t) = \varphi_1(t)$ and $x_2(t) = \varphi_2(t)$ are given for $t \in [-\tau, 0]$. If

$$r_1 a_{21} - r_2 a_{11} > 0, \tag{26}$$

then system (25) has a positive equilibrium points $E^* = (x_1^*, x_2^*)$, where

$$x^*_1 = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \qquad x^*_2 = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \tag{27}$$

Let $X_1 = x_1 - x_1^*$, $X_2 = x_2 - x_2^*$. We then obtain (25) as follows:

$$\begin{cases} {}^{C}D_t^{b(\alpha_1)} X_1(t) = (X_1(t) + x_1^*)[-a_{11} X_1(t-\tau) - a_{12} X_2(t-\tau)], \\ \\ {}^{C}D_t^{b(\alpha_2)} X_2(t) = (X_2(t) + x_2^*)[a_{21} X_1(t-\tau) - a_{22} X_2(t-\tau)]. \end{cases} \tag{28}$$

The linearization of system (28) at $(0,0)$ is linear system:

$$\begin{cases} {}^{C}D_t^{b(\alpha_1)} X_1(t) = k_1 X_1(t-\tau) - k_2 X_2(t-\tau), \\ \\ {}^{C}D_t^{b(\alpha_2)} X_2(t) = k_3 X_1(t-\tau) - k_4 X_2(t-\tau), \end{cases} \tag{29}$$

where

$$k_1 = -a_{11} x_1^*, \qquad k_2 = -a_{12} x_1^*, \qquad k_3 = a_{21} x_2^*, \qquad k_4 = -a_{22} x_2^*. \tag{30}$$

Clearly, the characteristic matrix this system is

$$\Delta(s) = \begin{pmatrix} B_1(s) - k_1 e^{-\tau s} & k_2 e^{-\tau s} \\ k_3 e^{-\tau s} & B_2(s) - k_4 e^{-\tau s} \end{pmatrix}, \tag{31}$$

and the characteristic equation is

$$B_1(s) B_2(s) - k_4 B_1(s) e^{-\tau s} - k_1 B_2(s) e^{-\tau s} + k_1 k_4 e^{-2\tau s} - k_2 k_3 e^{-2\tau s} = 0. \tag{32}$$

Now, we consider the following special case of system (28)

$$\begin{cases} {}^{C}D_t^{b(\alpha_1)} x_1(t) = x_1(t)[1 - x_1(t-\tau) - x_2(t-\tau)], \\ \\ {}^{C}D_t^{b(\alpha_2)} x_2(t) = x_2(t)[-1 + 2 x_1(t-\tau) - x_2(t-\tau)], \end{cases} \tag{33}$$

with initial values $x_1(t) = 0.2$ and $x_2(t) = 0.3$ which has a positive equilibrium points $E^* = (2/3, 1/3)$. For analyzing system (33), we compute $I_{n_{b(\alpha)}}(A - T)$ in the case that various density function. The results are shown in Table **1**.

Table 1

Stability analysis of system (33) for various density function

$b_i(\alpha)=\delta(\alpha-q_i)\quad i=1,2$			$b_i(\alpha)=\delta(\alpha-q_{1i})+\delta(\alpha-q_{2i})\quad i=1,2$		
τ	$q=(q_1,q_2)$	$I_{n_{b(\alpha)}}(A-T)$	τ	$\begin{cases}(q_{11},q_{12})\\(q_{21},q_{22})\end{cases}$	$I_{n_{b(\alpha)}}(A-T)$
1.98	$(4/5,4/5)$	(4,4,0)	0.5	$\begin{cases}(1/3,2/5)\\(1/3,1/3)\end{cases}$	(1,0,0)
0.65	(1,1)	(0,2,0)	0.75	$\begin{cases}(0.75,0.4)\\(0.65,0.4)\end{cases}$	(1,0,0)
1.5	(0.95,0.95)	(1,0,0)	1.75	$\begin{cases}(0.95,0.85)\\(0.95,0.85)\end{cases}$	(0,2,0)

Since system (33) needs to be solved numerically for the reconciling our results are given in Table **1**, a suitable numerical method needs to be selected. H. Rezazadeh et al has presented a Grunwald-Letnikovs method for solving delay differential equations of fractional order [24]. Fig. **1** indicates that system (33) with parameters: $q_1=q_2=4/5$, when $\tau=1.98$ is unstable. Fig. **2** shows system (33) has a unique positive equilibrium $E^*=(2/3,1/3)$ that this equilibrium is asymptotically stable when $q_1=q_2=1$, $\tau=0.65$. Figure **3** demonstrates that system (33) with parameters: $q_1=0.95, q_2=0.85$, when $\tau=1.5$ is unstable.

Figure **4, 5** indicate that system (33) with the assumptions mentioned Table **1** is unstable when $\tau=0.5$, $\tau=0.75$. Figure **6** shows system (33) has a unique positive equilibrium $E^*=(2/3,1/3)$ that this equilibrium is asymptotically stable when $q_{11}=0.95, q_{12}=0.85, q_{21}=0.95, q_{22}=0.85$ and $\tau=1.75$.

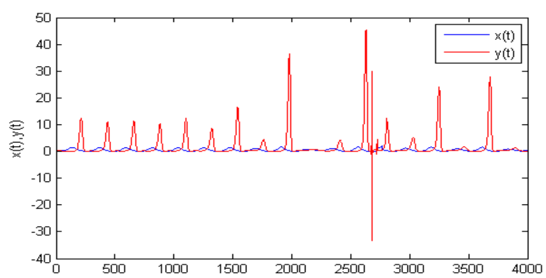

Fig. 1. The numerical approximations of system (33) when $\tau=1.98$ and $(q_1,q_2)=(4/5,4/5)$.

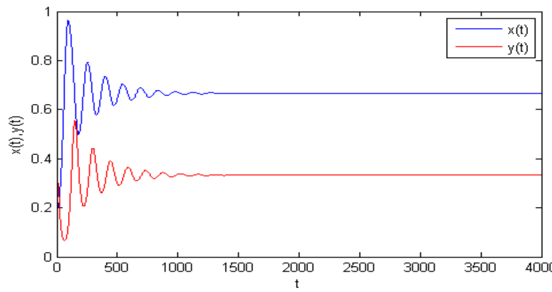

Fig. 2. The numerical approximations of system (33) when $\tau = 0.65$ and $(q_1, q_2) = (1, 1)$.

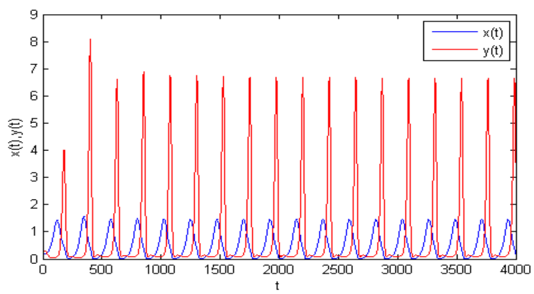

Fig. 3. The numerical approximations of system (33) when $\tau = 1.5$ and $(q_1, q_2) = (0.95, 0.85)$.

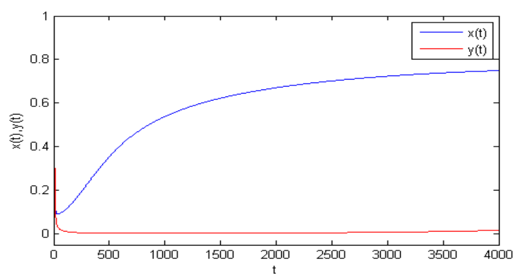

Fig. 4. The numerical approximations of system (33) when $\tau = 0.5$ and $(q_{11}, q_{12}, q_{21}, q_{22}) = (1/3, 2/5, 1/3, 1/3)$.

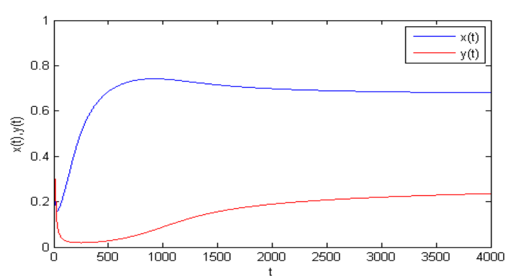

Fig. 5. The numerical approximations of system (33) when $\tau = 0.75$ and $(q_{11}, q_{12}, q_{21}, q_{22}) = (0.75, 0.4, 0.65, 0.4)$.

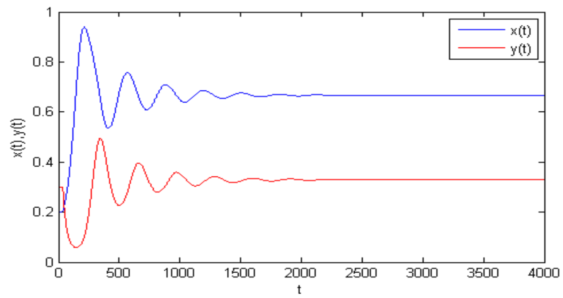

Fig. 6. The numerical approximations of system (33) when $\tau = 1.75$ and $(q_{11}, q_{12}, q_{21}, q_{22}) =$ $(0.95, 0.85, 0.95, 0.85)$.

5. Conclusions

In this paper, we introduced the distributed order fractional system with multiple time delays. Then the asymptotical stability for such systems has been investigated. We generalize the inertia and characteristics polynomial concepts of such a system with respect to the nonnegative density function. Numerical simulations were coincident with results of Table **1** described in the previous Section. Although this paper just focuses on the systems with $\alpha \in (0,1]$, the higher order systems can be discussed based on the analysis of this paper. This will be the investigation goal of future works. All numerical results are obtained using Matlab 7.8.

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