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DETERMINANTS WITH BERNOULLI POLYNOMIALS AND THE RESTRICTED PARTITION FUNCTION

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Let $r \geq 1$ be an integer, $\mathbf{a} = (a_1, \ldots, a_r)$ a vector of positive integers and let $D \geq 1$ be a common multiple of $a_1, \ldots, a_r$. We study two natural determinants of order rD with Bernoulli polynomials and we present connections with the restricted partition function $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ to $\sum_{j=1}^{r} a_j x_j = n$ with $x_1 \geq 0, \ldots, x_r \geq 0$.

Keywords: restricted partition function, Bernoulli polynomials, Bernoulli Barnes numbers.

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1. Introduction

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ of $\sum_{i=1}^{r} a_i x_i = n$ with $x_i \geq 0$. Let D be a common multiple of $a_1, \ldots, a_r$. The restricted partition function $p_{\mathbf{a}}(n)$ was studied extensively in the literature, starting with the works of Sylvester [13] and Bell [3]. Popoviciu [10] gave a precise formula for $r = 2$. Recently, Bayad and Beck [2, Theorem 3.1] proved an explicit expression of $p_{\mathbf{a}}(n)$ in terms of Bernoulli-Barnes polynomials and the Fourier Dedekind sums, in the case that $a_1, \ldots, a_r$ are are pairwise coprime.

Let D be a common multiple of $a_1, \ldots, a_r$. In [7], we reduced the computation of $p_{\mathbf{a}}(n)$ to solving the linear congruence $a_1 j_1 + \cdots + a_r j_r \equiv n (\bmod D)$ in the range $0 \leq j_1 \leq \frac{D}{a_1} - 1, \ldots, 0 \leq j_r \leq \frac{D}{a_r} - 1$. In [8], we proved that if a determinant $\Delta_{r,D}$, see (2.5), which depends only on r and D, with entries consisting in values of Bernoulli polynomials is nonzero, then $p_{\mathbf{a}}(n)$ can be computed in terms of values of Bernoulli polynomials and Bernoulli Barnes numbers. In the second section, we outline several construction and results from [8].

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In the third section, we study the polynomial

$$F_{r,D}(x_1,\ldots,x_D) := \begin{vmatrix} \frac{B_1(x_1)}{1} & \cdots & \frac{B_1(x_D)}{1} & \cdots & \frac{B_r(x_1)}{r} & \cdots & \frac{B_r(x_D)}{r} \\ \frac{B_2(x_1)}{2} & \cdots & \frac{B_2(x_D)}{2} & \cdots & \frac{B_{r+1}(x_1)}{r+1} & \cdots & \frac{B_{r+1}(x_D)}{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{B_{rD}(x_1)}{rD} & \cdots & \frac{B_{rD}(x_D)}{rD} & \cdots & \frac{B_{rD+r-1}(x_1)}{rD+r-1} & \cdots & \frac{B_{rD+r-1}(x_D)}{rD+r-1} \end{vmatrix},$$

which is related to $\Delta_{r,D}$ by the identity

$$\Delta_{r,D} = (-1)^{\frac{rD(rD+r)}{2}} D^{\frac{rD(rD+r-2)}{2}} \cdot F_{r,D}(\frac{D-1}{D},\ldots,\frac{1}{D},0).$$

In Theorem 3.1 we prove that

$$F_{1,D}(x_1,\ldots,x_D) = \frac{1}{D!}\prod_{1\leq i<j\leq D}(x_j-x_i)\sum_{t=0}^{D}(-1)^t\frac{E_{D,D-t}(x_1,\ldots,x_D)}{t+1},$$

where $E_{D,0}(x_1,\ldots,x_D)=1$, $E_{D,1}(x_1,\ldots,x_D)=x_1+\cdots+x_D$ etc. are the *elementary symmetric polynomials*. In Proposition 3.3, we prove that

$$F_{r,D}(x_1,\ldots,x_D) = \prod_{1\leq i<j\leq D}(x_j-x_i)^r \cdot G_{r,D}(x_1,\ldots,x_D),$$

where $G_{r,D}(x_1,\ldots,x_D)$ is a symmetric polynomial, hence $\Delta_{r,D}\neq 0$ if and only if $G_{r,D}(\frac{D-1}{D},\ldots,0)\neq 0$.

In the last section, we propose another approach to the initial problem, studied in [8], of computing $p_{\mathbf{a}}(n)$ in terms of values of Bernoulli polynomials and Bernoulli Barnes numbers. In formula (4.3) we show that

$$\sum_{v=0}^{D-1}(-D)^m d_{\mathbf{a},m}(v) = \frac{(-1)^{r-1}D^{m+1}}{m!(r-1-m!)}B_{r-1-m}(\mathbf{a}),\ (\forall)0\leq m\leq r-1.$$

Seeing $d_{\mathbf{a},m}(v)$'s as indeterminates and considering also the identities

$$\sum_{m=0}^{r-1}\sum_{v=1}^{D} d_{\mathbf{a},m}(v)D^{n+m}\frac{B_{n+m+1}(\frac{v}{D})}{n+m+1} = \frac{(-1)^{r-1}n!}{(n+r)!}B_{r+n}(\mathbf{a})-\delta_{0n},\ (\forall)0\leq n\leq rD-r-1,$$

we obtain a system of rD linear equations with a determinant $\bar{\Delta}_{r,D}$. In Remark 4.1 we note that if $\bar{\Delta}_{r,D}\neq 0$, then $d_{\mathbf{a},m}(v)$, $0\leq m\leq r-1$, $1\leq v\leq D$, are the unique solutions of the above system. We consider the polynomial $\bar{F}_{r,D}\in\mathbb{Q}[x_1,\ldots,x_D]$ defined by

$$\bar{F}_{r,D}(x_1,\ldots,x_D) := \begin{vmatrix} 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\ B_1(x_1) & \cdots & B_1(x_D) & \cdots & \frac{B_r(x_1)}{r} & \cdots & \frac{B_r(x_D)}{r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{B_{rD-r}(x_1)}{rD-r} & \cdots & \frac{B_{rD-r}(x_D)}{rD-r} & \cdots & \frac{B_{rD-1}(x_1)}{rD-1} & \cdots & \frac{B_{rD-1}(x_D)}{rD-1} \end{vmatrix}.$$

We have that $\bar{\Delta}_{r,D} = (-D)^{D\binom{r}{2}+\binom{rD-r}{2}}\bar{F}_{r,D}(\frac{D-1}{D},\ldots,\frac{1}{D},0)$. In formula (4.9) we show that

$$\bar{F}_{r,D}(x_1,\ldots,x_D) = (-1)^{(D+1)\binom{r}{2}}\left(\prod_{1\leq i<j\leq D}(x_j-x_i)^r\right)G_{r,D}(x_1,\ldots,x_D),$$

where $G_{r,D}$ is a symmetric polynomial with $\deg G_{r,D} \leq r\binom{rD-r}{2} + D\binom{r}{2} - r\binom{D}{2}$.

Using the methods of Olson [9], in Proposition 4.1 we prove that for any $D \geq 1$ we have

$$(1)\ F_{1,D}(x_1,\ldots,x_D) = \frac{1}{(D-1)!}\prod_{1\leq i<j\leq D}(x_j-x_i),\ (2)\ \bar{\Delta}_{1,D} = \frac{1!2!\cdots(D-2)!}{(-D)^D}.$$

By our computer experiments in Singular [6], we expect that the following formula holds

$$\bar{F}_{r,2}(x_1,x_2) = (-1)^{\binom{r}{2}}\frac{[1!2!\cdots(r-1)!]^3}{r!(r+1)!\cdots(2r-1)!}(x_2-x_1)^r\prod_{j=0}^{r-1}((x_2-x_1)^2-j^2)^{r-j},\ (\forall)r\geq 1,$$

some justifications being noted in Remark 4.2. Also, we propose a formula for $\bar{F}_{2,D}$, see Conjecture 4.2, but we are unable to "guess" a formula for $\bar{F}_{r,D}$ in general.

2. **Preliminaries**

Let $\mathbf{a} := (a_1,a_2,\ldots,a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$,

$$p_{\mathbf{a}}(n) := \#\{(x_1,\ldots,x_r)\in\mathbb{N}^r\ :\ \sum_{i=1}^r a_ix_i = n\},\ (\forall)n\geq 0.$$

Let D be a common multiple of $a_1,\ldots,a_r$. Bell [3] has proved that $p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $r-1$, with the period D, i.e.

$$p_{\mathbf{a}}(n) = d_{\mathbf{a},r-1}(n)n^{r-1}+\cdots+d_{\mathbf{a},1}(n)n+d_{\mathbf{a},0}(n),\ (\forall)n\geq 0, \tag{2.1}$$

where $d_{\mathbf{a},m}(n+D) = d_{\mathbf{a},m}(n)$, $(\forall)0 \leq m \leq r-1, n \geq 0$, and $d_{\mathbf{a},r-1}(n)$ is not identically zero. The *Barnes zeta* function associated to $\mathbf{a}$ and $w > 0$ is

$$\zeta_{\mathbf{a}}(s,w) := \sum_{n=0}^{\infty}\frac{p_{\mathbf{a}}(n)}{(n+w)^s},\ \operatorname{Re}s > r,$$

see [1] and [12] for further details. It is well known that $\zeta_{\mathbf{a}}(s,w)$ is meromorphic on $\mathbb{C}$ with poles at most in the set $\{1.\ldots,r\}$. We consider the function

$$\zeta_{\mathbf{a}}(s) := \lim_{w\searrow 0}(\zeta_{\mathbf{a}}(s,w)-w^{-s}). \tag{2.2}$$

In [7, Lemma 2.6] we proved that

$$\zeta_{\mathbf{a}}(s) = \frac{1}{D^s}\sum_{m=0}^{r-1}\sum_{v=1}^{D}d_{\mathbf{a},m}(v)D^m\zeta(s-m,\frac{v}{D}), \tag{2.3}$$

where $\zeta(s,w) := \sum_{n=0}^{\infty} \frac{1}{(n+w)^s}$, $\operatorname{Re} s > 1$, is the *Hurwitz zeta* function. The *Bernoulli numbers* B_j are defined by

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!},$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ and $B_n = 0$ if n is odd and greater than 1. The *Bernoulli polynomials* are defined by

$$\frac{ze^{xz}}{(e^z - 1)} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

They are related with the Bernoulli numbers by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

The *Bernoulli-Barnes polynomials* are defined by

$$\frac{z^r e^{xz}}{(e^{a_1 z} - 1) \cdots (e^{a_r z} - 1)} = \sum_{j=0}^{\infty} B_j(x; \mathbf{a}) \frac{z^j}{j!}.$$

The *Bernoulli-Barnes numbers* are defined by

$$B_j(\mathbf{a}) := B_j(0; \mathbf{a}) = \sum_{i_1 + \cdots + i_r = j} \binom{j}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r} a_1^{i_1 - 1} \cdots a_r^{i_r - 1}.$$

In [8, Formula (2.9)] we proved that

$$\sum_{m=0}^{r-1} \sum_{v=1}^{D} d_{\mathbf{a},m}(v) D^{n+m} \frac{B_{n+m+1}(\frac{v}{D})}{n+m+1} = \frac{(-1)^{r-1} n!}{(n+r)!} B_{r+n}(\mathbf{a}) - \delta_{0n}, \ (\forall) n \in \mathbb{N}, \quad (2.4)$$

where $\delta_{0n} = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1 \end{cases}$ is the *Kronecker symbol*. Given values $0 \leq n \leq rD - 1$ in (1.9) and seeing $d_{\mathbf{a},m}(v)$'s as indeterminates, we obtain a system of linear equations with the determinant $\Delta_{r,D} :=$

$$\begin{vmatrix} \frac{B_1(\frac{1}{D})}{1} & \cdots & \frac{B_1(1)}{1} & \cdots & D^{r-1} \frac{B_r(\frac{1}{D})}{r} & \cdots & D^{r-1} \frac{B_r(1)}{r} \\ D \frac{B_2(\frac{1}{D})}{2} & \cdots & D \frac{B_2(1)}{2} & \cdots & D^r \frac{B_{r+1}(\frac{1}{D})}{r+1} & \cdots & D^r \frac{B_{r+1}(1)}{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D^{rD-1} \frac{B_{rD}(\frac{1}{D})}{rD} & \cdots & D^{rD-1} \frac{B_{rD}(1)}{rD} & \cdots & D^{rD+r-2} \frac{B_{rD+r-1}(\frac{1}{D})}{rD+r-1} & \cdots & D^{rD+r-2} \frac{B_{rD+r-1}(1)}{rD+r-1} \end{vmatrix} \quad (2.5)$$

Using basic properties of determinants and the fact that

$$B_n(1-x) = (-1)^n B_n(x) \text{ for all } n \geq 0,$$

it follows that

$$\Delta_{r,D} = C \cdot \begin{vmatrix} \frac{B_1(\frac{D-1}{D})}{1} & \cdots & \frac{B_1(0)}{1} & \cdots & \frac{B_r(\frac{D-1}{D})}{r} & \cdots & \frac{B_r(0)}{r} \\ \frac{B_2(\frac{D-1}{D})}{2} & \cdots & \frac{B_2(0)}{2} & \cdots & \frac{B_{r+1}(\frac{D-1}{D})}{r+1} & \cdots & \frac{B_{r+1}(0)}{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{B_{rD}(\frac{D-1}{D})}{rD} & \cdots & \frac{B_{rD}(0)}{rD} & \cdots & \frac{B_{rD+r-1}(\frac{D-1}{D})}{rD+r-1} & \cdots & \frac{B_{rD+r-1}(0)}{rD+r-1} \end{vmatrix}, \quad (2.6)$$

where $C = (-1)^{\frac{rD(rD+r)}{2}} D^{\frac{rD(rD+r-2)}{2}}$.

Proposition 2.1. *(See* [8, Proposition 2.1] *and* [8, Corollary 2.2]*)*
With the above notations, if $\Delta_{r,D} \neq 0$, then

$$d_{\mathbf{a},m}(v) = \frac{\Delta_{r,D}^{m,v}}{\Delta_{r,D}}, \ (\forall) 1 \leq v \leq D, 0 \leq m \leq r-1,$$

where $\Delta_{r,D}^{m,v}$ is the determinant obtained from $\Delta_{r,D}$, as defined in (2.5)*, by replacing the* $(mD+v)$*-th column with the column* $(\frac{(-1)^{r-1}n!}{(n+r)!}B_{n+r}(\mathbf{a}) - \delta_{n0})_{0 \leq n \leq rD-1}$. *Consequently,*

$$p_{\mathbf{a}}(n) = \frac{1}{\Delta_{r,D}} \sum_{m=0}^{r-1} \Delta_{r,D}^{m,v} n^m, \ (\forall) n \in \mathbb{N}.$$

Proof. The first part follows from the Cramer rule applied to the system (2.4). The second part is a consequence of the first part and (2.1). □

Remark 2.1. In [8] it was conjectured that $\Delta_{r,D} \neq 0$ for any $r, D \geq 1$. An affirmative answer was given in the case $r = 1$, $r = 2$ and $D = 1$. In the general case, an equivalent form was given in [8, Corollary 2.14], which reduced the problem to show that a $r \times r$ determinant is non zero. In the next section we tackle this problem from another point of vue, by studying a polynomial $F_{r,D}$ is D indeterminates with the property that $\Delta_{r,D} = F_{r,D}(\frac{D-1}{D}, \ldots, \frac{1}{D}, 0)$.

3. **Determinants with Bernoulli polynomials**

Let $r, D \geq 1$ be two integers. We consider the polynomial

$$F_{r,D}(x_1, \ldots, x_D) := \begin{vmatrix} \frac{B_1(x_1)}{1} & \cdots & \frac{B_1(x_D)}{1} & \cdots & \frac{B_r(x_1)}{r} & \cdots & \frac{B_r(x_D)}{r} \\ \frac{B_2(x_1)}{2} & \cdots & \frac{B_2(x_D)}{2} & \cdots & \frac{B_{r+1}(x_1)}{r+1} & \cdots & \frac{B_{r+1}(x_D)}{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{B_{rD}(x_1)}{rD} & \cdots & \frac{B_{rD}(x_D)}{rD} & \cdots & \frac{B_{rD+r-1}(x_1)}{rD+r-1} & \cdots & \frac{B_{rD+r-1}(x_D)}{rD+r-1} \end{vmatrix} \quad (3.1)$$

According to (2.6) and (3.1), using the notations from the previous section, we have that

$$\Delta_{r,D} = (-1)^{\frac{rD(rD+r)}{2}} D^{\frac{rD(rD+r-2)}{2}} \cdot F_{r,D}(\frac{D-1}{D}, \ldots, \frac{1}{D}, 0). \quad (3.2)$$

Lemma 3.1. *For any $r \geq 1$ we have that*

$$\Delta := \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{r} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r} & \frac{1}{r+1} & \cdots & \frac{1}{2r-1} \end{vmatrix} = \frac{[1!2!\cdots(r-1)!]^3}{r!(r+1)!\cdots(2r-1)!}.$$

Proof. We let

$$\Delta_\ell := \begin{vmatrix} \frac{r!}{\ell} & \frac{r!}{\ell+1} & \cdots & \frac{r!}{r} \\ \frac{(r+1)!}{\ell+1} & \frac{(r+1)!}{\ell+2} & \cdots & \frac{(r+1)!}{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2r-\ell)!}{r} & \frac{(2r-\ell)!}{r+1} & \cdots & \frac{(2r-\ell)!}{2r-\ell} \end{vmatrix}.$$

Note that $\Delta = r!(r+1)!\cdots(2r-1)!\Delta_1$. We have $\Delta_r = (r-1)!$. For $1 \leq \ell < r$, we have

$$\Delta_\ell = (r-1)! \begin{vmatrix} \frac{r!}{\ell} & \cdots & \frac{r!}{r-1} & 1 \\ \frac{(r+1)!}{\ell+1} & \cdots & \frac{(r+1)!}{r} & \frac{r!}{(r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2r-\ell)!}{r} & \cdots & \frac{(2r-\ell)!}{2r-\ell-1} & \frac{(2r-\ell-1)!}{(r-1)!} \end{vmatrix}.$$

Multiplying the first line accordingly and adding to the next lines in order to obtain zeroes on the last column, it follows that

$$\Delta_\ell = (r-\ell)! \det\left(\tfrac{k(r+k-1)!(r-s)}{s+k}\right)_{\substack{\ell \leq s \leq r-1 \\ 1 \leq k \leq r-\ell}} =$$

$$= (r-1)! \tfrac{((r-\ell)!)^2}{\ell\cdots(r-1)} \Delta_{\ell+1} = ((r-\ell)!)^2 \ell! \Delta_{\ell+1},$$

hence the induction step is complete. □

Proposition 3.1. *We have that*

$$F_{r,1}(x) = \frac{[1!2!\cdots(r-1)!]^3}{r!(r+1)!\cdots(2r-1)!} x^{r^2} + \text{ terms of lower degree.}$$

Proof. We have $B_n(x) = x^n +$ terms of lower order, hence the result follows from Lemma 3.1. □

Proposition 3.2. *For $r = 1$ and $D \geq 1$ we have that:*

1. *There exists a symetric polynomial $G_{1,D}(x_1, \ldots, x_D)$ of degree D such that*
$$F_{1,D}(x_1, \ldots, x_D) = \prod_{1 \leq i < j \leq D} (x_j - x_i) G_{1,D}(x_1, \ldots, x_D).$$
2. $G_{1,D}(x_1, \ldots, x_D) = \frac{1}{D!} x_1 x_2 \cdots x_D +$ *terms of lower degree.*
3. $G_{1,D}(0, \ldots, 0) = \frac{(-1)^D}{(D+1)!}$.

Proof. (1) From (3.1) it follows that

$$F_{1,D}(x_1, \ldots, x_D) = \frac{1}{D!} \begin{vmatrix} B_1(x_1) & B_1(x_2) & \cdots & B_1(x_D) \\ B_2(x_1) & B_2(x_2) & \cdots & B_2(x_D) \\ \vdots & \vdots & \vdots & \vdots \\ B_D(x_1) & B_D(x_2) & \cdots & B_D(x_D) \end{vmatrix}. \tag{3.3}$$

Moreover, for any permutation $\sigma \in S_D$, we have that

$$F_{1,D}(x_{\sigma(1)},\ldots,x_{\sigma(D)}) = \varepsilon(\sigma)F_{1,D}(x_1,\ldots,x_D). \tag{3.4}$$

Since

$$(x_j - x_i)|B_\ell(x_j) - B_\ell(x_i),\ (\forall)1 \leq \ell \leq D,\ 1 \leq i < j \leq D,$$

from (3.3) and (3.4) it follows that

$$F_{1,D}(x_1,\ldots,x_D) = G_{1,D}(x_1,\ldots,x_D) \cdot \prod_{1\leq i<j\leq D}(x_j - x_i), \tag{3.5}$$

where $G_{1,D} \in \mathbb{Q}[x_1,\ldots,x_D]$ is a symmetrical polynomial of degree D.

(2) The homogeneous component of highest degree of $F_{1,D}$ is

$$\frac{1}{D!}\begin{vmatrix} x_1 & x_2 & \cdots & x_D \\ x_1^2 & x_2^2 & \cdots & x_D^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^D & x_2^D & \cdots & x_D^D \end{vmatrix} = \frac{x_1\cdots x_D}{D!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_D \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{D-1} & x_2^{D-1} & \cdots & x_D^{D-1} \end{vmatrix} = \frac{x_1\cdots x_D}{D!}\prod_{1\leq i<j\leq D}(x_j - x_i),$$

hence $G_{1,D}(x_1,\ldots,x_D) = \frac{1}{D!}x_1\cdots x_D +$ terms of lower order.

(3) For any integers $j \geq 0$ and $1 \leq n \leq D$, we let

$$L_j(x_1,\ldots,x_n) := \text{ the sum of all monomials of degree } j \text{ in } x_1,\ldots,x_n,$$

i.e. $L_1(x_1,\ldots,x_n) = x_1 + \cdots + x_n$, $L_2(x_1,\ldots,x_n) = x_1^2 + \cdots + x_n^2 + x_1x_2 + \cdots + x_{n-1}x_n$, etc. It is easy to check that

$$L_j(x_1,\ldots,x_{n-2},x_n) - L_j(x_1,\ldots,x_{n-1}) = (x_n - x_{n-1})L_{j-1}(x_1,\ldots,x_n). \tag{3.6}$$

We let

$$B_\ell(x_1,x_k) := \frac{B_j(x_k) - B_j(x_1)}{x_k - x_1},\ (\forall)1 < k \leq D, \ell \geq 1.$$

Inductively, for $1 < j \leq k \leq D$ and $\ell \geq 1$, we define

$$B_\ell(x_1,\ldots,x_{j-1},x_k) := \frac{B_\ell(x_1,\ldots,x_{j-2},x_k) - B_\ell(x_1,\ldots,x_{j-1})}{x_k - x_{j-1}}. \tag{3.7}$$

We prove by induction on $j \geq 1$ that

$$B_\ell(x_1,\ldots,x_{j-1},x_k) = \sum_{t=0}^{\ell-j+1}\binom{\ell}{t+j-1}B_{\ell-j+1-t}L_t(x_1,\ldots,x_{j-1},x_k),\ (\forall)1 \leq \ell \leq D. \tag{3.8}$$

Indeed, since $B_\ell(x) = \sum_{t=0}^{\ell}\binom{\ell}{t}B_{\ell-t}x^t$, it follows that (3.8) holds for $j = 1$. Now, assume that $j \geq 2$. From the induction hypothesis, (3.7), (3.6) and (3.8) it follows that

$$B_\ell(x_1,\ldots,x_{j-1},x_k) = \sum_{t=1}^{\ell-j+2}\tbinom{\ell}{t+j-2}B_{\ell-j+2-t} \cdot \tfrac{L_t(x_1,\ldots,x_{j-2},x_k) - L_t(x_1,\ldots,x_{j-1})}{x_k - x_{j-1}} =$$

$$= \sum_{t=1}^{\ell-j+2}\tbinom{\ell}{t+j-2}B_{\ell-j+2-t}L_{t-1}(x_1,\ldots,x_{j-1},x_k) =$$

$$= \sum_{t=0}^{\ell-j+1}\tbinom{\ell}{t+j-1}B_{\ell-j+1-t}L_t(x_1,\ldots,x_{j-1},x_k),$$

hence the induction step is complete. Using standard properties of determinants, from (3.3) it follows that

$$F_{1,D}(x_1,\ldots,x_D)=\frac{1}{D!}\prod_{2\leq j\leq D}(x_j-x_1)\begin{vmatrix} B_1(x_1) & 1 & \cdots & 1 \\ B_2(x_1) & B_2(x_1,x_2) & \cdots & B_2(x_1,x_D) \\ \vdots & \vdots & \vdots & \vdots \\ B_D(x_1) & B_D(x_1,x_2) & \cdots & B_D(x_1,x_D) \end{vmatrix}=$$

$$=\cdots=\frac{1}{D!}\prod_{1\leq i<j\leq D}(x_j-x_i)\begin{vmatrix} B_1(x_1) & B_1(x_1,x_2) & \cdots & B_1(x_1,\ldots,x_D) \\ B_2(x_1) & B_2(x_1,x_2) & \cdots & B_2(x_1,\ldots,x_D) \\ \vdots & \vdots & \vdots & \vdots \\ B_D(x_1) & B_D(x_1,x_2) & \cdots & B_D(x_1,\ldots,x_D) \end{vmatrix}, \quad (3.9)$$

hence the last determinant is $D!\cdot G_D(x_1,\ldots,x_D)$. Note that (3.8) implies that

$$B_\ell(x_1,\ldots,x_j)=0,\ (\forall)1\leq \ell\leq j-2\leq D-2, B_\ell(x_1,\ldots,x_{\ell+1})=1,\ (\forall)1\leq\ell\leq D-1. \quad (3.10)$$

From (3.9) and (3.10) it follows that $F_{1,D}(x_1,\ldots,x_D)=\frac{1}{D!}\prod_{1\leq i<j\leq D}(x_j-x_i)\cdot$

$$\cdot\begin{vmatrix} B_1(x_1) & 1 & 0 & \cdots & 0 \\ B_2(x_1) & B_2(x_1,x_2) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{D-1}(x_1) & B_{D-1}(x_1,x_2) & \cdots & B_{D-1}(x_1,\ldots,x_{D-1}) & 1 \\ B_D(x_1) & B_D(x_1,x_2) & \cdots & B_D(x_1,\ldots,x_{D-1}) & B_D(x_1,\ldots,x_D) \end{vmatrix}. \quad (3.11)$$

Also, from (3.8), we have $B_\ell(0,\ldots,0)=\binom{\ell}{j-1}B_{\ell-j+1}$, hence, from (3.5) and (3.11), we get

$$M_D:=D!G_{1,D}(0,\ldots,0)=\begin{vmatrix} B_1 & 1 & 0 & \cdots & 0 \\ B_2 & \binom{2}{1}B_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{D-1} & \binom{D-1}{1}B_{D-2} & \cdots & \binom{D-1}{D-2}B_1 & 1 \\ B_D & \binom{D}{1}B_{D-1} & \cdots & \binom{D}{D-2}B_2 & \binom{D}{D-1}B_1 \end{vmatrix}. \quad (3.12)$$

Since M_D is the determinant of a lower Hessenberg matrix, according to [4, pag.222,Theorem], we have the recursive relation

$$M_D=\binom{D}{D-1}B_1M_{D-1}+\sum_{\ell=1}^{D-1}(-1)^{D-\ell}\binom{D}{D+1-\ell}B_{D+1-\ell}M_{\ell-1},\ (\forall)D\geq 1, \quad (3.13)$$

where $M_0:=1$. We prove that

$$M_D=\frac{(-1)^D}{D+1},(\forall)D\geq 1, \quad (3.14)$$

using induction on $D\geq 1$. For $D=1$ we have $M_1=B_1=-\frac{1}{2}$, hence the (3.14) holds. If $D\geq 2$ then from induction hypothesis and (3.14) it follows that

$$M_D = (-1)^{D-1}B_1 + \sum_{\ell=1}^{D-1}(-1)^{D-\ell}\binom{D}{D-\ell+1}B_{D+1-\ell}\frac{(-1)^{\ell-1}}{\ell} = (-1)^{D-1}\sum_{\ell=1}^{D}\frac{1}{\ell}\binom{D}{\ell-1}B_{D+1-\ell}. \tag{3.15}$$

Since $\binom{D+1}{\ell} = \frac{D+1}{\ell}\binom{D}{\ell-1}$, $(\forall)1 \leq \ell \leq D$, from (3.15) it follows that

$$M_D = \frac{(-1)^{D-1}}{D+1}\sum_{\ell=1}^{D}\binom{D+1}{\ell}B_{D+1-\ell} = \frac{(-1)^{D-1}}{D+1}\left(\sum_{\ell=1}^{D+1}\binom{D+1}{\ell}B_{D+1-\ell} - 1\right) \tag{3.16}$$

On the other hand $\sum_{\ell=1}^{D+1}\binom{D+1}{\ell}B_{D+1-\ell} = B_{D+1}(1) - B_{D+1}(0) = 0$, hence (3.16) completes the induction step. Therefore, we proved (3.14) and thus $G_D(0,\ldots,0) = \frac{(-1)^D}{(D+1)!}$, as required. □

For any integer $n \geq 1$, we denote $E_{n,0}(x_1,\ldots,x_n) := 1$, $E_{n,1}(x_1,\ldots,x_n) := x_1+\cdots+x_n$, $\ldots$, $E_{n,n}(x_1,\ldots,x_n) := x_1x_2\cdots x_n$, the *elementary symmetric polynomials* in $\mathbb{Q}[x_1,\ldots,x_n]$.

Theorem 3.1. *With the above notations, we have that*

$$F_{1,D}(x_1,\ldots,x_D) = \frac{1}{D!}\prod_{1\leq i<j\leq D}(x_j - x_i)\sum_{t=0}^{D}(-1)^t\frac{E_{D,D-t}(x_1,\ldots,x_D)}{t+1}.$$

Proof. We use induction on $D \geq 1$. For $D = 1$ we have

$$F_{1,1}(x_1) = B_1(x) = x_1 - \frac{1}{2} = E_{1,1}(x_1) - \frac{E_{1,0}(x_1)}{2},$$

hence the required formula holds. For $D \geq 2$, from (3.3) it follows that

$$F_{1,D}(x_1,\ldots,x_D) = \frac{1}{D}\sum_{k=1}^{D}(-1)^{D+k}B_D(x_k)F_{1,D-1}(x_1,\ldots,\widehat{x_k},\ldots,x_D), \tag{3.17}$$

where $\widehat{x_k}$ means that the variable x_k is omitted. From the induction hypothesis and (3.17) it follows that $F_{1,D}(x_1,\ldots,x_D) =$

$$= \frac{1}{D!}\sum_{k=1}^{D}(-1)^{D+k}B_D(x_k)\prod_{\substack{1\leq i<j\leq D\\ i,j\neq k}}(x_j-x_i)\sum_{\ell=0}^{D-1}(-1)^\ell\frac{E_{D-1,D-1-\ell}(x_1,\ldots,\widehat{x_k},\ldots,x_D)}{\ell+1}. \tag{3.18}$$

The relation (3.18) is equivalent to $\frac{D!F_{1,D}(x_1,\ldots,x_D)}{\prod_{1\leq i<j\leq D}(x_j-x_i)} =$

$$= \sum_{k=1}^{D}(-1)^{D-1}\frac{1}{\prod_{j\neq k}(x_j - x_k)}B_D(x_k)\sum_{\ell=0}^{D-1}(-1)^\ell\frac{E_{D-1,D-1-\ell}(x_1,\ldots,\widehat{x_k},\ldots,x_D)}{\ell+1}. \tag{3.19}$$

From (3.19), in order to complete the proof it is enough to show that

$$\sum_{t=0}^{D}(-1)^t\frac{E_{D,D-t}(x_1,\ldots,x_D)}{t+1} = \sum_{k=1}^{D}\sum_{\ell=0}^{D-1}\frac{(-1)^{D-1-\ell}B_D(x_k)E_{D-1,D-1-\ell}(x_1,\ldots,\widehat{x_k},\ldots,x_D)}{(\ell+1)\prod_{j\neq k}(x_j - x_k)}. \tag{3.20}$$

Since $B_D(x_k)=\sum_{s=0}^D \binom{D}{s} B_{D-s}x^s$, it follows that (3.20) is equivalent to

$$(-1)^t\frac{E_{D,D-t}(x_1,\ldots,x_D)}{t+1}=\sum_{k=1}^{D}\sum_{\ell=0}^{\min\{t,D-1\}}\frac{(-1)^{D-1-\ell}\binom{D}{t-\ell}B_{t-\ell}x_k^{D-t+\ell}E_{D-1,D-1-\ell}(x_1,\ldots,\widehat{x_k},\ldots,x_D)}{(\ell+1)\prod_{j\neq k}(x_j-x_k)}, \tag{3.21}$$

for any $0\leq t\leq D$. Since, by Proposition 3.2(3), we have that

$$\frac{D!F_{1,D}(x_1,\ldots,x_D)}{\prod_{1\leq i<j\leq D}(x_j-x_i)}|_{x_1=\cdots=x_D=0}=\frac{(-1)^D}{D+1}=\frac{(-1)^D E_{D,0}(x_1,\ldots,x_D)}{D+1},$$

it is enough to prove (3.21) for $0\leq t\leq D-1$. Similarly, by Proposition 3.2(2) we can dismiss the case $t=0$. Assume in the following that $1\leq t\leq D-1$. As the both sides in (3.21) are symmetric polynomials, it is enough to prove that (3.21) holds when we evaluate it in $x_{D-t+1}=\cdots=x_D=0$. Moreover, in this case, $E_{D-1,D-1-\ell}(x_1,\ldots,\widehat{x_k},\ldots,x_D)=0$ for any $\ell<t$. Therefore, (3.21) is equivalent to

$$(-1)^t\frac{x_1\cdots x_{D-t}}{t+1}=\sum_{k=1}^{D-t}\frac{(-1)^t x_k^D x_1\cdots\widehat{x_k}\cdots x_{D-t}}{(t+1)\prod_{j\neq k,\ j\leq D-t}(x_k-x_j)x_k^t},$$

hence it is equivalent to $\sum_{k=1}^{D-t}\frac{x_k^{D-t-1}}{\prod_{j\neq k,\ j\leq D-t}(x_k-x_j)}=1$, which can be easily proved by expanding a Vandermonde determinant of order $D-t$. □

Corollary 3.1. *We have that*

$$\Delta_{1,D}=(-1)^{\frac{D(D+1)}{2}}\frac{(D-1)!(D-2)!\cdots 1!}{D!}\sum_{t=0}^{D}(-1)^t\frac{E_{D,D-t}(\frac{D-1}{D},\ldots,\frac{1}{D},0)}{t+1}.$$

Proof. From (3.2) and Theorem 3.1 it follows that

$$\Delta_{1,D}=(-1)^{\frac{D(D+1)}{2}}D^{\frac{D(D-1)}{2}}\frac{1}{D!}\prod_{1\leq i<j\leq D}\left(\frac{j-i}{D}\right)\sum_{t=0}^{D}(-1)^t\frac{E_{D,D-t}(\frac{D-1}{D},\ldots,\frac{1}{D},0)}{t+1}. \tag{3.22}$$

On the other hand

$$\prod_{1\leq i<j\leq D}\left(\frac{j-i}{D}\right)=\frac{(D-1)!(D-2)!\cdots 1!}{D^{\frac{D(D-1)}{2}}}, \tag{3.23}$$

hence, from (3.22) and (3.23) we get the required result. □

Unfortunately, in the general, it seems to be very difficult to give an exact formula for $F_{r,D}(x_1,\ldots,x_D)$. We prove the following generalization of Proposition 3.2(1):

Proposition 3.3. *For any integers $r,D\geq 1$, there exists a symmetric polynomial $G_{r,D}$ of degree $\leq r^2\binom{D+1}{2}-r\binom{D}{2}$ such that*

$$F_{r,D}(x_1,\ldots,x_D)=\prod_{1\leq i<j\leq D}(x_j-x_i)^r G_{r,D}(x_1,\ldots,x_D),$$

where, with the notations from (3.7), we have that $G_{r,D}(x_1,\ldots,x_D) =$

$$= \begin{vmatrix} \frac{B_1(x_1)}{1} & \cdots & \frac{B_1(x_1,\ldots,x_D)}{1} & \cdots & \frac{B_r(x_1)}{r} & \cdots & \frac{B_r(x_1,\ldots,x_D)}{r} \\ \frac{B_2(x_1)}{2} & \cdots & \frac{B_2(x_1,\ldots,x_D)}{2} & \cdots & \frac{B_{r+1}(x_1)}{r+1} & \cdots & \frac{B_{r+1}(x_1,\ldots,x_D)}{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{B_{rD}(x_1)}{rD} & \cdots & \frac{B_{rD}(x_1,\ldots,x_D)}{rD} & \cdots & \frac{B_{rD+r-1}(x_1)}{rD+r-1} & \cdots & \frac{B_{rD+r-1}(x_1,\ldots,x_D)}{rD+r-1} \end{vmatrix}.$$

Proof. Using standard properties of determinants, as in the proof of formula (3.9), we get the required decomposition. The fact that $G_{r,D}(x_1,\ldots,x_D)$ is symmetric follows from the identity $F_{r,D}(x_{\sigma(1)},\ldots,x_{\sigma(D)}) = \varepsilon(\sigma)^r F_{r,D}(x_1,\ldots,x_r),\ (\forall)\sigma \in S_D$ and the decomposition $F_{r,D}(x_1,\ldots,x_D) = \prod_{1\leq i<j\leq D}(x_j - x_i)^r G_{r,D}(x_1,\ldots,x_D)$. ☐

4. **An approach to compute** $p_{\mathbf{a}}(n)$

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. Let D be a common multiple of $a_1,\ldots,a_r$. Using the notations and definitions from the second section, according to [7, Proposition 2.4] and (2.3), the function $\zeta_{\mathbf{a}}(s)$ is meromorphic in the whole complex plane with poles at most in the set $\{1,\ldots,r\}$ which are all simple with residues

$$R_{m+1} = Res_{s=m+1}\zeta_{\mathbf{a}}(s) = \frac{1}{D}\sum_{v=0}^{D-1} d_{\mathbf{a},m}(v),\ (\forall)0 \leq m \leq r-1. \tag{4.1}$$

On the other hand, according to [7, Theorem 2.10] or [11, Formula (3.9)] and (2.2), we have that

$$R_{m+1} = \frac{(-1)^{r-1-m}}{m!(r-1-m!)} B_{r-1-m}(a_1,\ldots,a_r),\ (\forall)0 \leq m \leq r-1. \tag{4.2}$$

It follows that

$$\sum_{v=0}^{D-1}(-D)^m d_{\mathbf{a},m}(v) = \frac{(-1)^{r-1}D^{m+1}}{m!(r-1-m!)} B_{r-1-m}(\mathbf{a}),\ (\forall)0 \leq m \leq r-1. \tag{4.3}$$

On the other hand, from (2.4) it follows that

$$\sum_{m=0}^{r-1}\sum_{v=1}^{D} d_{\mathbf{a},m}(v) D^{n+m}\frac{B_{n+m+1}(\frac{v}{D})}{n+m+1} = \frac{(-1)^{r-1}n!}{(n+r)!}B_{r+n}(\mathbf{a}) - \delta_{0n},\ (\forall)0 \leq n \leq rD-r-1. \tag{4.4}$$

If we see $d_{\mathbf{a},m}(v)$ as indeterminates, (4.3) and (4.4) form a system of linear equations with the determinant $\bar{\Delta}_{r,D} :=$

$$= \begin{vmatrix} 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & (-D)^{r-1} & \cdots & (-D)^{r-1} \\ B_1(\frac{1}{D}) & \cdots & B_1(1) & \cdots & \frac{D^{r-1}B_r(\frac{1}{D})}{r} & \cdots & \frac{D^{r-1}B_r(1)}{r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{D^{rD-r-1}B_{rD-r}(\frac{1}{D})}{rD-r} & \cdots & \frac{D^{rD-r-1}B_{rD-r}(1)}{rD-r} & \cdots & \frac{D^{rD-2}B_{rD-1}(\frac{1}{D})}{rD-1} & \cdots & \frac{D^{rD-2}B_{rD-1}(1)}{rD-1} \end{vmatrix}. \tag{4.5}$$

From (4.5) and the identity $B_n(1-x) = (-1)^n B_n(x)$ it follows that

$$\bar{\Delta}_{r,D} = (-D)^{D\binom{r}{2}+\binom{rD-r}{2}} \cdot \begin{vmatrix} 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\ B_1(\frac{D-1}{D}) & \cdots & B_1(0) & \cdots & \frac{B_r(\frac{D-1}{D})}{r} & \cdots & \frac{B_r(0)}{r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{B_{rD-r}(\frac{D-1}{D})}{rD-r} & \cdots & \frac{B_{rD-r}(0)}{rD-r} & \cdots & \frac{B_{rD-1}(\frac{D-1}{D})}{rD-1} & \cdots & \frac{B_{rD-1}(0)}{rD-1} \end{vmatrix}. \tag{4.6}$$

Remark 4.1. Similarly to Proposition 2.1, if $\bar{\Delta}_{r,D} \neq 0$, then $d_{\mathbf{a},m}(v)$, $0 \leq m \leq r-1$, $1 \leq v \leq D$ are the solutions of the system of linear equations consisting in (4.3) and (4.4).

Now, we consider the polynomial $\bar{F}_{r,D} \in \mathbb{Q}[x_1, \ldots, x_D]$ defined as

$$\bar{F}_{r,D}(x_1, \ldots, x_D) := \begin{vmatrix} 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\ B_1(x_1) & \cdots & B_1(x_D) & \cdots & \frac{B_r(x_1)}{r} & \cdots & \frac{B_r(x_D)}{r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{B_{rD-r}(x_1)}{rD-r} & \cdots & \frac{B_{rD-r}(x_D)}{rD-r} & \cdots & \frac{B_{rD-1}(x_1)}{rD-1} & \cdots & \frac{B_{rD-1}(x_D)}{rD-1} \end{vmatrix} \tag{4.7}$$

From (4.6) and (4.7) it follows that

$$\bar{\Delta}_{r,D} = (-D)^{D\binom{r}{2}+\binom{rD-r}{2}} \bar{F}_{r,D}(\frac{D-1}{D}, \ldots, \frac{1}{D}, 0). \tag{4.8}$$

Note that if $D=1$ then (4.5) and (4.7) implies

$$\bar{\Delta}_{r,1} = (-D)^{\binom{r}{2}} \text{ and } \bar{F}_{r,1}(x_1, \ldots, x_r) = 1,$$

therefore, in the following we assume $D \geq 2$.

Using elementary operations in (4.7) and the notations (3.7) it follows that $(-1)^{(D+1)\binom{r}{2}} \bar{F}_{r,D}(x_1, \ldots, x_D) =$

$$= \prod_{2\leq j\leq D} (x_D - x_1)^r \begin{vmatrix} B_1(x_1,x_2) & \cdots & B_1(x_1,x_D) & \cdots & \frac{B_r(x_1,x_2)}{r} & \cdots & \frac{B_r(x_1,x_D)}{r} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{B_{rD-r}(x_1,x_2)}{rD-r} & \cdots & \frac{B_{rD-r}(x_1,x_D)}{rD-r} & \cdots & \frac{B_{rD-1}(x_1,x_2)}{rD-1} & \cdots & \frac{B_{rD-1}(x_1,x_D)}{rD-1} \end{vmatrix} =$$

$$\prod_{1\leq i<j\leq D} (x_j - x_i)^r \begin{vmatrix} B_1(x_1,x_2) & \cdots & B_1(x_1,\ldots,x_D) & \cdots & \frac{B_r(x_1,x_2)}{r} & \cdots & \frac{B_r(x_1,\ldots,x_D)}{r} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{B_{rD-r}(x_1,x_2)}{rD-r} & \cdots & \frac{B_{rD-r}(x_1,\ldots,x_D)}{rD-r} & \cdots & \frac{B_{rD-1}(x_1,x_2)}{rD-1} & \cdots & \frac{B_{rD-1}(x_1,\ldots,x_D)}{rD-1} \end{vmatrix} \tag{4.9}$$

We denote the last determinant in (4.9) with $\bar{G}_{r,D}(x_1, \ldots, x_D)$ and we note that $\bar{G}_{r,D}$ is a symmetric polynomial with

$$\deg(\bar{G}_{r,D}) \leq r\binom{rD-r}{2} + D\binom{r}{2} - r\binom{D}{2}.$$

Proposition 4.1. *For any $D \geq 2$ we have that*

$$(1)\ F_{1,D}(x_1,\ldots,x_D) = \frac{1}{(D-1)!}\prod_{1\leq i<j\leq D}(x_j-x_i),\ (2)\ \bar{\Delta}_{1,D} = \frac{1!2!\cdots(D-2)!}{(-D)^D}.$$

Proof. (1) Using the method from [9, Page 262], we get

$$\bar{F}_{1,D}(x_1,\ldots,x_D) = \frac{1}{(D-1)!}\begin{vmatrix} 1 & \cdots & 1 \\ B_1(x_1) & \cdots & B_1(x_D) \\ \vdots & \vdots & \vdots \\ B_{D-1}(x_1) & \cdots & B_{D-1}(x_D)\end{vmatrix} =$$

$$= \frac{1}{D!}\begin{vmatrix} B_0 & 0 & \cdots & 0 \\ \binom{2}{1}B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \binom{D-1}{1}B_{D-1} & \binom{D-1}{2}B_2 & \cdots & B_0\end{vmatrix}\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_D \\ \vdots & \vdots & \vdots \\ x_1^{D-1} & \cdots & x_D^{D-1}\end{vmatrix} = \frac{1}{(D-1)!}\prod_{1\leq i<j\leq D}(x_j-x_i). \tag{4.10}$$

(2) The last identity follows from (1), (4.8) and (4.9). □

Remark 4.2. *For $D=2$, according to (4.9) we have that*

$$\bar{F}_{r,2}(x_1,x_2) = (-1)^{\binom{r}{2}}(x_2-x_1)^r\bar{G}_{r,2}(x_1,x_2),\ \textit{where} \tag{4.11}$$

$$\bar{G}_{r,2}(x_1,x_2) = \begin{vmatrix} B_1(x_1,x_2) & \frac{B_2(x_1,x_2)}{2} & \cdots & \frac{B_r(x_1,x_2)}{r} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{B_r(x_1,x_2)}{r} & \frac{B_{r+1}(x_1,x_2)}{r+1} & \cdots & \frac{B_{2r-1}(x_1,x_2)}{2r-1}\end{vmatrix}. \tag{4.12}$$

On the other hand, according to (3.8), we have that

$$B_k(x_1,x_2) = \sum_{t=0}^{k-1}\binom{k}{t+1}B_{k-1-t}\sum_{s=0}^{t}x_1^{t-s}x_2^t,\ (\forall)1\leq k\leq 2r-1. \tag{4.13}$$

In particular, from (4.12) and (4.13) it follows that

$$\bar{G}_{r,2}(x_1,0) = \begin{vmatrix} 1 & \frac{x_1}{2} & \cdots & \frac{x^{r-1}}{r} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x_1^{r-1}}{r} & \frac{x_1^r}{r+1} & \cdots & \frac{x_1^{2r-2}}{2r-1}\end{vmatrix} + \textit{ terms of lower degree.} \tag{4.14}$$

From Lemma 3.1 *and (4.14) it follows that*

$$\bar{G}_{r,2}(x_1,0) = \frac{[1!2!\cdots(r-1)!]^3}{r!(r+1)!\cdots(2r-1)!}x_1^{r(r-1)} + \textit{ terms of lower degree.}$$

Our computer experiments in Singular [6] and Remark 4.2 yield us to the following:

Conjecture 4.1. *For any $r\geq 1$, it holds that*

$$\bar{F}_{r,2}(x_1,x_2) = (-1)^{\binom{r}{2}}\frac{[1!2!\cdots(r-1)!]^3}{r!(r+1)!\cdots(2r-1)!}(x_2-x_1)^r\prod_{j=0}^{r-1}((x_2-x_1)^2-j^2)^{r-j}.$$

We checked Conjecture 4.1 for $r \leq 4$ and we believe that the formula holds in general. Our computer experiments in SINGULAR [6] yield us to:

Conjecture 4.2. *For any $D \geq 2$, it holds that*

$$\bar{F}_{2,D}(x_1,\ldots,x_D) = K(D) \prod_{1\leq i<j\leq D} (x_j - x_i)^2 \sum_{1\leq i<j\leq D} ((x_j - x_i)^2 - 1),$$

where $K(D) \in \mathbb{Q}$. Moreover, $K(D) \neq 0$, hence $\bar{\Delta}_{2,D} \neq 0$.

We checked Conjecture 4.2 for $D \leq 4$ and we believe it is true in general. Unfortunately, we are not able to "guess" a general formula for $\bar{F}_{r,D}$, the situation being difficult even for $D = r = 3$ as $\bar{G}_{3,3}$ is an irreducible polynomial of degree 18.

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