APPLICATION OF THE HOMOTOPY ANALYSIS METHOD IN APPROXIMATION OF CONVOLUTIONS STOCHASTIC DISTRIBUTIONS

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This paper describes application of the homotopy analysis (HA) technique in approximation of infinity convolutions of mixed stochastic distributions, which usually do not have a closed form. The main result is based on HA approximations of their characteristics functions, as the Fourier-Stieltjes transforms of the appropriate distributions.

Keywords: Homotopy analysis, characteristic functions, approximations, contaminated distributions, convolutions.

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1. Introduction

The homotopy analysis method (HAM) proposed by Liao [1]-[3] is a general approximate analytic approach which is using to obtain series solutions of nonlinear equations of various types. In recent years, the HAM has become the subject of extensive studies [4]-[6]. It has been applied to solving the various types of nonlinear differential equations [7, 8], partial differential equations [9]-[11] or integral equations [12, 13].

Let us notice that HAM has found significant application in solving problems mainly in the physical sciences. On the other hand, the importance of HAM in stochastic theory can be seen for calculations of the options pricing subordinate with so-called stochastic volatility (SV) model [14]. Moreover, in [15] has been introduced the discrete-time SV model, named Split-SV model. It is shown that stochastic distribution of this model is an infinity convolution of mixed stochastic distributions, which does not have a closed form. In this paper will be describe some possibilities to finding approximations of these types of stochastic distributions, by using the HAM. The main focus, similarly as in [16], will be on the HAM approximations of Fourier-Stieltjes transforms of appropriate distributions, i.e., their characteristics functions (CFs).

2. Stochastic assumptions

Firstly, we observe the autoregressive (AR) sequence of random variables (RVs) defined by the recurrent relation

$$\Delta_t = a\,\Delta_{t-1} + \eta_t, \quad t \in \mathbb{Z}. \tag{1}$$

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Here, a is some constant value and (η_t) is the sequence of independent identically distributed (i.i.d.) RVs which satisfies

$$P\{\eta_t \neq 0\} = p \in (0,1), \qquad \forall t \in \mathbb{Z}.$$

In some practical interpretations, Eq.(1) describes some linear system with input Δ_{t-1} and output Δ_t, at time $t \in \mathbb{Z}$. Thus, the sequence (η_t) is "optional" noise, because it affects the outputs of the system (1) only partially, with probability $p \in (0,1)$. This sequence can be formally written as the multiplicative decomposition $\eta_t = \xi_t q_t$, similarly as in [17]-[19]. Here, (ξ_t) is the "true" noise, i.e. the $(0, \delta^2)$ i.i.d. sequence of RVs, with some absolutely-continuous (usually, Gaussian) distribution. On the other hand, (q_t) is the so-called *Noise-Indicator*, i.e., the sequence of RVs mutually independent of (ξ_t), defined as

$$q_t = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases}, \qquad 0 < p < 1.$$

According to the aforementioned facts, the distribution function (DF) of the noise variable η_t, $t \in \mathbb{Z}$ can be found by using the conditional probability

$$\begin{aligned} F_\eta(x) &:= P\{\eta_t < x\} = \sum_{j=0,1} P\{\eta_t < x \mid q_t = j\}\, P\{q_t = j\} \\ &= p\, P\{\xi_t < x\} + (1-p)\, P\{X_0 < x\} \\ &= p\, F_\xi(x) + (1-p)\, F_0(x). \end{aligned}$$

Here, $F_\xi(x)$ denotes the DF of RV ξ_t, and

$$F_0(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

is the DF of $X_0 \overset{as}{=} 0$. Therefore, the DF $F_\eta(x)$ is a mixture of the continuous DF $F_\xi(x)$ and the discrete type DF $F_0(x)$ of the RV X_0, almost surely concentrated at $x = 0$. For these reasons, the distribution of η_t is usually called *the Contaminated Distribution (CD)*. Namely, the function $F_\eta(x)$ is continuous almost everywhere, i.e., the only point of discontinuity is $x = 0$, where the jump has the value $1 - p$ (Fig. 1). Due to this fact, some standard stochastic procedures cannot be applied in researching the properties of the RVs (Δ_t) .

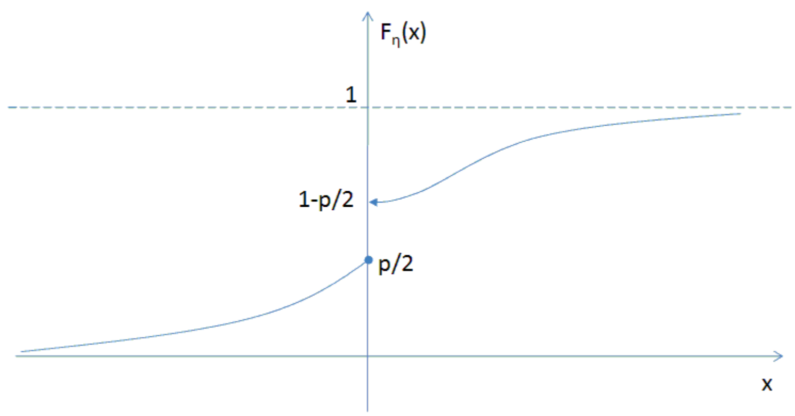

FIGURE 1. The distribution function of random variable with contaminated distribution.

In the following, we assume that the non-triviality and stationarity condition $0 < |a| < 1$ of the sequence (Δ_t) is satisfied. Thus, for any $t \in \mathbb{Z}$ and $k \in \mathbb{N}$ we can express Δ_t on $\Delta_{t-j},\ j = 1, \ldots, k$ as

$$\Delta_t = \sum_{j=0}^{k-1} a^j \xi_{t-j}\, q_{t-j-1} + a^k \Delta_{t-k}.$$

It is obvious that, when $k \to \infty$ and $|a| < 1$, we have

$$\Delta_t = \sum_{j=0}^{\infty} a^j \xi_{t-j}\, q_{t-j-1}, \tag{2}$$

where the above sum converges in mean-square and almost surely.

Let us now consider the DF of RVs Δ_t, denoted by $F_\Delta(x)$. Since the series (Δ_t) is stationary, it is clear that function $F_\Delta(x)$ does not depend on time parameter $t \in \mathbb{Z}$. Another application of conditional probability give to us

$$F_\Delta(x) = (G \otimes F_\eta)(x) = p\,(G \otimes F_\xi)(x) + (1-p)G(x),$$

where

$$G(x) := P\{a\Delta_t < x\} = \begin{cases} 1 - F_\Delta\left(\frac{x}{a}\right) & ,\quad a \in (-1, 0) \\ F_\Delta\left(\frac{x}{a}\right) & ,\quad a \in (0, 1) \end{cases},$$

and "$\otimes$" denotes the convolution of appropriate distribution functions, i.e.,

$$(G \otimes F_\eta)(x) = \int_{-\infty}^{+\infty} G(x-u) F_\eta(du) = \int_{-\infty}^{+\infty} G(x-u)\,[pF_\xi + (1-p)F_0]\,(du).$$

In order to find the unique expression for $F_\Delta(x)$, we find the explicit expression for the characteristic function (CF) of RVs $\eta_t = \xi_t q_t$, as

$$\begin{aligned} \mathcal{C}_\eta(u) &:= \int_{-\infty}^{+\infty} e^{iux} F_\eta(dx) = p \int_{x \neq 0} e^{iux} F_\xi(dx) + (1-p)e^{it0} \\ &= 1 + p\,(\mathcal{C}_\xi(u) - 1)\,, \end{aligned} \tag{3}$$

where $\mathcal{C}_\xi(u)$ is the (known) CF of RVs ξ_t. As RVs Δ_t, according to Eq.(2), are sums of uncorrelated RVs $a^j \eta_{t-j}$, $j = 0, 1, 2, \ldots$, their CF is

$$\mathcal{C}_\Delta(u) := \prod_{j=0}^{\infty} \mathcal{C}_\eta\left(a^j u\right) = \prod_{j=0}^{\infty} \left[1 + p\left(\mathcal{C}_\xi(a^j u) - 1\right)\right]. \tag{4}$$

Finally, from the Lévy's convergence theorem, the equality

$$F_\Delta(x) = \lim_{n\to\infty} \bigotimes_{j=0}^{n} \Big[p\, G_j + (1-p)F_0\Big](x) \tag{5}$$

holds, where $G_j(x)$ is the DF of $a^j \xi_t$ and, according to the Corollary 3.2 in [20], convolutions on the right side of Eq.(5) uniformly converge, when $n \to \infty$.

3. Non-homotopy approximation of $\mathcal{C}_\Delta(u)$

The previous stochastic analysis shows that the DF of Δ_t represents (in limit sense) the infinite convolution of DFs of mixed CD variables $a^j \eta_{t-j}$, where $j = 0, 1, 2, \ldots$ In that way, the analytic expression of $F_\Delta(x)$, as well as the appropriate CF $\mathcal{C}_\Delta(u)$, do not have a closed form. Therefore, they should be approximated with some numerical methods.

For this purpose, by using the fact that $\mathcal{C}_\Delta(u)$ is the Fourier-Stieltjes transform of probability density of RV Δ_t, we can apply the various approximation methods of $\mathcal{C}_\Delta(u)$. Firstly, let us consider the Laplace approximation for the logarithms of functions

$$f_j(u) = 1 + p\left[\mathcal{C}_\xi(a^j u) - 1\right], \qquad j = 0, 1, 2, \dots$$

which have a local maximum $f_j(0) = 1$. Thus, we find that

$$\ln \mathcal{C}_\Delta(u) = \lim_{k\to\infty} \sum_{j=0}^{k-1} \ln f_j(u) = \lim_{k\to\infty} \sum_{j=0}^{k-1} \left[\frac{(\ln f_j)''(0)}{2}\, u^2 + o_j(u^2) \right],$$

where $o_j(u^2)$ are infinitely small values of higher order than u^2 when $u \to 0$. In the case of Gaussian noise $\xi_t : \mathcal{N}(0, \delta^2)$, the last equality becomes

$$\ln \mathcal{C}_\Delta(u) = \lim_{k\to\infty} \sum_{j=0}^{k-1} \left[-\frac{p\, a^{2j} \delta^2\, u^2}{2} + o_j(u^2) \right],$$

and it can be easily shown that for an arbitrary but fixed $u \in \mathbb{R}$, values $o_j(u^2)$ are infinitely small values of higher order than a^{2j}, when $j \to \infty$. Therefore, we can write $o_j(u^2) = o\left(a^{2j}\right)$. Under the assumption that $k \to \infty$, and with previous notations, we have that

$$\sum_{j=0}^{k-1} \ln f_j(u) = -\frac{p\,\delta^2\, u^2}{2} \cdot \frac{1 - a^{2k}}{1 - a^2} + \sum_{j=0}^{k-1} o\left(a^{2j}\right),$$

which implies

$$\ln \mathcal{C}_\Delta(u) = -\frac{p\,\delta^2\, u^2}{2(1 - a^2)} + \lim_{k\to\infty} \sum_{j=0}^{k-1} o\left(a^{2j}\right). \tag{6}$$

Note that the first term in Eq.(6) corresponds to the logarithm of CF of Gaussian $\mathcal{N}\left(0, \frac{p\,\delta^2}{1-a^2}\right)$ distribution. Thus, the CF of Δ_t described by Eq.(4) differs from CF of the aforementioned Gaussian distribution by the multiplicative value

$$\exp\left[\lim_{k\to\infty} \sum_{j=0}^{k-1} o\left(a^{2j}\right)\right] \neq 1.$$

Fig. 2 (panel left) shows graphs of CFs of the CD variables $a^j \eta_{t-j}$, CF of the RV with Gaussian $\mathcal{N}\left(0, \frac{p\,\delta^2}{1-a^2}\right)$ distribution, and CF of the RV Δ_t. For all of them, we took $a = p = 0.5$ and $\delta = 1$. As can be easily seen, the CF of Δ_t is "between" two other classes of functions which emphasizes the specificity of DF of RVs Δ_t. On the other hand, in the right panel of Fig. 2 are shown standard Taylor's approximations of various order of the function $\mathcal{C}_\Delta(u)$.

4. Approximation of $\mathcal{C}_\Delta(u)$ with HAM

Firstly, let us notice that, in accordance to Eq.(3), the Eq.(4) can be rewritten as

$$\begin{aligned} \mathcal{C}_\Delta(u) &= \left[1 + p\left(\mathcal{C}_\xi(u) - 1\right)\right] \prod_{j=0}^{\infty} \left[1 + p\left(\mathcal{C}_\xi(a^{j+1}u) - 1\right)\right] \\ &= \mathcal{C}_\eta(u)\, \mathcal{C}_\Delta(au), \end{aligned} \tag{7}$$

where $\mathcal{C}_\eta(u) = (1 - p) + p\,\mathcal{C}_\xi(u)$ is the (known) CF of CD variables $\eta_t = \xi_t q_t$. In that way, Eq.(7) can be interpreted as the equation on unknown function $\mathcal{C}_\Delta(u)$, with the well-known

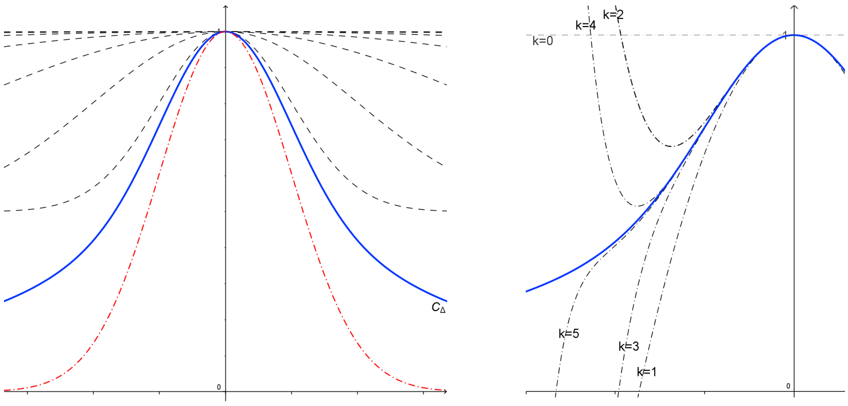

FIGURE 2. Panel left: CFs of CD variables (dashed lines), Gaussian distribution (dot-dashed line) and RVs Δ_t (solid line). Panel right: Standard Taylor's approximations of order $2k$, $k = 0, 1, 2 \ldots, 5$ (dashed lines) of the CF $\mathcal{C}_\Delta(u)$ (solid line).

initial condition $\mathcal{C}_\Delta(0) = 1$, which is valid for an arbitrary CF. In order to apply the HA method, we construct *the homotopy equation*:

$$(1-\alpha)L\left[\widetilde{\mathcal{C}}_\Delta(u;\alpha)\right] = \alpha\, h(u)\, N\left[\widetilde{\mathcal{C}}_\Delta(u;\alpha)\right], \tag{8}$$

where $\alpha \in (0,1)$ is the embedding parameter, $h(u) \neq 0$ is the auxiliary function,

$$L\left[\widetilde{\mathcal{C}}_\Delta(u;\alpha)\right] := \widetilde{\mathcal{C}}_\Delta(u;\alpha) - \widetilde{\mathcal{C}}_\Delta(au;\alpha)$$

is the *linear part*, and

$$N\left[\widetilde{\mathcal{C}}_\Delta(u;\alpha)\right] := \mathcal{C}_\eta(u)\,\widetilde{\mathcal{C}}_\Delta(au;\alpha) - \widetilde{\mathcal{C}}_\Delta(u;\alpha)$$

is the *non-linear ("true") part* of Eq.(8). It is easily to see that, when $\alpha = 0$, the unique solution of Eq.(8) is the CF of the RV $X_0 \overset{as}{=} 0$, i.e., $\mathcal{C}_\Delta(u;0) \equiv 1$. We notice this solution, usually called *an initial solution (approximation)*, with $\upsilon_0(u) := \widetilde{\mathcal{C}}_\Delta(u;0)$. On the other hand, remark that, when $\alpha = 1$, the Eq.(8) is equivalent to Eq.(7), with the "main" solution $\widetilde{\mathcal{C}}_\Delta(u;1) \equiv \mathcal{C}_\Delta(u)$.

The basic assumption of HAM is that the solution of homotopy equation can be expressed as the power series in α:

$$\widetilde{\mathcal{C}}_\Delta(u;\alpha) := \sum_{k=0}^{\infty} \alpha^k \upsilon_k(u), \tag{9}$$

where

$$\upsilon_k(u) := \frac{1}{k!} \cdot \frac{\partial^k \widetilde{\mathcal{C}}_\Delta(u;\alpha)}{\partial \alpha^k}\Big|_{\alpha=0} \tag{10}$$

are terms in Taylor series of the function $\widetilde{\mathcal{C}}_\Delta(u;p)$ with respect to α. Assuming that the auxiliary function $h(u)$ is chosen so that the series in Eq.(9) converges at $\alpha = 1$, the solution of the Eq.(7) will be

$$\mathcal{C}_\Delta(u) = \lim_{\alpha\to 1^-} \widetilde{\mathcal{C}}_\Delta(u;\alpha) = 1 + \sum_{k=1}^{\infty} \upsilon_k(u). \tag{11}$$

On the other hand, according to Taylor's expansions of CFs

$$\mathcal{C}_\Delta(u)=\sum_{k=0}^{\infty}\frac{(iu)^k}{k!}E\left(\Delta_t^k\right), \qquad \mathcal{C}_\Delta(au)=\sum_{k=0}^{\infty}\frac{(iau)^k}{k!}E\left(\Delta_t^k\right)$$

we can assume that $\upsilon_k(au)=a^k\upsilon_k(u)$, for any $k=0,1,2,\dots$, i.e.,

$$\widetilde{\mathcal{C}}_\Delta(au;\alpha)=\sum_{k=0}^{\infty}(a\alpha)^k\upsilon_k(u). \tag{12}$$

Therefore, by substituting Eq.(9) and Eq.(12) in the homotopy Eq.(8), we obtain

$$L\left[\sum_{k=0}^{\infty}\alpha^k\upsilon_k(u)\right]-\alpha\,L\left[\sum_{k=0}^{\infty}\alpha^k\upsilon_k(u)\right]=\alpha\,h(u)\,N\left[\sum_{k=0}^{\infty}\alpha^k\upsilon_k(u)\right]. \tag{13}$$

Now, by differentiating Eq.(13) k times with respect to α, and putting $\alpha=0$, we obtain

$$\begin{aligned} L\left[\upsilon_1(u)\right] &= h(u)\,N\left[\upsilon_0(u)\right],\\ k!\Big\{L\left[\upsilon_k(u)\right]-L\left[\upsilon_{k-1}(u)\right]\Big\} &= k\,h(u)\,\frac{\partial^{\,k-1}N\left[\widetilde{\mathcal{C}}_\Delta(u;\alpha)\right]}{\partial\,\alpha^{k-1}}\Bigg|_{\alpha=0}, \quad k=2,3,\dots \end{aligned}$$

These equalities, for an arbitrary $k=1,2,\dots$, can be rewritten, equivalently,

$$\begin{aligned}(1-a^k)\upsilon_k(u)-(1-a^{k-1})\upsilon_{k-1}(u) &= \frac{h(u)}{(k-1)!}\Big[a^{k-1}\mathcal{C}_\eta(u)\frac{\partial^{\,k-1}\widetilde{\mathcal{C}}_\Delta(au;\alpha)}{\partial\,\alpha^{k-1}}\\ &\quad -\frac{\partial^{\,k-1}\widetilde{\mathcal{C}}_\Delta(u;\alpha)}{\partial\,\alpha^{k-1}}\Big]\Big|_{\alpha=0}.\end{aligned}$$

Finally, according to the last equality and the Eq.(10), we have:

$$\begin{aligned}\upsilon_0(u) &\equiv 1,\\ \upsilon_1(u) &= h(u)\left[\mathcal{C}_\eta(u)-1\right],\\ \upsilon_k(u) &= \frac{1-h(u)+a^{k-1}\left(h(u)\,\mathcal{C}_\eta(u)-1\right)}{1-a^k}\,\upsilon_{k-1}(u), \quad k=2,3,\dots,\end{aligned}$$

i.e., we obtain the explicit expressions of $\upsilon_k(u)$:

$$\begin{aligned}\upsilon_0(u) &\equiv 1,\\ \upsilon_1(u) &= h(u)\left[\mathcal{C}_\eta(u)-1\right],\\ \upsilon_k(u) &= h(u)\left[\mathcal{C}_\eta(u)-1\right]\prod_{j=2}^{k}\frac{1-h(u)+a^{j-1}\left(h(u)\,\mathcal{C}_\eta(u)-1\right)}{1-a^j}, \quad k=2,3,\dots\end{aligned}$$

Let us remark that for obtained functions $\upsilon_k(u)$, on the condition of stationarity $0<|a|<1$ and for fixed but an arbitrary $u\in\mathbb{R}$, the radius of convergence of the power series in Eq.(9) is

$$r(u)=\lim_{k\to\infty}\left|\frac{\upsilon_{k-1}(u)}{\upsilon_k(u)}\right|=\lim_{k\to\infty}\frac{1-a^k}{\left|1-h(u)+a^{k-1}\left[h(u)\,\mathcal{C}_\eta(u)-1\right]\right|}=\frac{1}{|1-h(u)|}.$$

Thus, this power series uniformly converges on $\alpha\in(-r(u),r(u))$. According to Abel's theorem (see, for instance [21]), on the condition $r(u)\geq 1$ or, equivalently, $0\leq h(u)\leq 2$, function $\widetilde{\mathcal{C}}_\Delta(u;\alpha)$ is continuous from the left at $\alpha = 1$. Then, the solution of Eq.(8) will be obtain as it is given in the Eq.(11), i.e.,

$$
\begin{aligned}
(14)\quad \mathcal{C}_\Delta(u) &:= \lim_{\alpha\to 1^-} \widetilde{\mathcal{C}}_\Delta(u;\alpha) \\
&= 1 + h(u)\left[\mathcal{C}_\eta(u) - 1\right]\left[1 + \sum_{k=2}^{\infty}\prod_{j=2}^{k} \frac{1 - h(u) + a^{j-1}\left(h(u)\,\mathcal{C}_\eta(u) - 1\right)}{1 - a^j}\right].
\end{aligned}
$$

Notice that, according to Eq.(14), for an arbitrary $k = 0, 1, 2, \ldots$ we can compute the approximate solutions $\widehat{\mathcal{C}}^{(k)}_\Delta(u)$, as

$$
\begin{aligned}
\widehat{\mathcal{C}}^{(0)}_\Delta(u) &\equiv 1, \\
\widehat{\mathcal{C}}^{(k)}_\Delta(u) &= \widehat{\mathcal{C}}^{(k-1)}_\Delta(u) + \upsilon_k(u), \quad k = 1, 2, \ldots
\end{aligned}
$$

In the following, we give HAM approximations of CFs for some specific choices of auxiliary functions $h(u)$, where the parameters are $a = p = 0.5$, and the noise ξ_t has Gaussian $\mathcal{N}\left(0, \delta^2\right)$ distribution with $\delta = 1$. In this case, CFs of RVs ξ_t and η_t are, respectively, $\mathcal{C}_\xi(u) = e^{-u^2/2}$ and $\mathcal{C}_\eta(u) = (e^{-u^2/2} + 1)/2$.

- Auxiliary function $h(u) = \left[\mathcal{C}_\eta(u)\right]^{-1}$:

$$
\begin{aligned}
\upsilon_0(u) &\equiv 1 \\
\upsilon_1(u) &= \frac{4}{3}\left(\frac{e^{-\frac{u^2}{2}} - 1}{e^{-\frac{u^2}{2}} + 1}\right) \\
\upsilon_2(u) &= \frac{4^3}{3\cdot 15}\left(\frac{e^{-\frac{u^2}{2}} - 1}{e^{-\frac{u^2}{2}} + 1}\right)^2 \\
&\vdots \\
\upsilon_k(u) &= \frac{4^{k(k+1)/2}}{3\cdot 15\cdots(4^k - 1)}\left(\frac{e^{-\frac{u^2}{2}} - 1}{e^{-\frac{u^2}{2}} + 1}\right)^k
\end{aligned}
$$

- Auxiliary function $h(u) = 1$:

$$
\begin{aligned}
\upsilon_0(u) &\equiv 1 \\
\upsilon_1(u) &= \frac{2}{3}\left(e^{-\frac{u^2}{2}} - 1\right) \\
\upsilon_2(u) &= \frac{2^2}{3\cdot 15}\left(e^{-\frac{u^2}{2}} - 1\right)^2 \\
&\vdots \\
\upsilon_k(u) &= \frac{2^k}{3\cdot 15\cdots(4^k - 1)}\left(e^{-\frac{u^2}{2}} - 1\right)^k
\end{aligned}
$$

- Auxiliary function $h(u) = \mathcal{C}_\eta(u)$:

$$\begin{aligned}
\upsilon_0(u) &\equiv 1\\
\upsilon_1(u) &= \frac{b(u)^* - 2\left(e^{-\frac{u^2}{2}} - 1\right)}{3}\\
\upsilon_2(u) &= \frac{b(u) - 8\left(e^{-\frac{u^2}{2}} - 1\right)}{15}\cdot \upsilon_1(u)\\
&\vdots\\
\upsilon_k(u) &= \frac{b(u) - \frac{4^k}{2}\left(e^{-\frac{u^2}{2}} - 1\right)}{4^k - 1}\cdot \upsilon_{k-1}(u)
\end{aligned}$$

$^*b(u) = \left(e^{-\frac{u^2}{2}} - 1\right)^2 + 4\left(e^{-\frac{u^2}{2}} + 1\right)$

- Auxiliary function $h(u) = [\mathcal{C}_\eta(u)]^2$:

$$\begin{aligned}
\upsilon_0(u) &\equiv 1\\
\upsilon_1(u) &= \frac{d(u)^{**} - \left(e^{-\frac{u^2}{2}} - 1\right)^2 - 4\left(e^{-\frac{u^2}{2}} - 1\right)}{3}\\
\upsilon_2(u) &= \frac{d(u) - 4\left(e^{-\frac{u^2}{2}} - 1\right)^2 - 16\left(e^{-\frac{u^2}{2}} - 1\right)}{15}\cdot \upsilon_1(u)\\
&\vdots\\
\upsilon_k(u) &= \frac{d(u) - 4^{k-1}\left(e^{-\frac{u^2}{2}} - 1\right)^2 - 4^k\left(e^{-\frac{u^2}{2}} - 1\right)}{4^k - 1}\cdot \upsilon_{k-1}(u)
\end{aligned}$$

$^{**}d(u) = \frac{1}{2}\left(e^{-\frac{u^2}{2}} - 1\right)^3 + 3\left(e^{-\frac{u^2}{2}} - 1\right)^2 + 6\left(e^{-\frac{u^2}{2}} - 1\right)$

As an illustration, Fig. 3 shows graphics of the CF of RVs Δ_t, $t \in \mathbb{Z}$, as well as its first several approximations, obtained for the aforementioned choices of auxiliary functions $h(u)$. It is easily seen that approximations of $\mathcal{C}_\Delta(u)$ are better in the region around of the origin $u = 0$. This is in accordance to the fact that $u = 0$ is the point of maxima and, in general, it contains the most of informations around this point. Finally, notice that the best approximation has been obtained with the auxiliary function $h(u) \equiv 1$ (panel right above). In this case, the series in Eq.(9) has a maximum radius of convergence $r(u) = [1 - h(u)]^{-1} = +\infty$. We point out that this is also the case when HAM reduces to the so-called *Homotopy Perturbation Method (HPM)*, introduced by He [22]-[24]. Similarly as HAM, this approximation method was successfully applied in solving various, mainly physically-based problems (see, for instance [25]-[27]).

5. **Final remarks**

As it is well-known, by using the inverse Fourier-Stieltjes transform, it can be obtain the expression of density distribution of RVs Δ_t:

$$f_\Delta(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixu}\mathcal{C}_\Delta(u)du,$$

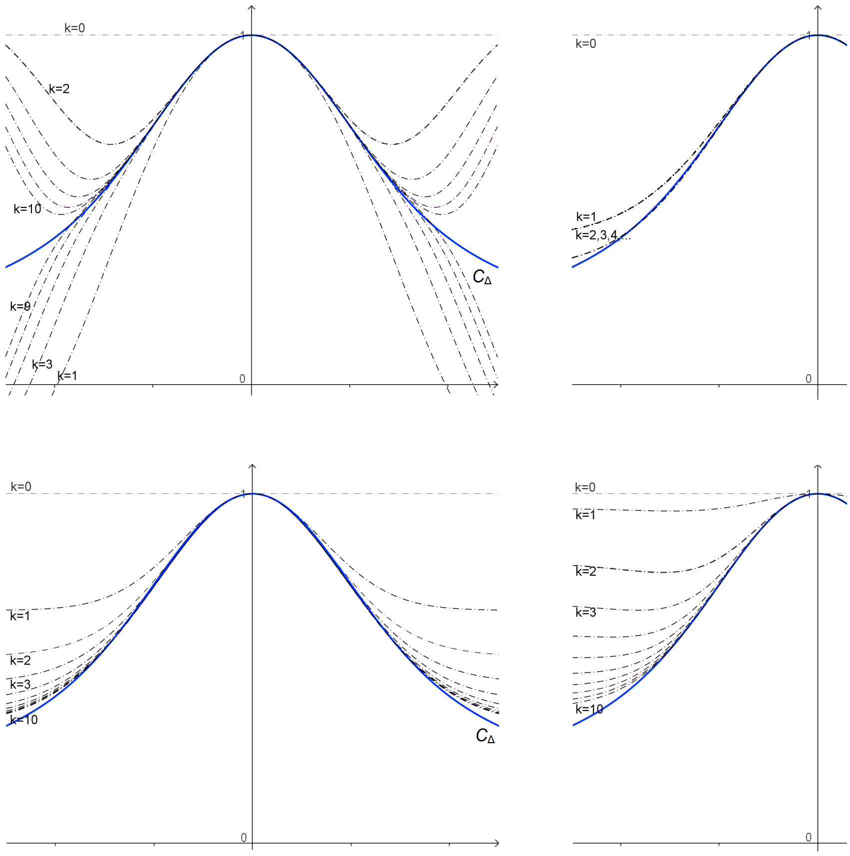

FIGURE 3. Graphics of CFs $\mathcal{C}_\Delta(u)$ (solid lines) and its various HA approximations (dashed lines).

as well as the appropriate DF:

$$F_\Delta(x) := \int_{-\infty}^{x} f_\Delta(y)dy.$$

According to these, as well as HAM approximations $\widehat{\mathcal{C}}_\Delta^{(k)}(u)$ of the CF $\mathcal{C}_\Delta(u)$, we can compute approximations of the density distribution of Δ_t:

$$\begin{aligned}\widehat{f}_\Delta^{(0)}(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixu}\, \widehat{\mathcal{C}}_\Delta^{(0)}(u)du = \begin{cases} +\infty, & x=0; \\ 0, & x \neq 0. \end{cases} \\ \widehat{f}_\Delta^{(k)}(x) &= \widehat{f}_\Delta^{(k-1)}(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixu}\, \upsilon_k(u)du, \quad k=1,2,\ldots.\end{aligned}$$

Here, the function $\widehat{f}_\Delta^{(0)}(x)$ represents the so-called unit impulse, i.e., the density function of the RV $X_0 \overset{as}{=} 0$.

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