A FINITE-RANK APPROACH TO THE NEWTON-STAR METHOD FOR SOLVING NONLINEAR EQUATIONS

by Hamza Guebbai[1], Nada Bouazila[2], Mohamed Zine Aissaoui[3] and Ridha Dida[4]

In this paper, we study the local convergence of the Finite-Rank Newton-Star (FRNS) method for approximating solutions to nonlinear equations in infinite-dimensional spaces. By discretizing the Newton-Star sequence, defined in a Hilbert space, we replace the inverse operator used in traditional Newton-Kantorovich and Newton-type methods with the adjoint operator, simplifying the process and reducing computational complexity. We provide convergence proofs under specific conditions and apply the method to solve a nonlinear Fredholm-Hermite integral equation using Green's kernel. Our numerical results demonstrate the efficiency and robustness of the (FRNS) method. Notably, it allows for initial guesses farther from the solution compared to Newton-Kantorovich, making it more flexible and applicable to a broader range of problems. This study highlights the potential of the (FRNS) method for practical applications in infinite-dimensional settings.

Keywords: Newton-Star method, Nonlinear equations, Finite-Rank approximation, Adjoint operator, Iterative methods

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1. Introduction

The resolution of nonlinear equations is a fundamental problem in many scientific disciplines. These equations are typically of the form:

$$G(V) = 0, \tag{1}$$

where G is a nonlinear operator defined on a Hilbert space $(\mathcal{X}, <.,.>)$ with the norm $\|.\|_{\mathcal{X}} = \sqrt{<.,.>}$ and 0 is the neutral element of $\mathcal{X}$. Finding exact solutions to these equations is often impractical, necessitating the use of iterative methods to obtain approximations[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The Newton-Kantorovich method, based on the iteration:

$$V_{k+1} = V_k - (G'(V_k))^{-1} G(V_k),$$

is widely used to solve these problems in Banach spaces. However, this method relies on computing the inverse of the Fréchet derivative operator $G'(V_k)$, which can be complex and computationally expensive, particularly in infinite-dimensional spaces.

[1]Professor, Laboratoire de Mathématiques Appliquées et de Modélisation, Université 8 Mai 1945 Guelma, B.P. 401, Guelma, 24000, Algeria

[2]Assistant Professor, Laboratoire de Mathématiques Appliquées et de Modélisation, Université 8 Mai 1945 Guelma, B.P. 401, Guelma, 24000, Algeria

[3]Professor, Laboratoire de Mathématiques Appliquées et de Modélisation, Université 8 Mai 1945 Guelma, B.P. 401, Guelma, 24000, Algeria

[4]Assistant Professor, Laboratoire de Mathématiques Appliquées et de Modélisation, Université 8 Mai 1945 Guelma, B.P. 401, Guelma, 24000, Algeria

When the equation is defined in an infinite-dimensional space, manipulating operators becomes more complicated and resource-intensive. The use of finite-rank approximations helps reduce the complexity of the problem, making the computations more tractable. In this context, our finite-rank Newton-Star method plays a crucial role. By replacing the inverse operator with the adjoint operator and applying a finite-rank approximation, we significantly simplify the process of obtaining approximate solutions while maintaining convergence to the solution.

Several recent works have sought to improve the convergence and applicability of these methods. For instance, Chuiko et al.[2] studied the conditions for the existence of solutions to weakly nonlinear periodic boundary-value problems using the Newton-Kantorovich method in the case of parametric resonance. Their work enhances the classification of critical and non-critical cases and proposes an iterative algorithm based on the generalized Newton-Kantorovich theorem for these complex situations.

Moreover, Amaral et al.[3] extended the solvability results of generalized equations in Banach spaces using a Kantorovich-like approach, introducing a Hölder-type condition to improve convergence analysis. Similarly, Regmi et al.[4] developed Kantorovich-type results for generalized equations without additional conditions, leading to finer Lipschitz constants and more precise convergence analysis.

Jaiswal[5] recently analyzed the semi-local convergence of a fourth-order Newton-type scheme, based on the assumption that a generalized Lipschitz condition is satisfied by the operator's first derivative. These results contribute to a better theoretical understanding of Newton schemes in Banach spaces and their application to integral equations.

In this context, we propose a new approach based on the Newton-Star method. Instead of using the inverse operator, we replace it with the adjoint operator, simplifying the process of obtaining a numerical approximation and making it more practical for infinite-dimensional applications. This sequence is defined as follows:

$$V_{k+1} = V_k - \alpha_k (G'(V_k))^* G(V_k), \quad \alpha_k \in \mathbb{R}_+^*, \tag{2}$$

where $(G'(V_k))^*$ represents the adjoint operator of the Fréchet derivative of G at the point V_k. Bouazila et al [1] introduced this new sequence (2) and demonstrated its convergence by assuming that equation (1) admits a unique solution $V_\infty \in \Omega$ an open set of $\mathcal{X}$, $G'(.)$ is δ-Lipschitzian over Ω and

$$\exists m > 0,\ \forall W \in \mathcal{X},\ \| G'(V_\infty)W \|_{\mathcal{X}} \geq m \| W \|_{\mathcal{X}} . \tag{3}$$

Using the philosophy developed by Grammont et al[10], we are going to construct the discretized version of the Newton-Star sequence (2) in the following form :

$$V_{k+1}^n = V_k^n - \alpha_{n,k} (G_n'(V_k^n))^* G(V_k^n), \tag{4}$$

where G_n' is a finite-rank approximation of the operator G'.

To ensure the convergence of this method, certain hypotheses must be verified. We formulate the following set of hypotheses to ensure the convergence of the Finite-Rank-Newton-Star (FRNS) (4):

$$\sup_{n\geq 1} \| G'_n(V_\infty) \| = M < +\infty, \tag{5}$$

$$\exists \delta > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ G'_n(.) \text{ is } \delta - \text{Lipschitzian, i.e.,}$$

$$\forall W_1, W_2 \in \mathcal{X},\ \| G'_n(W_1) - G'_n(W_2) \| \leq \delta \| W_1 - W_2 \|_{\mathcal{X}}, \tag{6}$$

$$\exists m > 0,\ \forall W \in \mathcal{X},\ \| G'(V_\infty)W \|_{\mathcal{X}} \geq m \| W \|_{\mathcal{X}}, \tag{7}$$

$$\lim_{n\to+\infty} \| \left(G'(V_\infty) - G'_n(V_\infty)\right)^* G'(V_\infty) \| = 0, \tag{8}$$

where, $\| \ . \ \|$ is the norm of the Banach space $BL(\mathcal{X})$, the space of bounded linear operator defined on $\mathcal{X}$ to itself, given by:

$$\forall T \in BL(\mathcal{X}),\ \| T \| = \sup \left\{ \| TW \|_{\mathcal{X}} : \| W \|_{\mathcal{X}} = 1 \right\}.$$

The objective of this study is to demonstrate the local convergence of this finite-rank Newton-Star method. We use the hypotheses (5,6,7,8) to prove this convergence and apply the theoretical results to solve a nonlinear Fredholm-Hermite integral equation with a Green's kernel. The numerical results show the efficiency of this new approach and confirm its convergence under practical conditions.

2. Main result

We are interesting in the numerical approximation of the unique solution $V_\infty \in \Omega$ of the equation (1), i.e.

$$G(V_\infty) = 0, \tag{9}$$

where $G : \Omega \subset \mathcal{X} \to \mathcal{X}$ is supposed to be Fréchet differentiable and to verify (7).

To define our new (FRNS) sequence, we introduce, for $n \geq 1$, the finite rank linear operators $G'_n(.)$ supposed to verify $(5-8)$. This family of operators can be easily constructed by assuming the Bouazila et al[1] hypothesis: $G'(.)$ is δ-Lipschitzian over Ω, and using the Projection approach[12]

$$G'_n(.)^P = \pi_n G'(.),$$

the Galerkien approch [12]

$$G'_n(.)^G = \pi_n G'(.)\pi_n,$$

or the Sloan approach [12]

$$G'_n(.)^S = G'(.)\pi_n,$$

where, $\pi_n : \mathcal{X} \to \mathcal{X}_n = [e_1, e_2, ..., e_n] \subset \mathcal{X}$ such that

$$\pi_n^2 = \pi_n,\ \forall W \in \mathcal{X},\ \lim_{n\to+\infty} \| W - \pi_n W \|_{\mathcal{X}} = 0.$$

It is clear that the three previous approaches verify the hypotheses $(5, 6)$.

Now, using Theorem 4.1 from[12] and supposing that the operator $G'(V_\infty)$ is compact, we easily deduce that Projection , Galerkien and Sloan approaches verify (8). At this stage we have just given a practical analytical framework, in which our family of hypotheses $(5-8)$ is easily applicable.

We introduce the (FRNS) sequence, for a fixed $n \in \mathbb{N}$, by

$$\begin{cases} V_0^n \text{ chosen in } \Omega, \\ V_{k+1}^n = V_k^n - \alpha_{n,k}(G'_n(V_k^n))^* G(V_k^n),\ k \geq 0. \end{cases}$$

Before studying the convergence of the sequence (FRNS), we prove the following lemma,

Lemma 2.1. *For $d > 0$, $\lambda \in]0,1[$ and $\xi \in]\lambda, 1[$, we define the sequence $\{u_k\}_{k\geq 0}$ by*

$$\begin{aligned} u_0 &= \frac{-1+\sqrt{1+4d(\xi-\lambda)}}{2d}, \\ u_{k+1} &= \lambda u_k + u_k^2 + d u_k^3, \;\; k \geq 0. \end{aligned}$$

Then, for all $k \geq 0$,

$$u_k \leq \frac{-1+\sqrt{1+4d(\xi-\lambda)}}{2d}\xi^k \rightarrow_{k\rightarrow+\infty} 0.$$

Proof. It is clear that $f(x) = \lambda x + x^2 + dx^3$ is increasing over $\left[0, \frac{-1+\sqrt{1+4d(1-\lambda)}}{2d}\right]$ and

$$f(0) = 0, \;\; f\left(\frac{-1+\sqrt{1+4d(1-\lambda)}}{2d}\right) = \frac{-1+\sqrt{1+4d(1-\lambda)}}{2d}.$$

Then, $\{u_k\}_{k\geq 0}$ is monotonic and included in $\left[0, \frac{-1+\sqrt{1+4d(1-\lambda)}}{2d}\right]$. But,

$$u_1 - u_0 = (\lambda - 1)u_0 + u_0^2 + d u_0^3 = -(1-\xi)\frac{-1+\sqrt{1+4d(\xi-\lambda)}}{2d} < 0,$$

which means that $\{u_k\}_{k\geq 0}$ is decreasing and for all $k \geq 0$

$$u_{k+1} = u_k(\lambda + u_k + d u_k^2) \leq u_k(\lambda + u_0 + d u_0^2) \leq \xi u_k.$$

Then, for all $k \geq 0$,

$$u_k \leq \frac{-1+\sqrt{1+4d(\xi-\lambda)}}{2d}\xi^k \rightarrow_{k\rightarrow+\infty} 0.$$

□

The previous lemma is used to show the convergence of our new sequence:

Theorem 2.1. *Let $n \geq 1$ chosen big enough such that*

$$|| \, (G'(V_\infty) - G'_n(V_\infty))^* G'(V_\infty) \, || \leq \mu,$$

let $\sigma \in \left]0, (\frac{m}{||G'(V_\infty)||})^2\right[$ such that $(1-\sigma+\mu) < 1$ and suppose that for all $k \geq 0$,

$$\frac{\sigma}{m^2} < \alpha_{n,k} < \frac{1}{|| \, G'(V_\infty) \, ||^2}.$$

Then, $\exists C > 0$,

$$|| \, V_0^n - V_\infty \, ||_{\mathcal{X}} < C \Rightarrow \lim_{k\rightarrow+\infty} || \, V_k^n - V_\infty \, ||_{\mathcal{X}} = 0.$$

Proof. For all $k \geq 0$, we have

$$\begin{aligned} V_{k+1}^n - V_\infty &= V_k^n - V_\infty - \alpha_{n,k}(G'_n(V_k^n))^*(G(V_k^n) - G(V_\infty)) \\ &= V_k^n - V_\infty - \alpha_{n,k}(G'_n(V_k^n))^*(G'(V_\infty)(V_k^n - V_\infty) + o(|| \, V_k^n - V_\infty \, ||_{\mathcal{X}}^2). \end{aligned}$$

Then, $\exists \eta > 0$,

$$\begin{aligned} || \, V_{k+1}^n - V_\infty \, ||_{\mathcal{X}} &\leq || \, I - \alpha_{n,k}(G'_n(V_k^n))^* G'(V_\infty) \, || \, || \, V_k^n - V_\infty \, ||_{\mathcal{X}} \\ &+ \eta \, || \, \alpha_{n,k}(G'_n(V_k^n))^* \, || \, || \, V_k^n - V_\infty \, ||_{\mathcal{X}}^2 . \end{aligned}$$

But,

$$\begin{aligned} \| \alpha_{n,k}(G_n'(V_k^n))^* \| &\leq \| G'(V_\infty) \|^{-2} \| G_n'(V_k^n) - G_n'(V_\infty) + G_n'(V_\infty) \| \\ &\leq \| G'(V_\infty) \|^{-2} (\delta \| V_k^n - V_\infty \|_{\mathfrak{X}} + M). \end{aligned}$$

And,

$$\begin{aligned} \| I - \alpha_{n,k}(G_n'(V_k^n))^* G'(V_\infty) \| &= \| I - \alpha_{n,k}(G'(V_\infty) - (G'(V_\infty) - G_n'(V_k^n)))^* G'(V_\infty) \| \\ &\leq \| I - \alpha_{n,k}(G'(V_\infty))^* G'(V_\infty) \| + | \alpha_{n,k} | H_k^n, \end{aligned}$$

where,

$$\begin{aligned} H_k^n &= \| (G'(V_\infty) - G_n'(V_k^n))^* G'(V_\infty) \| \\ &\leq \| (G'(V_\infty) - G_n'(V_\infty))^* G'(V_\infty) \| + \| (G'(V_\infty) - G_n'(V_k^n))^* G'(V_\infty) \| \\ &\leq \mu + \delta \| G'(V_\infty) \| \| V_k^n - V_\infty \|_{\mathfrak{X}} . \end{aligned}$$

We have,

$$\begin{aligned} \| I - \alpha_{n,k}(G'(V_\infty))^* G'(V_\infty) \| &= \sup_{\|U\|_{\mathfrak{X}}=1} |< (I - \alpha_{n,k}(G'(V_\infty))^* G'(V_\infty))U, U >| \\ &= \sup_{\|U\|_{\mathfrak{X}}=1} | 1 - \alpha_{n,k} \| (G'(V_\infty))U \|_{\mathfrak{X}}^2 | . \end{aligned}$$

But, for all $k \geq 0$ and all $U \in \mathfrak{X}$, $\| U \|_{\mathfrak{X}} = 1$,

$$0 < \sigma < \alpha_{n,k} m^2 \leq \alpha_{n,k} \| (G'(V_\infty))U \|_{\mathfrak{X}}^2 \leq \alpha_{n,k} \| (G'(V_\infty)) \|^2 < 1.$$

Then, for all $k \geq 0$,

$$\| I - \alpha_{n,k}(G'(V_\infty))^* G'(V_\infty) \| \leq 1 - \sigma.$$

Finally, we obtain for all $k \geq 0$,

$$\begin{aligned} \| V_{k+1}^n - V_\infty \|_{\mathfrak{X}} &\leq (1 - \sigma + \mu + \delta \| G'(V_\infty) \| \| V_k^n - V_\infty \|_{\mathfrak{X}}) \| V_k^n - V_\infty \|_{\mathfrak{X}} \\ &+ \| G'(V_\infty) \|^{-2} (\delta \| V_k^n - V_\infty \|_{\mathfrak{X}} + M)\eta \| V_k^n - V_\infty \|_{\mathfrak{X}}^2 \\ &\leq (1 - \sigma + \mu) \| V_k^n - V_\infty \|_{\mathfrak{X}} \\ &+ (\delta \| G'(V_\infty) \| + \eta M \| G'(V_\infty) \|^{-2}) \| V_k^n - V_\infty \|_{\mathfrak{X}}^2 \\ &+ \delta\eta \| G'(V_\infty) \|^{-2} \| V_k^n - V_\infty \|_{\mathfrak{X}}^3 . \end{aligned}$$

Let $\kappa > 0$ large as possible to obtain

$$\{W \in \mathfrak{X} : \| W - V_\infty \|_{\mathfrak{X}} < \kappa\} \subset \Omega,$$

and $V_0^n \in \Omega$ such that

$$\| V_0^n - V_\infty \|_{\mathfrak{X}} < C = \min \left\{ \kappa, \frac{-1 + \sqrt{1 + 4d(\xi - \lambda)}}{2d(\delta \| G'(V_\infty) \| + \eta M \| G'(V_\infty) \|^{-2})} \right\},$$

where, $\lambda = 1 - \sigma + \mu$, $\xi \in]\lambda, 1[$ and

$$d = (\delta \| G'(V_\infty) \| + \eta M \| G'(V_\infty) \|^{-2}) \delta\eta \| G'(V_\infty) \|^{-2} .$$

Then, for all $k \geq 0$,

$$(\delta \| G'(V_\infty) \| + \eta M \| G'(V_\infty) \|^{-2}) \| V_k^n - V_\infty \|_{\mathfrak{X}} \leq u_k,$$

where, $\{u_k\}_{k \geq 0}$ is the same sequence as in the previous lemma. Which achieves the proof. □

3. Numerical test

To illustrate the numerical application of our new (FRNS) sequence we consider the following Fredholm-Hermite integral equation: For all $t \in [0,1]$,

$$V(t) = \int_0^1 g(t,s)V^2(s)ds + f(t), \tag{10}$$

with green kernel g, defined by

$$(s,t) \in [0,1] \times [0,1] \mapsto g(s,t) := \begin{cases} s(1-t), & s \leq t, \\ t(1-s), & t \leq s. \end{cases}$$

Our study is done on the Hilbert space $\mathcal{X} = L^2(0,1)$ equipped with its usual scalar product and norm: For all $V, W \in \mathcal{X}$

$$\begin{aligned} < V, W > &= \int_0^1 V(s)W(s)ds, \\ \| V \|_{\mathcal{X}} &= \sqrt{\int_0^1 V(s)^2 ds}. \end{aligned}$$

We rewrite (10) for $t \in [0,1]$ as

$$G(V)(t) := V(t) - \int_0^1 g(t,s)V^2(s)ds - f(t) = 0. \tag{11}$$

If we fix $f(t) = -\frac{5}{48}t^4 - \frac{1}{24}t + 5$, then the exact solution of (11) is $V_\infty(t) = t$.

G is Fréchet differentiable on $\mathcal{X}$ with

$$\forall V \in \mathcal{X},\ G'(V) = I - T(V),$$

and $T(V)$ is a linear operator given for all $W \in \mathcal{X}$ by

$$\forall t \in [0,1],\ T(V)W(t) = 2\int_0^1 g(t,s)V(s)W(s)ds.$$

Using $\delta = 1$ and the fact that $T(.)$ is compact, we conclude that hypotheses $(5-8)$ are obtained for the following approximations:

$$\begin{aligned} G_n'(.)^P &= I - \pi_n T(.), \\ G_n'(.)^G &= I - \pi_n T(.)\pi_n, \\ G_n'(.)^S &= I - T(.)\pi_n, \end{aligned}$$

where π_n is given for a subdivision $n \geq 2,\ h = (n-1)^{-1},\ t_j = (j-1)h,\ 1 \leq j \leq n$, and for all $W \in \mathcal{X}$ by

$$\begin{aligned} \pi_n W &= \sum_{j=2}^{n} < W, e_j > e_j, \\ e_j(t) &= \frac{1}{\sqrt{h}} \begin{cases} 1, & t_{j-1} \leq t \leq t_j, \\ 0, & \text{otherwise}, \end{cases} \\ < W, e_j > &= \frac{1}{\sqrt{h}} \int_{t_{j-1}}^{t_j} x(s)ds. \end{aligned}$$

Table 1 shows that the three techniques are similar when taking $V_0(t) = t - a,\ a = 1$ and $n = 5$ Now to compare our method with the Finite Rank Newton-Kantorovich method (N-K)[10] with Galerkien projection, we fix $k = 5,\ n = 5$ and we vary a. The results are in table 2.

Methods	k=3	k=5	k=10
Projection	3.29e-04	1.55e-05	7.12e-07
Galerkien	4.59e-04	1.02e-05	6.62e-07
Sloan	2.89e-04	2.02e-05	5.53e-07

Table 1. The errors obtained by using the (FRNST) with different projection methods.

Methods	a=0.2	a=1	a=20
Projection	4.39e-07	1.55e-05	8.11e-04
Galerkien	3.32e-07	1.02e-05	9.01e-04
Sloan	3.89e-07	2.02e-05	7.78e-04
N-K	2.91e-09	NaN	NaN

Table 2. (FRNST) vs (N-K)

4. Conclusion

In this study, we introduced an innovative method, the Finite-Rank Newton-Star (FRNS), to solve nonlinear equations in infinite-dimensional spaces. By replacing the inverse operator with the adjoint operator and using finite-rank approximations, we simplified the solution process while ensuring convergence under specific conditions. Our theoretical results, supported by numerical tests on a Fredholm-Hermite integral equation, demonstrated the efficiency and robustness of this approach.

Compared to the Newton-Kantorovich (N-K) method, our approach has a key advantage: it works for initial points that are farther from the desired solution, whereas NK requires an initial guess close to the solution for convergence. This flexibility makes our method particularly suited to a wider range of complex problems. As a result, FRNS proves to be more robust and versatile, offering significant potential for practical applications.

Looking forward, the development of a fixed slope version of our method could represent a further enhancement. This variant would aim to stabilize the slopes used in the iterations, potentially improving convergence speed and reducing the requirements for initial approximations. Such an extension could be particularly beneficial for problems where the derivative of the nonlinear operator is highly sensitive.

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