INTERIOR APPROXIMATE CONTROLLABILITY OF A CLASS OF REACTION-DIFFUSION EQUATIONS OF SEMILINEAR PARABOLIC TYPE

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We prove in this paper a new result about approximate controllability of a semilinear parabolic equation. This equation is given by $y_t = a\Delta y + 1_\omega u + f(t,y,u)$, $x \in \Omega \subset \mathbb{R}^n (n \geq 1)$, $t \in (0,T]$, *with* $y = 0$ *on* $(0,T) \times \partial\Omega$, $y(0,x) = y^0(x)$, $x \in \Omega$, *where* Ω *is a bounded domain in* $\mathbb{R}^n$ *(*$n \in \mathbb{N}^*$*),* 1_ω *is the characteristic function of a subset* $\omega \subset \Omega$, u *is the distributed control.* $f : [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow f(t,y,u) \in \mathbb{R}$ *is a nonlinear function locally Lipschitz in* y *uniformly in* t *on bounded intervals and there are two bounded positive functions* b_0, $b_1 : [0,T] \longrightarrow \mathbb{R}$ *such that* $|f(t,y,u)| \leq b_1(t)\,|u| + b_0(t)$ *for all* y, $u \in \mathbb{R}$, $|z|$, $|u| \geq R$.

Keywords: compact semigroups, fixed point, existence of solutions, approximate controllability, semilinear parabolic equation.
MSC2020: 93B05, 35K58, 93C10.

1. Introduction

This paper is concerned with the interior approximate controllability of the following reaction-diffusion of semilinear parabolic type equation

$$y_t(t,x) = a\Delta y(t,x) + 1_\omega u(t,x) + f(t,y,u(t,x)),\ (t,x) \in (0,T] \times \Omega, \tag{1.1.a}$$

$$y(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \tag{1.1.b}$$

$$y(0,x) = y^0(x), \quad x \in \Omega, \tag{1.1.c}$$

where $a > 0$ is a real constant, Ω is a bounded open set in $\mathbb{R}^n, y^0 \in L^2(\Omega)$, $\omega \subset \Omega$ is an open nonempty subset of Ω, 1_ω denotes the characteristic function of ω, u is a control function belonging to $L^2(0,T;L^2(\Omega))$ and the nonlinear function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is locally Lipschitz in y uniformly in t on bounded intervals and there are two positive bounded functions b_0, $b_1 \in \mathcal{B}([0,T]\,;\mathbb{R})$ such that

$$|f(t,y,u)| \leq b_1(t)\,|u| + b_0(t), \tag{1.2}$$

for all y, $u \in \mathbb{R}$, $|y|$, $|u| \geq R$.

Definition 1.1. The system (1.1) is said to be approximately controllable on $[0,T]$ if for every y^0, $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the solution y of (1.1) corresponding to u verifies $\left\|y(T) - y^1\right\|_2 < \varepsilon$.

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The approximate controllability of the problem (1.1) was studied in [3] and [4] in the particular case where the nonlinear perturbation f is independent of the variables t and u

$$y_t(t,x) = \Delta y(t,x) + 1_\omega u(t,x) + f(y(t,x)), \ \ (t,x) \in (0,T] \times \Omega, \tag{1.3.a}$$

$$y(t,x) = 0, \ \ (t,x) \in (0,T) \times \partial\Omega, \tag{1.3.b}$$

$$y(0,x) = y^0(x), \ \ x \in \Omega, \tag{1.3.c}$$

and f is sublinear.

$$|f(y)| \leq a_1 |y| + a_0, \ \ a_0, \ a_1 \in \mathbb{R} \ \text{ are constants.} \tag{1.4}$$

The problem (1.1) was studied in [7] . It is proved that if there exist two real constants a_1, b_1 with $a_1 \neq -1$ such that

$$\sup_{(t,z,u)} |f(t,y,u) - a_1 y - b_1 u| < \infty \tag{1.5}$$

then, the equation is approximately controllable.

Recently, H. Leiva, N. Merentes and Sanchez proved in [8] the approximate controllability of the following problem

$$y_t(t,x) = \Delta y(t,x) + 1_\omega u(t,x) + f(t,y,u(t,x)), \ (t,x) \in (0,T] \times \Omega, \tag{1.6.a}$$

$$y(t,x) = 0, \ \ (t,x) \in (0,T) \times \partial\Omega, \tag{1.6.b}$$

$$y(0,x) = y^0(x), \ \ x \in \Omega \ , \tag{1.6.c}$$

if f is smooth enough and there are real constants $a_1, b_0, b_1 \in \mathbb{R}$, $R > 0$ and $1/2 \leq \beta < 1$ such that

$$|f(t,y,u) - a_1 y| \leq b_1 |u|^\beta + b_0. \tag{1.7}$$

For more information on basic results on exact and approximate controllability of parabolic equations, you can consult [10] and [11] and [5] and references therein.

2. Notations and preliminaries

Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $T > 0$ a real constant, we denote by
$C_0^\infty(\Omega)$: the space of infinitely continuously differentiable functions on Ω and compactly supported in Ω.
$L^2(\Omega) = \left\{ u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R} \ \text{ measurable function and } \int_\Omega |u(x)|^2 \, dx < \infty \right\}$, with norm

$$\|u\|_2 = \left\{ \int_\Omega |u(x)|^2 \, dx \right\}^{1/2} \tag{2.1}$$

$$L^2(0,T;L^2(\Omega)) := \left\{ \begin{array}{c} u : (0,T) \subset \mathbb{R} \longrightarrow L^2(\Omega) \\ \text{measurable function and } \displaystyle\int_0^T \|u(s)\|_2^2 \, ds = \int_0^T \int_\Omega |u(s,x)|^2 \, dxds < \infty \end{array} \right\},$$

with norm

$$\|u\|_{2,2} = \left\{ \int_0^T \int_\Omega |u(s,x)|^2 \, dxds \right\}^{1/2} = \left(\int_0^T \|u(t)\|_2^2 \, dt \right)^{1/2} \tag{2.2}$$

$C([0,T]\,;L^2(\Omega)) := \big\{u : [0,T] \longrightarrow L^2(\Omega)$ continuous function$\big\}$ with norm

$$\|u\|_{2,\infty} = \max_{t\in[0,T]} \|u(t)\|_2 \tag{2.3}$$

$\mathcal{B}(E;X) : = \{u : E \longrightarrow X \; : u$ is a bounded function on $E\}$, where E is a nonempty set and X a Banach space.
$B(v,r)$ the open ball of center v and radius r.
$B'(v,r)$ the closed ball of center v and radius r.
$\partial B(v,r)$ the boundary of $B(v,r)$.
$t_{\max}$ the maximal interval of existence.

To prove the approximate controllability of (1.1) we need to some known results.

Theorem 2.1 (Rothe's theorem [6], page 129). Let $E(\tau)$ be a Hausdorff topological vector space. Let $B \subset E$ be a closed convex subset such that the zero of E is contained in the interior of B. Let $\Phi : B \longrightarrow E$ be a continuous mapping with $\Phi(B)$ relatively compact in E and $\Phi(\partial B) \subset B$. Then there is a point $x^* \in B$ such that $\Phi(x^*) = x^*$.

Theorem 2.2 (cf. [12] on page 286 and 307, [1] on page 192). Let us consider the following classical boundary-eigenvalue problem for the laplacian

$$\begin{cases} -\Delta u = \lambda u, & \text{on } \Omega \\ \;\; u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a nonempty bounded open set in $\mathbb{R}^N$ and $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.
This problem has a countable system of eigenvalues $0 < c \le \lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots$ and $\lambda_j \to +\infty$ as $j \to \infty$ and
(i) All the eigenvalues λ_j have finite multiplicity m_j equal to the dimension of the corresponding eigenspace S_j.
(ii) Let $\{\varphi_{jk}\}_{k=1}^{m_j}$ a basis of the S_j for every j, then the eigenvectors $\{\varphi_{jk}\}_{k=1,j=1}^{m_j,\infty}$ form a complete orthonormal system in the space $L^2(\Omega)$. Hence for all $u \in L^2(\Omega)$ we have $u = \sum_{j=1}^{\infty}\sum_{k=1}^{m_j} \langle u, \varphi_{jk}\rangle \varphi_{jk}$. If we put $E_j u = \sum_{k=1}^{m_j} \langle u, \varphi_{jk}\rangle \varphi_{jk}$ then we get $u = \sum_{j=1}^{\infty} E_j u$.
Also, the eigenfunctions $\{\varphi_{jk}\}_{k=1,j=1}^{m_j,\infty} \subset C_0^\infty(\Omega) \cap H_0^1(\Omega)$.
(iii) For all $u \in D(-\Delta)$ we have $-\Delta u = \sum_{j=1}^{\infty} \lambda_j E_j u$.
(iv) The operator Δ generates an analytic semigroup S on $L^2(\Omega)$ defined by

$$S(t)u = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j u \quad \text{and} \quad \|S(t)\| \le e^{-\lambda_1 t}, \text{ for all } t \ge 0$$

3. **Main results**

Let :
$A : D(A) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ be the linear operator defined by $Au = -a\Delta u$.
$B_\omega : L^2(\Omega) \longrightarrow L^2(\Omega)$ the operator defined by $B_\omega u = 1_\omega u$. The operator B_ω is trivially linear and bounded.
$f : [0,T] \times L^2(\Omega) \times L^2(\Omega) \longrightarrow L^2(\Omega) \; : (t,y,u) \longrightarrow f(t,y,u)$ be a function.

Then the equation (1.1a) can be written in the form

$$y'(t) = -Ay(t) + B_\omega u(t) + f(t, y(t), u(t))$$

Proposition 3.1. Let $u \in L^2\left(0,T;L^2(\Omega)\right)$ be a fixed function and let f be continuous in t on $[0,T]$ and Lipschitz in y uniformly in t with constant $L = L(u) > 0$

$$\|f(t,v_1,u) - f(t,v_2,u)\|_2 \leq L\,\|v_1 - v_2\|_2\,, \tag{3.1}$$

for all $t \in [0,T]$ and all $v_1,\, v_2 \in L^2(\Omega)$
Then, the problem (1.1) has a unique solution $y \in L^2(0,T;L^2(\Omega))$.

Proof. Define the function $F : [0,T] \times L^2(\Omega) \longrightarrow L^2(\Omega)$ as $F(t,v) = B_\omega u + f(t,v,u)$ and the application $\Psi : C([0,T]\,;L^2(\Omega)) \longrightarrow C([0,T]\,;L^2(\Omega))$ by

$$\begin{aligned}\Psi(v)(t) &= S(t)y^0 + \int_0^t S(t-s)F(s,v(s))ds \\ &= S(t)y^0 + \int_0^t S(t-s)\left[B_\omega u(s) + f(s,v(s),u(s))\right]ds \end{aligned} \tag{3.2}$$

where S is the C_0-semigroup generated by the operator $-A$.
It folows from (3.2) that

$$\|\Psi(v_1)(t) - \Psi(v_2)(t)\|_2 \leq L\,\|v_1 - v_2\|_{\infty,2}\,, \tag{3.3}$$

for all $t \in [0,T]$ and all $v_1, v_2 \in L^2\left(0,T;L^2(\Omega)\right)$.
Then the theorem 1.2 on page 184 in the reference [9] is applicable, and the application Ψ admits a unique fixed point $u \in C([0,T]\,;L^2(\Omega))$, this fixed point is the solution (mild solution in reality) of the problem (1.1)

$$y(t) = S(t)y^0 + \int_0^t S(t-s)\left[B_\omega u(s) + f(s,v(s),u(s))\right]ds \tag{3.4}$$

As $C([0,T]\,;L^2(\Omega)) \subset L^2(0,T;L^2(\Omega))$, the solution is in $L^2(0,T;L^2(\Omega))$.

Proposition 3.2. Let $u \in L^2\left(0,T;L^2(\Omega)\right)$ be a fixed function and let $f : [0,T] \times L^2(\Omega) \times L^2(\Omega) \longrightarrow L^2(\Omega) : (t,v,u) \longrightarrow f(t,v,u)$ be a continuous function in t on $[0,T]$ and locally Lipschitz in v uniformly in t on bounded intervals, i.e; for every $t' \geq 0$ and every constant $c \in \mathbb{R}_+$ there exists a constant $L = L(c,t',u) \in \mathbb{R}_+$ such that

$$\|f(t,v_1,u) - f(t,v_2,u)\|_2 \leq L(c,t',u)\,\|v_1 - v_2\|_2\,, \tag{3.5}$$

for all $t \in [0,t']$ and all $v_1,\, v_2 \in B'(0,c)$.
where $B'(0,c)$ is the closed ball in $L^2(\Omega)$ with center 0 and radius c.

Then, there is a $t_{\max} \in\,]0,\infty[$ such that the problem (1.1) has a unique mild solution $y \in C([0,t_{\max}[\,;L^2(\Omega))$. Moreover, if $t_{\max} < \infty$ then $\lim\limits_{t\uparrow t_{\max}} \|y(t)\|_2 = \infty$.

Proof. The same proof as in Proposition 3.1, and apply the theorem 1.4 on page 185 in [9].

Theorem 3.3. Assume that the equation (1.1) admits a solution $y \in L^2(0,T;L^2(\Omega))$ where $0 < T < t_{\max}$ (for exemple if (3.1) or (3.2) is verified) and assume also that there are two positive bounded functions $b_0, b_1 \in \mathcal{B}([0,T];\mathbb{R})$ and a real constant $R > 0$ such that

$$|f(t,y,u)| \leq b_1(t)\,|u| + b_0(t) \tag{3.6}$$

for all $y, u \in \mathbb{R}, |y|, |u| \geq R$.

Then, we can construct a sequence of controls $(u_{\alpha_k})_{k\in\mathbb{N}} \subset L^2(0,T;L^2(\Omega))$ steering the system (1.1) from any initial state y^0 to an ε-neighborhood of a final state y^1 at time $T \in \mathbb{R}_+^*$ for every $\varepsilon \in \mathbb{R}_+^*$:

$$\lim_{k\to\infty} y_{\alpha_k} \equiv \lim_{k\to\infty} \left\{ e^{\gamma T} S(T)y^0 + G_1(u_{\alpha_k}) + G_2(u_{\alpha_k}) \right\} = y^1 \tag{3.7}$$

where y_{α_k} is the corresponding solution to the control u_{α_k} and $G_1, G_2 : L^2(0,T;L^2(\Omega)) \longrightarrow L^2(\Omega)$ are defined as

$$G_1 u = \int_0^T e^{-\gamma(T-s)} S(T-s) B_\omega u(s) ds, \tag{3.8}$$

$$G_2 u = \int_0^T e^{-\gamma(T-s)} S(T-s) g(s,y(s),u(s)) ds. \tag{3.9}$$

Whence, the system (1.1) is approximately controllable on $[0,T]$.

Proof. Our proof is inspired from the work in [8]. If we put for every real positive constant γ

$$g(t,u,w) = f(t,y,u) + \gamma y \tag{3.10}$$

then the equation (1.1a) become

$$y_t(t,x) = a\Delta y(t,x) - \gamma y(t,x) + 1_\omega u(t,x) + g(t,y(t,x),u(t,x)),\ (t,x) \in (0,T] \times \Omega \tag{3.11}$$

The solution y verify the integral equation

$$y(t) = e^{-\gamma t} S(t) y^0 + \int_0^t e^{-\gamma(t-s)} S(t-s) \left[B_\omega u(s) + g(s,y(s),u(s)) \right] ds, \tag{3.12}$$

where S is the C_0-semigroup generated by the operator $-A$, then we have from Theorem 2.2 (iv) that

$$S(t)u = \sum_{j=1}^{\infty} e^{-a\lambda_j t} E_j u \text{ and } \|S(t)\|_2 \leq e^{-a\lambda_1 t}, \text{ for all } t \geq 0. \tag{3.13}$$

If we put

$$Gu = G_1 u + G_2 u \tag{3.14}$$

From (3.8) and (3.9) into (3.12), the solution y is written as

$$y(T) = e^{-\gamma T} S(T) y^0 + Gu \tag{3.15}$$

Claim 1. We have

$$\|g(t,y(t),u(t)\|_2 \leq \sqrt{2}\gamma \|y(t)\|_2 + B_1 \|u(t)\|_2 + B_0 \tag{3.16}$$

where

$$B_1 = \sqrt{2} \sup_{s\in[0,T]} b_1(s), \quad B_0 = \sqrt{2\mu(\Omega)} \sup_{s\in[0,T]} b_0(s) \tag{3.17}$$

In fact, we have

$$\begin{aligned}\|g(t,y(t),u(t)\|_2^2 &= \int_\Omega |g(t,y(t,x),u(t,x)|^2\,dx\\ &\le \int_\Omega (\gamma\,|y(t,x)| + b_1(t)\,|u(t,x)| + b_0(t))^2\,dx\\ &\le 2\int_\Omega \left(\gamma^2\,|y(t,x)|^2 + b_1^2(t)\,|u(t,x|^2 + b_0^2(t)\right)dx\end{aligned}$$

then

$$\|g(t,y(t),u(t)\|_2 \le \sqrt{2}\gamma\,\|y(t)\|_2 + \sqrt{2}\sup_{s\in[0,T]} b_1(s)\,\|u(t)\|_2 + \sup_{s\in[0,T]} b_0(s)\sqrt{2\mu\,(\Omega)}$$

If we take $\gamma = 0$ we get

$$\|f(t,y(t),u(t)\|_2 \le B_1\,\|u(t)\|_2 + B_0 \tag{3.18}$$

Claim 2. Let $u \in L^2\left(0,T;L^2(\Omega)\right)$ a control, then the corresponding solution y verifies

$$\|y\|_{2,2} \le \sqrt{2}\;\left[\left\|y^0\right\| + B_0\sqrt{T} + B_2\sqrt{T}\,\|u\|_{2,2}\right] \tag{3.19}$$

In fact, the solution y verify (3.4)

$$y(t) = S(t)y^0 + \int_0^t S(t-s)\left[B_\omega u(s) + f(s,y(s),u(s))\right]ds$$

Then, by (3.18) we get

$$\|y(t)\|_2 \le \left\|u^0\right\| + B_0\sqrt{T} + B_2\int_0^t \|u(s)\|_2\,ds \tag{3.20}$$

where $B_2 = 1 + B_1$.

By Cauchy-Schwarz inequality we get $\int_0^t \|u(s)\|_2\,ds \le \sqrt{t}\left(\int_0^t \|u(s)\|_2^2\,ds\right)^{1/2}$, which implies

$$\left(\int_0^t \|u(s)\|_2\,ds\right)^2 \le t\int_0^t \|u(s)\|_2^2\,ds \le T\,\|u\|_{2,2}^2 \tag{3.21}$$

From (3.21) into (3.20) we get

$$\begin{aligned}\int_0^t \|y(t)\|_2^2\,dt &\le 2\;\left[\left(\left\|y^0\right\| + B_0\sqrt{T}\right)^2 + B_2^2\left(\int_0^t \|u(s)\|_2\,ds\right)^2\right]\\ &\le 2\left[\left(\left\|y^0\right\| + B_0\sqrt{T}\right)^2 + B_2^2 T\,\|u\|_{2,2}^2\right]\end{aligned}$$

Whence (3.19).

Claim 3. Let $C_0 = \left\| G_1^* \left(\alpha I + G_1 G_1^*\right)^{-1} \right\| := \left\| G_1^* \left(\alpha I + G_1 G_1^*\right)^{-1} \right\|_{L^2(\Omega)\to L^2(0,T;L^2(\Omega))}$.
Then, there exist two real constants $\gamma_0 > 0$ and $R > 0$ such that

$$\frac{\|G_2 u\|_2}{\|u\|_{2,2}} \leq \frac{1}{2C_0} \tag{3.22}$$

for all $\gamma \geq \gamma_0$ and $\|u\|_{2,2} \geq R$.

By Cauchy-Schwarz inequality and (3.16)

$$\begin{aligned}
\|G_2 u\|_2 &\leq \int_0^T e^{-(\gamma+a\lambda_1)(T-s)} \|g(s, y(s), u(s))\|_2 \, ds \\
&\leq \left(\int_0^T e^{-2(\gamma+a\lambda_1)(T-s)} ds\right)^{1/2} \left(\int_0^T \|g(s, v(s), u(s))\|_2^2 \, ds\right)^{1/2} \\
&\leq \left(\frac{-1}{2(\gamma+a\lambda_1)} e^{-2(\gamma+a\lambda_1)T} + \frac{1}{2(\gamma+a\lambda_1)}\right)^{1/2} \\
&\left(\int_0^T \left(\sqrt{2}\gamma \|y(s)\|_2 + B_1 \|u(s)\|_2 + B_0\right)^2 ds\right)^{1/2} \\
&\leq \left(\frac{-1}{2(\gamma+a\lambda_1)} e^{-2(\gamma+a\lambda_1)T} + \frac{1}{2(\gamma+a\lambda_1)}\right)^{1/2} \\
&\left(\int_0^T \left(4\gamma^2 \|y(s)\|_2^2 + 4B_1^2 \|u(s)\|_2^2 + 4B_0^2\right) ds\right)^{1/2} \\
&\leq \left(\frac{-1}{2(\gamma+a\lambda_1)} e^{-2(\gamma+a_1\lambda_1)T} + \frac{1}{2(\gamma+a\lambda_1)}\right)^{1/2} \left(2\gamma \|y\|_{2,2} + 2B_1 \|u\|_{2,2} + 2B_0\sqrt{T}\right)
\end{aligned} \tag{3.23}$$

From (3.19) into (3.23)

$$\|G_2 u\|_2 \leq \left(\frac{-1}{2(\gamma+a\lambda_1)} e^{2(-\gamma-a_1\lambda_1)T} + \frac{1}{2(\gamma+a\lambda_1)}\right)^{1/2} \left(B_3 \|u\|_{2,2} + B_4\right) \tag{3.24}$$

where $B_3 = 2\sqrt{2}\gamma B_2 \sqrt{T} + 2B_1$, $B_4 = 2\sqrt{2}\gamma \left(\left\|y^0\right\| + B_0\sqrt{T}\right) + 2B_0\sqrt{T}$.

As $\lim_{\gamma\to+\infty} \left(\frac{-1}{2(\gamma+a\lambda_1)} e^{-2(\gamma+a\lambda_1)T} + \frac{1}{2(\gamma+a\lambda_1)}\right)^{1/2} = 0$, then exists a real constant $\gamma_0 > 0$ such that, for all $\gamma \geq \gamma_0$

$$\left(\frac{-1}{2(\gamma+a\lambda_1)} e^{-2(\gamma+a\lambda_1)T} + \frac{1}{2(\gamma+a\lambda_1)}\right)^{1/2} \leq \frac{1}{4B_3 C_0}, \tag{3.24a}$$

and as $\lim_{\|u\|_{2,2}\to+\infty} \frac{1}{\|u\|_{2,2}} = 0$, then, there exists a real constant $R > 0$ big enough such that for all $\|u\|_{2,2} \geq R$

$$\frac{1}{\|u\|_{2,2}} \leq \frac{B_3}{B_4} \tag{3.24b}$$

Whence, for all for $\gamma \geq \gamma_0$ and all $\|u\|_{2,2} \geq R$ we get the relation (3.22) from (3.24a)-(3.24b) into (3.24) .

Claim 4. For each $v \in L^2(\Omega)$ fixed, define the family of nonlinear opperators $\{K_\alpha\}_{\alpha\in]0,1]}$ by $K_\alpha : L^2(0,T;L^2(\Omega)) \longrightarrow L^2(0,T;L^2(\Omega))$ and

$$K_\alpha(u) = G_1^*\left(\alpha I + G_1G_1^*\right)^{-1}(v - G_2(u)) \tag{3.25}$$

Then, there exist two real constants $0<\rho<1$ and $R_\alpha>0$ such that for all $\|u\|_{2,2} \geq R_\alpha$ we have

$$\frac{\|K_\alpha(u)\|_{2,2}}{\|u\|_{2,2}} \leq \rho \tag{3.26}$$

In fact, from the definition of K_α we have

$$\begin{aligned}\frac{\|K_\alpha(u)\|_{2,2}}{\|u\|_{2,2}} &\leq \frac{\left\|G_1^*\left(\alpha I + G_1G_1^*\right)^{-1} v\right\|_{2,2}}{\|u\|_{2,2}} + \frac{\left\|G_1^*\left(\alpha I + G_1G_1^*\right)^{-1} G_2(u)\right\|_{2,2}}{\|u\|_{2,2}} \\ &\leq \frac{\left\|G_1^*\left(\alpha I + G_1G_1^*\right)^{-1} v\right\|_{2,2}}{\|u\|_{2,2}} + \left\|G_1^*\left(\alpha I + G_1G_1^*\right)^{-1}\right\| . \frac{\|G_2(u)\|_{2,2}}{\|u\|_{2,2}}\end{aligned}$$

As $\lim\limits_{\|u\|_{2,2}\to\infty} \frac{\left\|G_1^*(\alpha I+G_1G_1^*)^{-1}v\right\|_{2,2}}{\|u\|_{2,2}} = 0$ and $\frac{\|G_2(u)\|_{2,2}}{\|u\|_{2,2}} \leq \frac{1}{2C_0}$ for $\gamma \geq \gamma_0$ and $\|u\|_{2,2} \geq R$ (from the claim 3), then (3.26) is verified.

Claim 5. For every $\alpha \in]0,1]$, the operator K_α admits a fixed point $u_\alpha \in L^2(0,T;L^2(\Omega))$: $K_\alpha(u_\alpha) = u_\alpha$.

In fact, from claim 4 : $K_\alpha(\partial B(0,R_\alpha)) \subset B(0,R_\alpha)$. Also, the C_0-semigroup S genrated by $-A = a\Delta$ in $L^2(\Omega)$ is compact ([2] on page 394), then G_2 and K_α are compacts. Applying theorem 2.1, there exists a fixed point u_α of K_α.

Claim 6. The family of fixed point $\{u_\alpha\}_{\alpha\in]0,1]}$ is bounded in $L^2(0,T;L^2(\Omega))$.

In fact, suppose the contrary; then, there exists a sequence $(u_{\alpha_k})_{k\in\mathbb{N}} \subset \{u_\alpha\}_{\alpha\in]0,1]}$ such that

$$\lim_{k\to\infty} \|u_{\alpha_k}\|_{2,2} = \infty \tag{3.27}$$

As $K_{\alpha_k}(u_{\alpha_k}) = u_{\alpha_k}$ we obtain that $\frac{\left\|K_{\alpha_k}(u_{\alpha_k})\right\|_{2,2}}{\left\|u_{\alpha_k}\right\|_{2,2}} = 1$, for all $k \in \mathbb{N}$, which contradicts the inequality (3.26).

Claim 7 . There exists a sequence $(G_2(u_{\alpha_k}))_{k\in\mathbb{N}} \subset \{G_2(u_\alpha)\}_{\alpha\in]0,1]}$ such that for every $v \in L^2(\Omega)$

$$\lim_{k\to\infty} \left\{\alpha_k \left(\alpha_k I + G_1G_1^*\right)^{-1}\left(v - G_2(u_{\alpha_k})\right)\right\} = 0 \tag{3.28}$$

In fact; according to claim 6 and Bolzano Weierstrass theorem, the set $(G_2(u_\alpha))_{\alpha\in]0,1]}$ admits a convergent subsequence $(G_2(u_{\alpha_k}))_{k\in\mathbb{N}}$ in $L^2(\Omega)$.
The rest of the proof is the same as in ([8] on page 6).

Finally; we have for all $k \in \mathbb{N}$

$$G_1(u_{\alpha_k}) + G_2(u_{\alpha_k}) = v - \alpha_k\left(\alpha_k I + G_1G_1^*\right)^{-1}\left(v - G_2(u_{\alpha_k})\right) \tag{3.29}$$

Really; from claim 5, for all $k \in \mathbb{N} : u_{\alpha_k} = K_{\alpha_k}(u_{\alpha_k})$, but from (3.25) we can write

$$u_{\alpha_k} = G_1^* \left(\alpha_{\alpha_k} I + G_1 G_1^*\right)^{-1} \left(v - G_2(u_{\alpha_k})\right) \tag{3.30}$$

Applying the operator G_1 on the expression (3.30) we get

$$\begin{aligned} G_1 u_{\alpha_k} &= G_1 G_1^* \left(\alpha I + G_1 G_1^*\right)^{-1} \left(v - G_2(u_{\alpha_k})\right) \\ &= \left[(\alpha_k I + G_1 G_1^*) - \alpha_k I\right] \left(\alpha I + G_1 G_1^*\right)^{-1} \left(v - G_2(u_{\alpha_k})\right) \\ &= v - G_2(u_{\alpha_k}) - \alpha_k \left(\alpha I + G_1 G_1^*\right)^{-1} \left(v - G_2(u_{\alpha_k})\right) \end{aligned}$$

Whence (3.29).
From (3.29) taking into account (3.28) we find that

$$\lim_{k\to\infty} \{G_1(u_{\alpha_k}) + G_2(u_{\alpha_k})\} = v, \quad \text{for all } v \in L^2(\Omega). \tag{3.31}$$

Now, let $y^1 \in L^2(\Omega)$ arbitrary and put $v = y^1 - e^{-\gamma T} S(T) y^0$ we get from (3.31) that

$$\lim_{k\to\infty} \{G_1(u_{\alpha_k}) + G_2(u_{\alpha_k})\} = y^1 - e^{-\gamma T} S(T) y^0,$$

whence

$$\lim_{k\to\infty} \left\{e^{-\gamma T} S(T) y^0 + G_1(u_{\alpha_k}) + G_2(u_{\alpha_k})\right\} = y^1,$$

which proves the approximate controllability of the equation (1.1).

At the end, it's worth to mention that the proof is similar to that of theorem 16 in [8] from claim 4.

Remark. By examining the proof of the theorem 3.3 we observe that it's also applicable to the case studied in [8] where there is a constant $\beta \in \left[\frac{1}{2}, 1\right[$ and two positive bounded functions $b_0, b_1 \in \mathcal{B}([0,T];\mathbb{R})$ such that $|f(t,y,u)| \leq b_1(t)|u|^\beta + b_0(t)$ for all $y, u \in \mathbb{R}$, $|y|, |u| \geq R$.

Perspectives. In our next work, we will try to apply these techniques to study :
1. Exact and null controllability of this type of equations.

2. Approximate controllabilty of systems of parabolic equations.

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