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EXISTENCE AND UNIQUENESS RESULTS FOR $\psi-$CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEMS INVOLVING THE $p-$LAPLACIAN OPERATOR.

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In this paper, we investigate the boundary value problems for nonlinear $\psi-$ Caputo fractional differential equations involving the p$-$Laplacian operator. We establish a new result on the existence and uniqueness of solutions by employing Banach fixed point theorem. As application, several examples are presented to illustrate the proposed result.

Keywords: $\psi-$fractional integral, $\psi-$Caputo fractional derivative, $p-$Laplacian operator, Banach fixed point theorem.

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. In recent years, fractional differential equations have emerged as a new branch of applied mathematics, due to the evolution of fractional calculus. This theory have attracted a considerable interest both in mathematics and in applications as material theory, transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, electromagnetic theory, thermodynamics, mechanics, geology, astrophysics, economics and control theory (see[2, 7, 8, 10, 14, 20, 21]). Basic issues related to the different fractional derivatives such as Riemann-Liouville [16], Caputo [4], Hilfer [17], Erdelyi-Kober [19] and Hadamard [3]. Indeed, fractional differential equations have been of great interest due to the intensive development of theory of fractional calculus itself and its applications. For example the fractional differential equations with p-Laplacian operator which model some phenomena from many applied fields such as turbulent filtration in porous media, blood flow problems, rheology, biology and propagations of mechanical waves in viscoelastic media. For many works on the studies of the existence and uniqueness results of fractional differential equations with the p-Laplacian operator, we refer the readers to the articles [1, 11, 12, 13, 18, 22] and the references therein.

Motivated by the above mentioned papers, we develop the theory of $p-$Laplacian fractional equations involving ψ-Caputo fractional derivatives. To be more precise, we establish

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the existence result the following $p-$Laplacian fractional equation:

$$\begin{cases} \left(\phi_p\left({}^{C}D_{0^+}^{\alpha,\psi}u(t)\right)\right)' = f(t,u(t)), & t\in\Delta=[0,T], \\ u(0)=\sigma_1 u(T), \quad u'(0)=\sigma_2 u'(T). \end{cases} \tag{1}$$

where $T>0$, ${}^{C}D_{0^+}^{\alpha,\psi}$ is the $\psi-$Caputo fractional derivative of order $\alpha\in(1,2)$ and ϕ_p is a $p-$Laplacian operator, i.e $\phi_p(t)=t^{p-1}$ such that $p-1>0$.

Our paper is organized as follows: In Section 2, we give some basic definitions and properties of $\psi-$fractional integral and $\psi-$Caputo fractional derivative which will be used in the rest of this paper. In Section 3, we establish the existence and uniqueness results for $p-$Laplacian fractional equation (1) by using some fixed point theorem. As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives, ψ-fractional integrals and $p-$Laplacian fractional equations, for more details we refer the reader to [5, 9, 15, 23].

Notations

• We denote by $C(\Delta,\mathbb{R})$ the space of continuous real-valued functions defined on Δ provided with the topology of the supremum norm

$$\| u \| = \sup_{t\in\Delta} | u(t) | .$$

• We denote by $L^1(\Delta,\mathbb{R})$ the space of Lebesgue integrable real-valued functions on Δ equipped with the norm

$$\| u \|_{L^1} = \int_\Delta | u(t) | \, dt.$$

Definition 2.1. [6] *Let $q>0$, $g\in L^1([\Delta,\mathbb{R})$ and $\psi\in C^n(\Delta,\mathbb{R})$ such that $\psi'(t)>0$ for all $t\in\Delta$.*
The ψ-Riemann-Liouville fractional integral at order α of the function g is given by

$$I_{0^+}^{\alpha,\psi}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)ds.$$

Definition 2.2. [6] *Let $\alpha>0$, $g\in C^{n-1}(\Delta,\mathbb{R})$ and $\psi\in C^n(\Delta,\mathbb{R})$ such that $\psi'(t)>0$ for all $t\in\Delta$.*
The $\psi-$Caputo fractional derivative at order α of the function g is given by

$${}^{C}D_{0^+}^{\alpha,\psi}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}g_\psi^{[n]}(s)ds,$$

Where

$$g_\psi^{[n]}(s)=\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n g(s) \quad and \quad n=[\alpha]+1.$$

And $[\alpha]$ denotes the integer part of the real number α.

Remark 2.1. *In particular, if $\alpha \in]0,1[$, then we have*

$$^{C}D_{0^+}^{\alpha,\psi}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}g'(s)ds.$$

and

$$^{C}D_{0^+}^{\alpha,\psi}g(t) = I_{0^+}^{1-\alpha,\psi}\left(\frac{g'(t)}{\psi'(t)}\right)$$

Proposition 2.1. [6] *Let $\alpha > 0$, if $g \in C^{n-1}(\Delta,\mathbb{R})$, then we have*

1) $^{C}D_{0^+}^{\alpha,\psi}I_{0^+}^{\alpha,\psi}g(t) = g(t)$.
2) $I_{0^+}^{\alpha,\psi}\,{}^{C}D_{0^+}^{\alpha,\psi}g(t) = g(t) - \sum_{k=0}^{n-1}\frac{g_{\psi}^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k$.
3) $I_{a^+}^{\alpha,\psi}$ *is linear and bounded from $C(\Delta,\mathbb{R})$ to $C(\Delta,\mathbb{R})$.*

Remark 2.2. [6] *If $1 < \alpha < 2$ and $g \in C^2(\Delta,\mathbb{R})$, then we have*

$$I_{0^+}^{\alpha,\psi}\,{}^{C}D_{0^+}^{\alpha,\psi}g(t) = g(t) - [c_0 + c_1(\psi(t)-\psi(0))]$$

where $c_0, c_1 \in \mathbb{R}$.

Some basic properties of the $p-$Laplacian operator which will be used in our paper are listed in the following lemma.

Lemma 2.1. [12] *Let $\phi_p : \mathbb{R} \to \mathbb{R}$ be a $p-$Laplacian operator defined by $\phi_p(x) = \mid x \mid^{p-2} x$, then we have the following basic properties:*

(1) *If $1 < p < 2$ and $x \neq 0$ then $(\phi_p(x))' = (p-1) \mid x \mid^{p-2}$.*
(2) *If $1 < p < 2$, $xy > 0$ and $\mid x \mid, \mid y \mid \geq l > 0$, then*

$$\mid \phi_p(x) - \phi_p(y) \mid \leq (p-1)l^{p-2} \mid x - y \mid .$$

(3) *If $p > 2$ and $\mid x \mid, \mid y \mid \leq L$, then*

$$\mid \phi_p(x) - \phi_p(y) \mid \leq (p-1)L^{p-2} \mid x - y \mid .$$

(4) *ϕ_p is is invertible such that $\phi_p^{-1}(x) = \phi_q(x)$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

3. Main result

In this section, we give our main existence and uniqueness result for the problem (1). We assume the following assumptions throughout the rest of our paper.

(H_1) There exists a real constant $k > 0$ such that

$$\mid f(t,u) - f(t,v) \mid \leq k \mid u - v \mid \quad for \quad all \quad t \in \Delta \quad and \quad u, v \in \mathbb{R}.$$

(H_2) There exists two constants $\gamma > 0$ and $0 < \rho < \frac{2}{2-q}$ such that

$$f(t,u) \geq \gamma\rho t^{\rho-1} \quad for\ all\ (t,u) \in \Delta \times \mathbb{R}.$$

Before we give the main result of our paper, we need to prove the following fundamental lemma.

Lemma 3.1. *Suppose that the function $u(t) \in C(\Delta,\mathbb{R})$, $\alpha \in (1,2)$ and $\sigma_1, \sigma_2 \in \mathbb{R}$. If $\sigma_1 \neq 1$ and $\sigma_2 \neq \dfrac{\psi'(0)}{\psi(T)}$ then $u(t)$ is a solution of the fractional differential equation (1) if and only if it satisfies the the following fractional integral equation*

$$u(t) = \frac{\sigma_1\sigma_2(\psi(T)-\psi(0))^2}{(1-\sigma_1)(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)}\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$

$$+\frac{\sigma_1}{(1-\sigma_1)\Gamma(\alpha)}\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$

$$+\frac{\sigma_2(\psi(T)-\psi(0))(\psi(t)-\psi(0)}{(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)}\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau.$$

Proof. Let $u(t)$ be a solution of the fractional differential equation (1).
By integrating (1), we obtain

$$\phi_p\left({}^CD_{0^+}^{\alpha,\psi}u(t)\right)-\phi_p\left({}^CD_{0^+}^{\alpha,\psi}u(0)\right)=\int_0^t f(s,u(s))ds,$$

since ${}^CD_{0^+}^{\alpha,\psi}u(0)=0$ then we have

$$\phi_p\left({}^CD_{0^+}^{\alpha,\psi}u(t)\right)=\int_0^t f(s,u(s))ds,$$

By using the inverse operator ϕ_q of the $p-$Laplacian operator ϕ_p we get

$${}^CD_{0^+}^{\alpha,\psi}u(t)=\phi_q\left(\int_0^t f(s,u(s))ds\right), \tag{2}$$

we apply the $\psi-$fractional integral $I_{0^+}^{\alpha,\psi}$ on both sides of (2) we have

$$I_{0^+}^{\alpha,\psi}\ {}^CD_{0^+}^{\alpha,\psi}u(t)=I_{0^+}^{\alpha,\psi}\ \phi_q\left(\int_0^t f(s,u(s))ds\right)$$

by using Remark 2.2 we obtain

$$u(t)=c_0+c_1(\psi(t)-\psi(0))+I_{0^+}^{\alpha,\psi}\ \phi_q\left(\int_0^t f(s,u(s))ds\right),$$

where $c_0,c_1\in\mathbb{R}$.
It follows that

$$u(t)=c_0+c_1(\psi(t)-\psi(0))+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau. \tag{3}$$

We can easily get also that

$$u'(t)=c_1\psi'(t)+\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left(\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau\right),$$

thus

$$u'(t)=c_1\psi'(t)+\frac{(\psi(t)-\psi(0))}{\Gamma(\alpha-1)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau.$$

We can deduce that

$$u(0)=c_0,$$
$$u'(0)=c_1\psi'(0),$$

and

$$u(T)=c_0+c_1(\psi(T)-\psi(0))+I_{0^+}^{\alpha,\psi}\ \phi_q\left(\int_0^T f(s,u(s))ds\right),$$

$$u'(T)=c_1\psi'(T)+\frac{(\psi(T)-\psi(0))}{\Gamma(\alpha-1)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau.$$

It follows that

$$c_1=\frac{\sigma_2(\psi(T)-\psi(0))}{(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau.$$

and

$$c_0=\frac{\sigma_1\sigma_2(\psi(T)-\psi(0))^2}{(1-\sigma_1)(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$
$$+\frac{\sigma_1}{(1-\sigma_1)\Gamma(\alpha)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau,$$

Substituting c_0 and c_1 in (3), we get

$$u(t)=\frac{\sigma_1\sigma_2(\psi(T)-\psi(0))^2}{(1-\sigma_1)(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$
$$+\frac{\sigma_1}{(1-\sigma_1)\Gamma(\alpha)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$
$$+\frac{\sigma_2(\psi(T)-\psi(0))(\psi(t)-\psi(0)}{(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)}\int_0^T\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau.$$

This completes the proof. □

We prove our main result by using the Banach fixed point theorem. For this purpose we set the following formulas

$$\omega_1=k(q-1)\gamma^{q-2}T^{\rho(q-2)+1}\left(\frac{\sigma_1\sigma_2(\psi(T)-\psi(0))^{\alpha+1}}{(1-\sigma_1)(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha)}+\frac{\sigma_1(\psi(T)-\psi(0))^{\alpha-1}}{(1-\sigma_1)(1-\alpha)\Gamma(\alpha)}\right).$$
$$\omega_2=k(q-1)\gamma^{q-2}T^{\rho(q-2)+1}\left(\frac{\sigma_2(\psi(T)-\psi(0))^{\alpha+1}}{(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha)}+\frac{(\psi(T)-\psi(0))^{\alpha-1}}{(1-\alpha)\Gamma(\alpha)}\right).$$

Theorem 3.1. *Suppose that $p>2$, $\sigma_1\neq 1$ and $\sigma_2\neq\dfrac{\psi'(0)}{\psi(T)}$. If the hypotheses (H_1) and (H_2) hold and $\omega_1+\omega_2<1$ then the $p-$Laplacian fractional equation (1) has a unique solution on Δ.*

Proof. By using the assumption (H_2) we can obtain

$$\int_0^t f(s,u(s))ds\geq\gamma t^\rho\quad for\ all\ (t,u)\in\Delta\times\mathbb{R}.$$

Let $u,v\in C(\Delta,\mathbb{R})$, then by using Lemma 2.1 and the assumption (H_1) we have the following estimation

$$\left|\phi_q\left(\int_0^t f(s,u(s))ds\right)-\phi_q\left(\int_0^t f(s,v(s))ds\right)\right|\leq(q-1)(\gamma t^\rho)^{q-2}\left|\int_0^t f(s,u(s))ds-\int_0^t f(s,v(s))ds\right|,$$

$$\left|\phi_q\left(\int_0^t f(s,u(s))ds\right)-\phi_q\left(\int_0^t f(s,v(s))ds\right)\right|\leq(q-1)(\gamma t^\rho)^{q-2}\int_0^t\left|f(s,u(s))ds-f(s,v(s))ds\right|,$$

$$\left|\phi_q\left(\int_0^t f(s,u(s))ds\right)-\phi_q\left(\int_0^t f(s,v(s))ds\right)\right| \leq (q-1)(\gamma t^{\rho})^{q-2}\int_0^t k \parallel u-v \parallel ds,$$

$$\left|\phi_q\left(\int_0^t f(s,u(s))ds\right)-\phi_q\left(\int_0^t f(s,v(s))ds\right)\right| \leq k(q-1)\gamma^{q-2}t^{\rho(q-2)+1} \parallel u-v \parallel . \quad (4)$$

Let $\mathcal{A}: C(\Delta,\mathbb{R}) \to C(\Delta,\mathbb{R})$ be the operator defined as follows:

$$\begin{aligned}\mathcal{A}u(t) = a(T)&\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau\\ +b(\sigma_1)&\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau\\ +c(t)&\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau\\ +d&\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\phi_q\left(\int_0^\tau f(s,u(s))ds\right)d\tau.\end{aligned}$$

where

$$a(T)=\frac{\sigma_1\sigma_2(\psi(T)-\psi(0))^2}{(1-\sigma_1)(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)},$$

$$b(\sigma_1)=\frac{\sigma_1}{(1-\sigma_1)\Gamma(\alpha)},$$

$$c(t))=\frac{\sigma_2(\psi(T)-\psi(0))(\psi(t)-\psi(0)}{(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha-1)},$$

and

$$d=\frac{1}{\Gamma(\alpha)}.$$

Now, we will show that the operator $\mathcal{A}$ is a contraction mapping.
Let $u,v \in C(\Delta,\mathbb{R})$, Then by hypothesis (H_2)

$$\begin{aligned}|\mathcal{A}u(t)-\mathcal{A}v(t)| \leq &\left|a(T)\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\left(\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right)d\tau\right|\\ &+\left|b(\sigma_1)\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\left(\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right)d\tau\right|\\ &+\left|c(t)\int_0^T \psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\left(\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right)d\tau\right|\\ &+\left|d\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\left(\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right)d\tau\right|.\end{aligned}$$

thus

$$\begin{aligned}|\mathcal{A}u(t)-\mathcal{A}v(t)| \leq &|a(T)|\int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right|\left|\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right|d\tau\\ &+|b(\sigma_1)|\int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\right|\left|\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right|d\tau\\ &+|c(t)|\int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right|\left|\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right|d\tau\\ &+|d|\int_0^t \left|\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\right|\left|\phi_q\left(\int_0^\tau f(s,u(s))ds\right)-\phi_q\left(\int_0^\tau f(s,v(s))ds\right)\right|d\tau.\end{aligned}$$

By using (4) we obtain

$$|\mathcal{A}u(t)-\mathcal{A}v(t)| \leq |a(T)| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right| \left|k(q-1)\gamma^{q-2}\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau$$

$$+|b(\sigma_1)| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\right| \left|k(q-1)\gamma^{q-2}\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau$$

$$+|c(t)| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right| \left|k(q-1)\gamma^{q-2}\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau$$

$$+|d| \int_0^t \left|\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\right| \left|k(q-1)\gamma^{q-2}\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau.$$

It follows that

$$|\mathcal{A}u(t)-\mathcal{A}v(t)| \leq \left|k(q-1)\gamma^{q-2}a(T)\right| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right| \left|\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau$$

$$+\left|k(q-1)\gamma^{q-2}b(\sigma_1)\right| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\right| \left|\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau$$

$$+\left|k(q-1)\gamma^{q-2}c(t)\right| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right| \left|\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau$$

$$+\left|k(q-1)\gamma^{q-2}d\right| \int_0^t \left|\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\right| \left|\tau^{\rho(q-2)+1}\right| \parallel u-v \parallel d\tau.$$

Since $\gamma > 0$ and $0 < \rho < \frac{2}{2-q}$ we get

$$|\mathcal{A}u(t)-\mathcal{A}v(t)| \leq \left(|a(T)| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right| d\tau\right) k(q-1)\gamma^{q-2}T^{\rho(q-2)+1} \parallel u-v \parallel$$

$$+\left(|b(\sigma_1)| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-1}\right| d\tau\right) k(q-1)\gamma^{q-2}T^{\rho(q-2)+1} \parallel u-v \parallel$$

$$+\left(|c(T)| \int_0^T \left|\psi'(\tau)(\psi(T)-\psi(\tau))^{\alpha-2}\right| d\tau\right) k(q-1)\gamma^{q-2}T^{\rho(q-2)+1} \parallel u-v \parallel$$

$$+\left(|d| \int_0^T \left|\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\right| d\tau\right) k(q-1)\gamma^{q-2}T^{\rho(q-2)+1} \parallel u-v \parallel .$$

Thus

$$|\mathcal{A}u(t)-\mathcal{A}v(t)| \leq \left(\frac{\sigma_1\sigma_2(\psi(T)-\psi(0))^{\alpha+1}}{(1-\sigma_1)(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha)} + \frac{\sigma_1(\psi(T)-\psi(0))^{\alpha-1}}{(1-\sigma_1)(1-\alpha)\Gamma(\alpha)}\right) k(q-1)\gamma^{q-2}T^{\rho(q-2)+1} \parallel u-v \parallel$$

$$+\left(\frac{\sigma_2(\psi(T)-\psi(0))^{\alpha+1}}{(\psi'(0)-\sigma_2\psi(T))\Gamma(\alpha)} + \frac{(\psi(T)-\psi(0))^{\alpha-1}}{(1-\alpha)\Gamma(\alpha)}\right) k(q-1)\gamma^{q-2}T^{\rho(q-2)+1} \parallel u-v \parallel .$$

Finally, we obtain

$$\| \mathcal{A}u - \mathcal{A}v \| \leq (\omega_1 + \omega_2) \| u - v \| .$$

By using the Banach contraction mapping principle, we can deduce that the operator $\mathcal{A}$ has a unique fixed point in the space $C(\Delta, \mathbb{R})$ which is the unique solution of the differential equation (1) defined on Δ.

□

4. **Illustrative example**

In this section we give an example to illustrate our main result.
Consider the following fractional differential equation:

$$\begin{cases} \left(\phi_{\frac{5}{2}}\left({}^{C}D_{0^+}^{\frac{3}{2},t^2} u(t)\right)\right)' = 3t^2 cos^2\left(\dfrac{\pi u(t)}{10} + \Theta\right), & t \in \Delta = [0,1], \\ u(0) = \dfrac{1}{10}u(1), \quad u'(0) = \dfrac{1}{10}u'(1). \end{cases} \tag{5}$$

where $\Theta \in \mathbb{R}$.
The fractional differential equation (5) can be regarded as fractional differential equation(1), where
$\alpha = \dfrac{3}{2}, \quad T = 1, \quad \psi(t) = t^2, p = \dfrac{5}{2} < 2, \quad \sigma_1 = \sigma_2 = \dfrac{1}{10}$ and
$f(t, u(t)) = 3t^2 cos^2\left(\dfrac{\pi u(t)}{10} + \Theta\right)$.
It is clear that $q = \dfrac{5}{3} < 2$ and $f \in C(\Delta \times \mathbb{R}, \mathbb{R})$.
We take $\gamma = 1, \quad \rho = 3$ and $k = \dfrac{\pi}{3}$ then we obtain $0 < \rho < \dfrac{2}{2-q} = 6$.

To prove the assumption (H_1), let $t \in \Delta$ and $u, v \in \mathbb{R}$, then we have

$$|f(t, u(t)) - f(t, v(t))| = \left| 3t^2 cos^2\left(\frac{\pi u(t)}{10} + \Theta\right) - 3t^2 cos^2\left(\frac{\pi v(t)}{10} + \Theta\right)\right|,$$

$$|f(t, u(t)) - f(t, v(t))| \leq \frac{\pi t^2}{3} |u(t) - v(t)|,$$

$$|f(t, u) - f(t, v)| \leq \frac{\pi}{3} |u - v|,$$

Thus, the assumption (H_2) in holds true with $k = \frac{\pi}{3}$.

It remains to verify the assumption (H_2). Let $t \in \Delta$ and $u \in \mathbb{R}$, then we have

$$\gamma \rho t^{\rho-1} = 3t^2 \leq 3t^2 cos^2\left(\frac{\pi u(t)}{10} + \Theta\right) = f(t, u).$$

Moreover, we have

$$\omega_1 + \omega_2 = \frac{2\pi}{27\Gamma(\frac{3}{2})} < 1.$$

Finally, all the conditions of Theorem 3.1 are satisfied, thus the $p-$Laplacian fractional problem (1) has a unique solution on $[0, 1]$.

5. Conclusion

In this article, we have considered the boundary value problem for $\psi-$Caputo fractional differential equations involving the $p-$Laplacian operator. As a first step, by applying the tools of fractional calculus and using some basic properties of $\psi-$Caputo fractional derivative and $\psi-$fractional integral, we build a general structure of solutions associated with our proposed model. Once the fixed point operator equation is available, the existence result is established by using Banach fixed point theorem. Finally, the investigation of the result has been illustrated by providing suitable examples.

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