ON (FUZZY) WEAK-ZERO GROUPOIDS AND (X, N)-ZERO GROUPOIDS

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In this paper, we generalize the left-zero semigroup by introducing two different algebras, called a weak-zero groupoid and an (X, N)-zero groupoid, respectively and describe some properties related to $Bin(X)$.1 Moreover, we fuzzify the notion of a weak-zero groupoid.

Keywords: (fuzzy) weak-zero groupoid, (X, N)-zero groupoid, $Bin(X)$, $ZBin(X)$, (left-, right-) similar, (dual-) N-groupoid.
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1. Introduction

In the study of groupoids $(X, *)$ defined on a set X, it has also proven useful to investigate the semigroups $(Bin(X), \square)$ where $Bin(X)$ is the set of all binary systems (groupoids) $(X, *)$ along with an associative product operation $(X, *)\square(X, \bullet) = (X, \square)$ such that $x\square y = (x * y) \bullet (y * x)$ for all $x, y \in X$. Thus, e.g., it becomes possible to recognize that the left-zero-semigroup $(X, *)$ with $x*y = x$ for all $x, y \in X$ acts as the identity of this semigroup [5]. H. F. Fayoumi [1] introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid. J. S. Han et al. [2] introduced the notion of hypergroupoids $(HBin(X), \square)$, and showed that $(HBin(X), \square)$ is a supersemigroup of the semigroup $(Bin(X), \square)$ via the identification $x \longleftrightarrow \{x\}$. They proved that $(HBin^*(X), \ominus, [\emptyset])$ is a BCK-algebra. For the references on BCK-algebras and BCI-algebras, we refer to [3], [4] and [6].

The notion of a fuzzy subset of a set was introduced by L. A. Zadeh [9]. His seminal paper in 1965 has opened up new insights and applications in a wide range of scientific fields. A. Rosenfeld [8] used the notion of a fuzzy subset to set

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down corner stone papers in several areas of mathematics. J. N. Mordeson and D. S. Malik [7] published a remarkable book, *Fuzzy commutative algebra*, presented a fuzzy ideal theory of commutative rings and applied the results to the solution of fuzzy intersection equations. The book included all the important work that has been done on L-subspaces of a vector space and on L-subfields of a field.

Given that the left-zero semigroup on the set X is the identity element of $(Bin(X), \Box)$ it is a question of interest in the study of the semigroups to identify related types of groupoids within the class $Bin(X)$ and identify these as themselves also having algebraic properties, including the property of being a subsemigroup of $(Bin(X), \Box)$ which identifies the type being described as representing a certain type of property. In the situation discussed below, we shall deal with weak-zero-groupoids and (X, N)-zero-groupoids as groupoid types of interest. As a by-product we will also consider a fuzzification of the notion of a weak-zero-groupoid.

2. **Preliminaries**

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. A groupoid $(X, *)$ is said to be a *left-zero-semigroup* (resp., *right-zero-semigroup*) if $x * y = x$ (resp., $x * y = y$) for all $x, y \in X$. Given elements $(X, *)$ and $(X, \bullet)$ of $Bin(X)$, define a product "$\Box$" on these groupoids as follows:

$$(X, *) \Box (X, \bullet) = (X, \Box)$$

where

$$x \Box y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1. [5] $(Bin(X), \Box)$ *is a semigroup, i.e., the operation "$\Box$" as defined in general is associative. Furthermore, the left- zero-semigroup is the identity for this operation.*

H. Fayoumi [1] introduced the notion of the center of the semigroup $Bin(X)$ as follows:

$$ZBin(X) := \{(X, \bullet) \in Bin(X) | (X, *)\Box(X, \bullet) = (X, \bullet)\Box(X, *), \forall (X, *)\}$$

She obtained several interesting properties.

Proposition 2.1. [1] *The left-zero semigroup and right-zero semigroup on X are both in $ZBin(X)$.*

Proposition 2.2. [1] *If $(X, \bullet) \in ZBin(X)$, then $x \bullet x = x$ for all $x \in X$.*

Theorem 2.2. [1] *If* $(X, \bullet) \in ZBin(X)$*, then* $x \neq y$ *implies that* $\{x, y\} = \{x \bullet y, y \bullet x\}$

Proposition 2.3. [1] *Let* $(X, *) \in ZBin(X)$*. If* $x \neq y$ *in* X*, then* $(\{x, y\}, \bullet)$ *is either a left-zero semigroup or a right-zero semigroup.*

Proposition 2.4. [1] $(\{x, y\}, \bullet)$ *is either a left-zero semigroup or a right-zero semigroup for any* $x \neq y$ *in* X*, then* $(X, \bullet) \in ZBin(X)$*.*

3. **Weak-zero groupoids.**

We shall consider a groupoid $(X, *)$ to be a *weak-left-zero groupoid* if $x*y = a*b$ implies $x = a$. Thus, if $(X, *)$ is a left-zero-semigroup then it is also a weak-left-zero groupoid. Similarly, a groupoid $(X, *)$ is said to be a *weak-right-zero groupoid* if $x * y = a * b$ implies $y = b$.

Example 3.1. *Let* $(X, *)$ *be a left-zero-semigroup, i.e.,* $x * y = x$ *for all* $x, y \in X$*. If* $x * y = a * b$*, then* $x = a$*, i.e., it is a weak-left-zero groupoid. Similary, every right-zero-semigroup* $(X, *)$*, i.e.,* $x * y = y$ *for all* $x, y \in X$*, is a weak-right-zero groupoid.*

Example 3.2. *Let* $\mathsf{N} = \{1, 2, 3, \cdots\}$ *be the set of all natural numbers and let* p, q *be distinct prime numbers. If we define* $x * y := p^x q^y$*, then* $1 * 1 = p^1 q^1 = pq$*. If we assume* $x * y = a * b$*, then* $p^x q^y = p^a q^b$ *and hence* $x = a$ *and* $y = b$*, proving that* $(\mathsf{N}, *)$ *is both a weak-left-zero groupoid and a weak-right-zero groupoid.*

A groupoid $(X, *)$ is said to be a *leftoid* for f if $x * y := f(x)$ for a map $f : X \to X$. The groupoid $(\mathsf{N}, *)$ in Example 3.2 is not a leftoid, since $x * y = p^x q^y$ is not a function of x alone.

Proposition 3.1. *Let* $(X, *)$ *be a leftoid for* f*. If* f *is one-one, then* $(X, *)$ *is a weak-left-zero groupoid.*

Proof. If $x * y = a * b$, then $f(x) = f(y)$. Since f is one-one, we obtain $x = a$, proving the proposition. □

Let $\mathsf{N} = \{1, 2, 3, \cdots\}$ be the set of all natural numbers. If we define $x * y := 2^x$ for all $x, y \in \mathsf{N}$, then $(\mathsf{N}, *)$ is not a left-zero-semigroup, but it is a weak-left-zero-semigroup, since the map $f(x) := 2^x$ is one-one.

Corollary 3.1. *Let* $(X, *)$ *be a leftoid for* f*. If* f *is the identity map, then* $(X, *)$ *is a weak-left-zero groupoid.*

A groupoid $(X,*)$ is said to be a *rightoid* for f for some map $f : X \to X$. We obtain the similar proposition.

Proposition 3.1′. *Let $(X,*)$ be a rightoid for f. If f is one-one, then $(X,*)$ is a weak-right-zero groupoid.*

Theorem 3.1. *Let $(X,*)$ be a finite weak-left-zero groupoid and let $y_0 \in X$ be a fixed element. If we define a map $f : X \to X$ by $f(x) := x * y_0$ for all $x \in X$, then $(X,*)$ is a leftoid for f.*

Proof. We claim that f is one-one. If $f(\alpha) = f(\beta)$ then $\alpha * y_0 = \beta * y_0$. Since $(X,*)$ is a weak-left-zero groupoid, we obtain $\alpha = \beta$. Since $|X| < \infty$, f is onto. Given $a, b \in X$, we let $a * b = u$ for some $u \in X$. Since f is a bijection, there exists $x \in X$ such that $u = f(x) = x * y_0$. It follows that $a * b = u = x * y_0$, which shows that $a = x$. Hence $a * b = u = f(x) = f(a)$, proving the theorem. □

We may obtain the similar result for weak-right-zero groupoids.

Theorem 3.1′. Let $(X,*)$ be a finite weak-right-zero groupoid and let $x_0 \in X$ be a fixed element. If we define a map $f : X \to X$ by $f(y) := x_0 * y$ for all $y \in X$, then $(X,*)$ is a rightoid for f.

Proposition 3.2. *Let $(X,*)$ be both a weak-left-zero groupoid and a weak-right-zero groupoid. If X is finite, then $|X| = 1$.*

Proof. Let $(X,*)$ be both a weak-left-zero groupoid and a weak-right-zero groupoid. Since $(X,*)$ is a weak-left-zero groupoid, by Theorem 3.1, $(X,*)$ is a leftoid for a bijective map $f : X \to X$. Similarly, since $(X,*)$ is a weak-right-zero groupoid, by Theorem 3.1′, $(X,*)$ is a rightoid for a bijective map $g : X \to X$. It follows that $x * y = f(x) = g(y)$ for all $x, y \in X$. Assume that $x_1 \neq x_2$ in X. Then $x_1 * y = f(x_1) = g(y)$ and $x_2 * y = f(x_2) = g(y)$ for all $y \in X$. This shows that $f(x_1) = f(x_2)$. Since f is a bijection, we obtain $x_1 = x_2$, a contradiction, proving the proposition. □

Theorem 3.2. *Let $(X,*), (X,\bullet) \in Bin(X)$ and let $(X,\Box) := (X,*)\Box(X,\bullet)$. Then the following hold:*

(i) *if $(X,*)$ and $(X,\bullet)$ are weak-left-zero groupoids, then $(X,\Box)$ is also a weak-left-zero groupoid,*
(ii) *if $(X,*)$ is a weak-left-zero groupoid and $(X,\bullet)$ is a weak-right-zero groupoid, then $(X,\Box)$ is a weak-right-zero groupoid,*
(iii) *if $(X,*)$ is a weak-right-zero groupoid and $(X,\bullet)$ is a weak-left-zero groupoid, then $(X,\Box)$ is a weak-right-zero groupoid,*
(iv) *if $(X,*)$ and $(X,\bullet)$ are weak-right-zero groupoids, then $(X,\Box)$ is a weak-left-zero groupoid.*

Proof. (i) Assume that $x\Box y = a\Box b$ for some $x, y, a, b \in X$. Then $(x*y)\bullet(y*x) = (a*b)\bullet(b*a)$. Since $(X,\bullet)$ is a weak-left-zero groupoid, we obtain $x*y = a*b$. Since $(X,*)$ is a weak-left-zero groupoid, we have $x = a$. The proofs of the others are similar to (i), and we omit them. $\Box$

Remark 3.1. *Theorem 3.2 suggests that weak-left-zero groupoids may be assigned a parity even (or 0) while right-left-zero groupoids may be assigned a parity odd (or 1). Theorem 3.2 then has the form (i)* $0\Box 0 = 0$*; (ii)* $0\Box 1 = 1$*; (iii)* $1\Box 0 = 1$ *and (iv)* $1\Box 1 = 1$*. Thus we obtain a version of addition mod 2 in this setting.*

A groupoid $(X,*)$ is said to be *right cancellative* if $x*y = z*y$, then $x = z$. Clearly, every weak-left-zero groupoid is right cancellative.

Proposition 3.3. *Let* $(X,*)$ *be a weak-left-zero groupoid. If* $(X,*)$ *is a semigroup, then it is a left-zero semigroup.*

Proof. If $(X,*)$ is a semigroup, then $(x*y)*z = x*(y*z)$ for all $x, y, z \in X$. Since $(X,*)$ is a weak-left-zero groupoid, we have $x*y = x$ for all $x, y \in X$, proving the proposition. $\Box$

Proposition 3.4. *Let* $(X,*,0)$ *be a* d/BCK*-algebra. If* $(X,*)$ *is a weak-left-zero groupoid, then* $|X| = 1$.

Proof. Given $x, y \in X$, we have $(x*x)*y = 0*x = 0 = y*y$. Since $(X,*)$ is a weak-left-zero groupoid, we obtain $y = x*x = 0$, proving the proposition. $\Box$

Note that the direct product of weak-left-zero groupoids is also a weak-left-zero groupoid, and a subgroupoid of a weak-left-zero groupoid is also a weak-left-zero groupoid.

A groupoid $(X,*)$ is said to be *left-similar* to a groupoid $(X,\bullet)$ if $x*y = a*b$, then $x\bullet y = a\bullet b$. In this case, $(X,\bullet)$ is said to be *right-similar* to $(X,*)$. Accordingly $(X,*)$ and $(X,\bullet)$ are *similar* if they are both left-similar and right-similar.

Example 3.3. *Given* $X := \mathbf{R}$*, the set of all real numbers, we define* $x*y := x+y$*, and* $x\bullet y := \lambda(x+y)$*, where* $\lambda \neq 0$ *and* $+$ *is the usual addition on* $\mathbf{R}$*. Then* $x+y = a+b$ *implies* $x\bullet y = a\bullet b$ *and conversely. Thus,* $(X,*)$ *and* $(X,\bullet)$ *are similar. If* $\lambda = 0$*, then* $x+y = z+b$ *implies* $x*y = 0(x=y) = 0(a+b) = a\bullet b$*. This shows that* $(X,*)$ *and* $(X,\bullet)$ *are left-similar, but not similar.*

Example 3.4. *If* $(X,*)$ *is commutative and left-similar to* $(X,\bullet)$*, then* $x*y = y*x$ *for all* $x, y \in X$ *and thus* $x\bullet y = y\bullet x$ *for all* $x, y \in X$ *as well, i.e.,* $(X,\bullet)$ *is commutative also.*

Example 3.5. *Clearly* $(\mathbf{R},+)$ *is a semigroup where* $+$ *is the usual addition on* $\mathbf{R}$*. If* $\lambda \neq 0, 1$*, then* $x*y := \lambda(x+y)$ *is not a semigroup. Indeed,* $(x*y)*z =$

$\lambda(\lambda(x+y)+z) = \lambda^2 x + \lambda^2 y + \lambda z$ and $x * (y * z) = \lambda(x + \lambda(y+z)) = \lambda x + \lambda^2 y + \lambda^2 z$ so that $\lambda \neq \lambda^2$ implies that $(\mathbf{R}, +)$ is not a semigroup even though $(\mathbf{R}, +)$ is similar to $(\mathbf{R}, *)$. This is clear since $\lambda \neq 0$ implies $\lambda^{-1}(x * y) = \lambda^{-1}(\lambda(x+y)) = x + y$.

Proposition 3.5. *If $(X, *)$ and $(Y, \bullet)$ are left-similar to $(X, \triangle)$ and (Y, ∇) respectively, then $(X, *) \times (Y, \bullet)$ is left-similar to $(X, \triangle) \times (Y, \nabla)$.*

Proof. Straightforward. $\square$

Proposition 3.6. *Let $(X, \bullet)$ be a weak-left-zero groupoid and let $(X, \square) := (X, *)\square(X, \bullet)$. Then $(X, \square)$ is left-similar to $(X, \bullet)$.*

Proof. Assume $x \square y = a \square b$. Then $(x * y) \bullet (y * x) = (a * b) \bullet (b * a)$. Since $(X, \bullet)$ is a weak-left-zero groupoid, we have $x * y = a * b$, proving the proposition. $\square$

Proposition 3.7. *Let $(X, \bullet)$ be a weak-right-zero groupoid and let $(X, \square) = (X, *) \square(X, \bullet)$. Then $(X, \square)$ is left-similar to $(X, *)^{\mathrm{op}} = (X, \diamond)$, i.e., $x * y = y \diamond x$ for all $x, y \in X$.*

Proof. Let $x \square y = a \square b$. Then $(x * y) \bullet (y * x) = (a * b) \bullet (b * a)$. Since $(X, \bullet)$ is a weak-right-zero groupoid, we obtain $y * x = b * a$, i.e., $x \diamond y = a \diamond b$, proving the proposition. $\square$

4. (X, N)-zero groupoids

Among mathematical objects, the simplest with regard to structure are the sets X. Perhaps the next step up as to complexity are the *nuclear sets* (X, N) consisting of a set X and a nucleus N contained in it. Thus $N \subset X$ is the structure of interest in the following.

Let X be a non-empty set and let $N \subset X$. Define a binary operation "$*$" on X by

$$x * y := \begin{cases} x & \text{if } x, y \in N, \\ y & \text{otherwise} \end{cases}$$

We denote it by $(X(N), *)$ and we call it a (X, N)*-zero groupoid*. If $N := X$, then $(X(N), *)$ is a left-zero semigroup, and if $N := \emptyset$, then $(X(N), *)$ is a right-zero semigroup. Note that $(X(N), *)$ need not be a semigroup except in the extreme cases $N = X$ and $N = \emptyset$. In fact, if $x, z \in N$ and $y \notin N$, then $(x * y) * z = y * z = z$, while $x * (y * z) = x * z = x$.

Proposition 4.1. *If $(X(N), *)$ is an (X, N)-zero groupoid, then*

$$(x * x) * y = x * (x * y)$$

for all $x, y \in X$.

Proof. Given $x, y \in X$, if $x, y \in N$, then $(x*x)*y = x*y = x$ and $x*(x*y) = x*x = x$. Otherwise, we have $(x * x) * y = x * y = y$ and $x * (x * y) = x * y = x$. □

Proposition 4.2. *Let $(X, *) \in Bin(X)$ and let $N \subseteq X$. If $(X(N), \Box) := (X(N), *) \Box (X(N), *)$, then it is a left-zero semigroup.*

Proof. Given $x, y \in X$, if $x, y \in N$, then $x \Box y = (x * y) * (y * x) = x * y = x$. Otherwise, we have $x \Box y = (x * y) * (y * x) = y * x = x$, proving the proposition. □

If $|X| = n < \infty$, then the number of groupoids $(X(N), *)$ is 2^n. Also, the number of groupoids satisfying $x * x = x, x * y \in \{x, y\}$ is 2^{n^2-n}, so that there are many such groupoids which are not of the type $(X(N), *)$ discussed here.

Proposition 4.3. *Let $(X, *) \in ZBin(X)$. If $(X, \Box) := (X, *) \Box (X, *)$, then $(X, \Box)$ is a left-zero semigroup.*

Proof. Since $(X, *) \in ZBin(X)$, by Proposition 2.3, $(\{x, y\}, *)$ is either a left-zero semigroup or a right-zero semigroup for all $x, y \in X$. It follows that either $x \Box y = (x * y) * (y * x) = x * y = x$ or $x \Box y = (x * y) * (y * x) = y * x = x$. Hence $(X, \Box)$ is a left-zero semigroup. □

Theorem 4.1. *Let $(X(M), *)$ be an (X, M)-zero groupoid and let $(X(N), \bullet)$ be an (X, N)-zero groupoid where $M, N \subseteq X$. If $(X, \Box) := (X(M), *) \Box (X(N), \bullet)$, then $(X, \Box) \in ZBin(X)$.*

Proof. Given $x, y \in X$, if $x, y \in M \setminus N$, then $x \Box y = (x * y) \bullet (y * x) = x \bullet y = y$ and $y \Box x = (y * x) \bullet (x * y) = y \bullet x = x$, i.e., $(\{x, y\}, \Box)$ is a right-zero semigroup. If $x, y \in N \setminus M$, then $x \Box y = (x * y) \bullet (y * x) = y \bullet x = y$ and $y \Box x = (y * x) \bullet (x * y) = x \bullet y = x$, i.e., $(\{x, y\}, \Box)$ is a right-zero semigroup. If either $x \in M \setminus N, y \in N \setminus M$ or $x \in N \setminus M, y \in M \setminus N$, then $x \Box y = (x*y) \bullet (y*x) = y \bullet x = x$ and $y \Box x = (y*x) \bullet (x*y) = x \bullet y = y$, i.e., $(\{x, y\}, \Box)$ is a left-zero semigroup. If $x \notin M \cup N$, $y \in M \cup N$, then $x \Box y = (x*y) \bullet (y*x) = y \bullet x = x$ and $y \Box x = (y*x) \bullet (x*y) = x \bullet y = y$, i.e., $(\{x, y\}, \Box)$ is a left-zero semigroup. If $x, y \notin M \cup N$, then $x \Box y = (x * y) \bullet (y * x) = y \bullet x = x$ and $y \Box x = (y * x) \bullet (x * y) = x \bullet y = y$, i.e., $(\{x, y\}, \Box)$ is a left-zero semigroup. By Proposition 2.4, $(X, \Box) \in ZBin(X)$. □

Corollary 4.1. *Let $(X(N), *)$ be an (X, N)-zero groupoid and let $(X(N^C), \bullet)$ be an (X, N^C)-zero groupoid. If $(X, \Box) := (X(N), *) \Box (X(N^C), \bullet)$, then $(X, \Box) \in ZBin(X)$.*

Proposition 4.4. *If $(X, *)$ is an (X, N)-zero groupoid, then $(X, *) \in ZBin(X)$.*

Proof. Given $x, y \in X$, if $x, y \in N$, then $x * y = x$ and $y * x = y$, i.e., $(\{x, y\}, *)$ is a left-zero semigroup. If $x, y \in N^C$, then $x * y = y$ and $y * x = x$, i.e., $(\{x, y\}, *)$ is a right-zero semigroup. If either $x \in N, y \in N^C$ or $x \in N^C, y \in N$, then $x * y = y, y * x = x$, i.e., $(\{x, y\}, *)$ is a right-zero semigroup. By Proposition 2.6, $(X, *) \in ZBin(X)$. □

Note that if $(X,*)$ is an (X,N)-zero groupoid, then $(X,\star) \in ZBin(X)$ where $x \star y := y * x$ for all $x, y \in X$, but it is not an (X,N)-zero groupoid. In fact, it is an (X,N^C)-zero groupoid. We give an example that the converse of Proposition 4.4 does not hold in general.

Example 4.1. *Let $X := \{a,b,c,d\}$ be a set with $M := \{a,b,c\}$ and $N := \{b,c,d\}$. Define two binary operations as follows:*

$*$	a	b	c	d
a	a	a	a	d
b	b	b	b	d
c	c	c	c	d
d	a	b	c	d

$\bullet$	a	b	c	d
a	a	b	c	d
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

Then $(X,*)$ is an (X,M)-zero groupoid and $(X,\bullet)$ is an (X,N)-zero groupoid. If we let $(X,\Box) := (X,*)\Box(X,\bullet)$, then we have the following table:

$\Box$	a	b	c	d
a	a	b	c	a
b	a	b	b	d
c	a	c	c	d
d	d	b	c	d

It is easy to see that $(X,\Box) \in ZBin(X)$, but is not an (X,K)-zero groupoid for any $K \subseteq X$ with $K \neq \emptyset$.

A groupoid $(X,*)$ is said to be an N-groupoid if $(N,*)$ is a subgroupoid of $(X,*)$ where $\emptyset \neq N \subseteq X$. We denote it by $(X,N,*)$ and we call it an N-*groupoid.* We denote the collection of all N-groupoids by $Bin(X,N)$. Clearly, every (X,N)-zero groupoid $(X(N),*)$ belongs to $Bin(X,N)$. Let $(X,*)$ be a left-zero semigroup. Then $(X,*)$ is an N-groupoid for any non-empty subset $N \subseteq X$, i.e., $(X,*) \in Bin(X,N)$. This means that the left-zero semigroup acts as the identity element of $Bin(X,N)$.

Proposition 4.5. *$(Bin(X,N),\Box)$ is a subsemigroup of $(Bin(X),\Box)$.*

Proof. Given $(X,N,*), (X,N,\bullet) \in Bin(X,N)$, we let $(X,N,\Box) := (X,N,*)\Box(X,N,\bullet)$. Given $x, y \in N$, we have $x\Box y = (x*y)\bullet(y*x) \in N \bullet N \subseteq N$. This shows that $(X,N,\Box) \in Bin(X,N)$. $\Box$

Let $(X,*) \in Bin(X)$ and let $(N,*)$ be a subgroupoid of $(X,*)$. Define a binary operation "$*_D$" on X by

$$x *_D y := \begin{cases} x * y & \text{if } x, y \in N, \\ y * x & \text{otherwise} \end{cases}$$

We call $(X, *_D)$ a *dual-N-groupoid* of $(X, *)$.

Proposition 4.6. *If $(X,*)$ is an (X,N)-zero groupoid, then its dual-N-groupoid $(X,*_D)$ is a left-zero semigroup.*

Proof. Given $x, y \in X$, if $x, y \in N$, then $x *_D y = x * y = x$. Otherwise, we have $x * y = y, y * x = x$. It follows that $x *_D y = y * x = x$, proving that $(X, *_D)$ is a left-zero semigroup. $\square$

Clearly $(X, *_D)$ is also an N-groupoid if $(X, *)$ is an N-groupoid.

Proposition 4.7. *Let $(X,\bullet) \in Bin(X,N)$ and let $(X,*)$ be an (X,N)-zero groupoid. If $(X,\square) := (X,*)\square(X,\bullet)$, then $(X,\square)$ is a dual-N-groupoid of $(X,\bullet)$.*

Proof. Given $x, y \in X$, if $x, y \in N$, then $x * y = x, y * x = y$ and hence $x \square y = (x * y) \bullet (y * x) = x \bullet y$. Otherwise, we have $x * y = y, y * x = x$ and hence $x \square y = (x * y) \bullet (y * x) = y \bullet x$. This shows that $(X, \square) = (X, \bullet_D)$. $\square$

Theorem 4.2. *Let $(X,\bullet) \in ZBin(X)$ and let $(X,*)$ be an (X,N)-zero groupoid. If $(X,\square) := (X,\bullet)\square(X,*)$, then $(X,\square)$ is a dual-N-groupoid of $(X,\bullet)$.*

Proof. Given $x, y \in X$, if $x, y \in N$, since $(X, \bullet) \in ZBin(X)$, by Theorem 2.3, we have $\{x, y\} = \{x \bullet y, y \bullet x\}$, i.e., $x \bullet y, y \bullet x \in N$. Since $(X, *)$ is an (X,N)-zero groupoid, we have $x \square y = (x \bullet y) * (y \bullet x) = x \bullet y$. Otherwise, since $(X, \bullet) \in ZBin(X)$, by Theorem 2.2, we have $\{x, y\} = \{x \bullet y, y \bullet x\}$, i.e., it is not true that $x \bullet y, y \bullet x \in N$. It follows that $x \square y = (x \bullet y) * (y \bullet x) = y \bullet x$, proving that $(X, \square) = (X, \bullet_D)$. $\square$

5. Some applications to fuzzy sets

Let $(X, *) \in Bin(X)$. A map $\mu : X \to [0,1]$ is called a *fuzzy weak-left-zero groupoid* if

$$\mu(x * y) \geq \mu(a * b) \quad \text{implies} \quad \mu(x) \geq \mu(a)$$

Example 5.1. *Suppose that $D : \mathbf{R} \to [0,1]$ is a non-decreasing function, e.g., the distribution function of a random variable. Also, suppose $f : (X,*) \to \mathbf{R}$ is any function such that $f(x*y) \geq f(a*b)$ implies $f(x) \geq f(a)$. Then $\mu(x) := D(f(x))$ yields $\mu(x*y) = D(f(x*y)) \geq D(f(a*b)) = \mu(a*b)$ yields $f(x*y) \geq f(a*b)$ and thus $f(x) \geq f(a)$, whence $\mu(x) \geq \mu(a)$ as well, i.e., μ is a fuzzy-weak-left-zero groupoid with respect to $(X,*)$.*

Example 5.2. *Let $(X,*)$ be a left-zero semigroup. Then every map $\mu : X \to [0,1]$ is a fuzzy-weak-left-zero groupoid. In fact, if $\mu(x*y) \geq \mu(a*b)$, since $(X,*)$ is a left-zero semigroup, we obtain $\mu(x) \geq \mu(a)$.*

Let $(X,*) \in Bin(X)$. A map $\mu : X \to [0,1]$ is called a *strict fuzzy weak-left-zero groupoid* if

$$\mu(x * y) = \mu(a * b) \quad \text{implies} \quad \mu(x) = \mu(a)$$

Clearly, if $(X,*)$ is a left-zero semigroup, then every map $\mu : X \to [0,1]$ is a strict fuzzy weak-left-zero groupoid. Note that every fuzzy weak-left-zero groupoid is a strict fuzzy weak-left-zero groupoid.

Let $(X,*) \in Bin(X)$ and let $\mu : X \to [0,1]$ be a fuzzy subset of X. We give some conditions:

(D_1) if $\mu(x) \geq \mu(a)$, then $\mu(x * y) \geq \mu(a * b)$ for all $y, b \in X$,
(D_2) if $\mu(x) = \mu(a)$, then $\mu(x * y) = \mu(a * b)$ for all $y, b \in X$.

Notice that if (D_1) holds, then $\mu(x) = \mu(a)$ implies $\mu(x * y) \geq \mu(a * b)$ and $\mu(a * b) \geq \mu(x * y)$, i.e., $\mu(x * y) = \mu(a * b)$, so that (D_1) implies the condition (D_2).

Proposition 5.1. *Let $(X,*) \in Bin(X)$ and let $\mu : X \to [0,1]$ be a fuzzy subset of X with (D_1). If $\mu(a) := \min_{x \in X} \mu(x)$, then $\mu(a * b) = \mu(a * c)$ for all $b, c \in X$.*

Proof. Since $\mu(a) \leq \mu(x)$ for all $x \in X$, by (D_1), we have $\mu(a * b) \leq \mu(x * c)$ for all $b, c, x \in X$. It follows that $\mu(a * b) \leq \mu(a * c)$ for all $b, c \in X$. If we exchange b with c, then we have $\mu(a * b) = \mu(a * c)$ for all $b, c \in X$. $\square$

Let $(X,*) \in Bin(X)$. A map $\mu : X \to [0,1]$ is called a *fuzzy weak-right-zero groupoid* if

$$\mu(x * y) \geq \mu(a * b) \quad \text{implies} \quad \mu(y) \geq \mu(b)$$

A map $\mu : X \to [0,1]$ is said to be a *fuzzy weak-zero* if

$$\mu(x * y) \geq \mu(a * b) \quad \text{implies} \quad \mu(x) \geq \mu(a), \;\; \mu(y) \geq \mu(b)$$

Theorem 5.1. *Let $(X,*), (X,\bullet) \in Bin(X)$ and let $(X,\square) := (X,*)\square(X,\bullet)$. Then the following conclusions hold:*

(i) *if μ is a fuzzy weak-left-zero groupoid of both $(X,*)$ and $(X,\bullet)$, it is also a fuzzy weak-left-zero groupoid of $(X,\square)$,*
(ii) *if μ is a fuzzy weak-left-zero groupoid of $(X,*)$ and a fuzzy weak-right-zero groupoid of $(X,\bullet)$, then it is a fuzzy weak-right-zero groupoid,*
(iii) *if μ is a fuzzy weak-right-zero groupoid of $(X,*)$ and a fuzzy weak-left-zero groupoid of $(X,\bullet)$, then it is a fuzzy weak-right-zero groupoid,*
(iv) *if μ is a fuzzy weak-right-zero groupoid of both $(X,*)$ and $(X,\bullet)$, it is a fuzzy weak-left-zero groupoid of $(X,\square)$.*

Proof. (i) If $\mu(x\Box y) \geq \mu(a\Box b)$ for some $x, y, a, b \in X$, then $\mu((x * y) \bullet (y * x)) \geq \mu((a * b) \bullet (b * a))$. Since μ is a fuzzy weak-left-zero groupoid of $(X, \bullet)$, we obtain $\mu(x * y) \geq \mu(a * b)$. Since μ is a fuzzy weak-left-zero groupoid of $(X, *)$, we have $\mu(x) \geq \mu(a)$. The proofs of the others are similar to (i), and we omit them. □

Let $(X, *) \in Bin(X)$. A map $\mu : X \to [0, 1]$ is called a *fuzzy weak-crossed-zero groupoid* if

$$\mu(x * y) \geq \mu(a * b) \quad \text{implies} \quad \mu(x) \geq \mu(a) \text{ or } \mu(y) \geq \mu(b)$$

Example 5.3. *In Example 5.1, if $f : (X, *) \to \mathbf{R}$ is any function such that $f(x*y) \geq f(a * b)$ implies $f(x) \geq f(a)$ or $f(y) \geq f(b)$. Then $D \circ f : (X, *) \to [0, 1]$ is a fuzzy weak-crossed-zero groupoid.*

Example 5.4. *Let $X := \mathsf{N}$, the set of all natural numbers, and let $x * y := x(x + y)$ for all $x, y \in X$. Define $f : X \to \mathbf{R}$ by $f(x) := x$. Assume $f(x * y) \geq f(a * b)$ and $f(x) \leq f(a)$. Then $x(x + y) \geq a(a + b)$ and $x \leq a$. It follows that $x + y \geq a + b$ and hence $y \geq a + b - x = (a - x) + b \geq b$, which shows that $f(y) \geq f(b)$. Define $\mu := D \circ f$, where D is a distribution function of a random variable. Then μ is a fuzzy weak-crossed-zero groupoid of $(X, *)$.*

Proposition 5.2. *Let μ be a fuzzy weak-crossed-zero groupoid of both $(X, *)$ and $(X, \bullet)$. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then μ is a fuzzy weak-crossed-zero groupoid of $(X, \Box)$.*

Proof. Assume $\mu(x\Box y) \geq \mu(a\Box b)$ and $\mu(x) < \mu(a)$. Then $\mu((x * y) \bullet (y * x))) \geq \mu((a*b) \bullet (b*a))$. Since μ is a fuzzy weak-crossed-zero groupoid of $(X, \bullet)$, we obtain either $\mu(x * y) \geq \mu(a * b)$ or $\mu(y * x) \geq \mu(b * a)$. Since μ is a fuzzy weak-crossed-zero groupoid of $(X, *)$, we have either $\mu(x) \geq \mu(a)$ or $\mu(y) \geq \mu(b)$. It follows that $\mu(y) \geq \mu(b)$, proving that μ is a fuzzy weak-crossed-zero groupoid of $(X, \Box)$. □

6. Conclusion.

In this paper, we consider elements of $(Bin(X), \Box)$ with similar properties, which acts as generalizations of the left-zero semigroup as well as the right-zero semigroup, viz, the weak-left-zero groupoids and the weak-right-zero semigroups, while still maintaining several of their properties to a considerable degree, especially with respect to those derived for the product $\Box$ in $(Bin(X), \Box)$. As another generalization of the left-zero semigroup, we introduce the notion of an (X, N)-zero groupoid and obtain several "parity" properties related to $ZBin(X)$. After having obtained information about these groupoids, we are in the future planning to proceed with the study of fuzzy subsets of X which have correspond properties to fuzzy subsets of X which have corresponding properties in the future.

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