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AN INVERSE PROBLEM FOR THE SIXTH-ORDER LINEAR BOUSSINESQ-TYPE EQUATION

He Yang[1]

In this paper, we consider an inverse problem for the sixth-order linear Boussinesq-type equation. Given some conditions of the data, we establish the proofs of existence and uniqueness for sufficiently small time. In addition, we introduce a numerical method to solve the inverse problem and present some numerical results.

Keywords: Boussinesq-type equation, Inverse problem.
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1. Introduction

The Boussinesq equation is a classical model to describe wave propagation with small amplitude and long waves. A lot of attention has been paid recently to study the sixth-order Boussinesq equation [1, 5, 6]. However, not much work has been done on the inverse problems for the sixth-order Boussinesq equation. An inverse problem for a fourth-order Boussinesq equation is discussed in [2], and inverse problems for hyperbolic and pseudo-hyperbolic equation are investigated in [3, 4]. In this paper, we consider an inverse problem for the sixth-order linear boussinesq-type equation. That is,

$$\begin{cases} u_{tt} = u_{xx} - \beta_1 u_{xxxx} + \beta_2 u_{xxxxxx} + a(t)u + f(x,t), \quad x \in (0,1), \quad t \in [0,T], \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le 1, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = u_{xxxx}(0,t) = u_{xxxx}(1,t) = 0, \end{cases} \tag{1}$$

with the additional condition

$$u(x_0,t) = h(t), \quad \text{for a fixed } x_0 \in (0,1), \ \forall t \in [0,T]. \tag{2}$$

Here $\beta_1, \beta_2, T > 0$ are three given constants, and we assume that $h(t) \neq 0$ for any $t \in [0,T]$. The functions f, ϕ, ψ and h are sufficiently smooth functions which will be made more precise later. In addition, the consistency conditions for ϕ and ψ will also follow in the next section. The inverse problem is described as follows: given data $\{f(x,t), \phi(x), \psi(x), h(t)\}$, and seek $\{u(x,t), a(t)\}$ such that (1) and (2) are satisfied. The purpose of this work is to prove the uniqueness and existence of the inverse problem (in Section 2), establish numerical methods and present some numerical results (in Section 3).

2. Existence and Uniqueness

2.1. Existence and Uniqueness

Based on equations (1), we assume the following consistency conditions:

$$\begin{cases} \phi(0) = \phi''(0) = \phi^{(4)}(0) = \phi(1) = \phi''(1) = \phi^{(4)}(1) = 0, \\ \psi(0) = \psi''(0) = \psi^{(4)}(0) = \psi(1) = \psi''(1) = \psi^{(4)}(1) = 0. \end{cases} \tag{3}$$

[1] Department of Mathematics, Augusta University, Augusta, GA 30912, USA, e-mail: hyang1@augusta.edu

One key component for the proof of the existence is equation (11). To derive it, we assume that all the series in (4), (7), (8), (9) and (10) converge and arbitrarily differentiable. Due to the homogenous boundary conditions, we look for solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{4}$$

where $X_n(x) = \sqrt{2}\sin(\mu_n x)$, $\mu_n = n\pi$ for $n = 1, 2, \ldots$ The time-dependent coefficients $u_n(t)$ satisfy

$$\begin{cases} u_n''(t) + b_n^2 u_n(t) = F_n(t; a, u), \\ u_n(0) = \phi_n, \quad u_n'(0) = \psi_n, \end{cases} \tag{5}$$

$b_n = \sqrt{\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6}$, $F_n(t; a, u) = a(t)u_n(t) + f_n(t)$, $f_n(t) = \sqrt{2}\int_0^1 f(x,t)\sin(\mu_n x)dx$, $\phi_n = \sqrt{2}\int_0^1 \phi(x)\sin(\mu_n x)dx$, $\psi_n = \sqrt{2}\int_0^1 \psi(x)\sin(\mu_n x)\ dx$ for all n.

We can solve the initial value problem (5) and obtain

$$u_n(t) = \phi_n\cos(b_n t) + \frac{\psi_n}{b_n}\sin(b_n t) + \frac{1}{b_n}\int_0^t F_n(s; a, u)\sin(b_n(t-s))ds, \quad \forall n \tag{6}$$

which leads to the solution

$$u(x,t) = \sqrt{2}\sum_{n=1}^{\infty}\left[\phi_n\cos(b_n t) + \tfrac{\psi_n}{b_n}\sin(b_n t) + \tfrac{1}{b_n}\int_0^t F_n(s; a, u)\sin(b_n(t-s))ds\right]\sin(\mu_n x). \tag{7}$$

We now consider the partial differential equation in (1) evaluated at $x = x_0$, and apply the condition (2) to get

$$h''(t) = u_{xx}(x_0,t) - \beta_1 u_{xxxx}(x_0,t) + \beta_2 u_{xxxxxx}(x_0,t) + a(t)h(t) + f(x_0,t).$$

We further substitute $u(x_0,t) = \sqrt{2}\sum_{n=1}^{\infty} u_n(t)\sin(\mu_n x)$ into the equation above, and obtain

$$a(t) = \tfrac{1}{h(t)}\left[h''(t) - f(x_0,t) + \sum_{n=1}^{\infty}(\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6)u_n(t)\sin(\mu_n x_0)\right]. \tag{8}$$

Here we have used the assumption that $h(t) \neq 0$. Note that the formulation of $u_n(t)$ in the equation above is given by (6). Let $\mathbf{z} = [u(x,t), a(t)]^T$ and $\mathbf{\Phi}(\mathbf{z}) = [\Phi_1(\mathbf{z}), \Phi_2(\mathbf{z})]^T$ where the functions Φ_1 and Φ_2 are defined as

$$\begin{aligned} \Phi_1 &= \sqrt{2}\sum_{n=1}^{\infty}\left[\phi_n\cos(b_n t) + \tfrac{\psi_n}{b_n}\sin(b_n t) + \tfrac{1}{b_n}\int_0^t F_n(s; a, u)\sin(b_n(t-s))ds\right]\sin(\mu_n x), \qquad (9) \\ \Phi_2 &= \tfrac{1}{h(t)}\Big[h''(t) - f(x_0,t) + \sum_{n=1}^{\infty}(\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6)\Big(\phi_n\cos(b_n t) + \tfrac{\psi_n}{b_n}\sin(b_n t) \\ &\quad + \tfrac{1}{b_n}\int_0^t F_n(s; a, u)\sin(b_n(t-s))ds\Big)\sin(\mu_n x_0)\Big]. \qquad (10) \end{aligned}$$

Therefore, we can see that the existence and uniqueness of the inverse problem (1)-(2) is equivalent to that of the equation

$$\mathbf{z} = \mathbf{\Phi}(\mathbf{z}). \tag{11}$$

Before we proceed with a proof, we first define some important spaces. Let $D_T = \{(x,t) : 0 < x < 1, 0 \leq t \leq T\}$,

$$B_{2,T}^7 = \left\{u(x,t) = \sqrt{2}\sum_{n=1}^{\infty} u_n(t)\sin(\mu_n x) : u_n(t) \in C[0,T],\ \sum_{n=1}^{\infty}\left(\mu_n^7\|u_n(\cdot)\|_{C[0,T]}\right)^2 < \infty\right\}$$

and

$$E_{2,T}^7 = B_{2,T}^7 \times C[0,T].$$

For $u \in B_{2,T}^7$, $z \in E_{2,T}^7$, we define the norm of u and z as $\|u\|_{B_{2,T}^7} = \left(\sum_{n=1}^{\infty}\left(\mu_n^7\|u_n(\cdot)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$, and $\|z\|_{E_{2,T}^7} = \|u\|_{B_{2,T}^7} + \|a(\cdot)\|_{C[0,T]}$.

Lemma 2.1. *$B_{2,T}^7$ and $E_{2,T}^7$ are Banach spaces.*

Proof. We only show that $B^7_{2,T}$ is a Banach space, and the proof for the space $E^7_{2,T}$ is similar. For any Cauchy sequence $\{u^m(x,t)\}_{m=0}^{\infty} \subset B^7_{2,T}$, let $u^m(x,t) = \sqrt{2}\sum_{i=1}^{\infty} u_i^m(t)\sin(\mu_n x)$. Then for any $\epsilon > 0$, there exists a positive integer N, such that for all $j, k > N$, there is

$$\|u^j - u^k\|_{B^7_{2,T}} = \left(\sum_{i=1}^{\infty}\left(\mu_i^7\|u_i^j - u_i^k\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} < \epsilon. \tag{12}$$

Since $\mu_i > 1$ for all $i \geq 1$, the inequality above leads to

$$\|u_i^j - u_i^k\|_{C[0,T]} < \epsilon, \quad \forall i \geq 1, \tag{13}$$

which implies that $\{u_i^m(t)\}_{m=0}^{\infty}$ is a Cauchy sequence. Thus for any $i \geq 1$, there exists $u_i(t) \in C[0,T]$ such that $\lim_{j\to\infty} u_i^j(t) = u_i(t)$. This is due to the fact that the space $C[0,T]$ is complete. Let $u(x,t) = \sqrt{2}\sum_{i=1}^{\infty} u_i(t)\sin(\mu_i x)$. From (12), we know that for any $l \geq 1$ there is

$$\lim_{j\to\infty}\sum_{i=1}^{l}\left(\mu_i^7\|u_i^j - u_i^k\|_{C[0,T]}\right)^2 = \sum_{i=1}^{l}\left(\mu_i^7\|u_i - u_i^k\|_{C[0,T]}\right)^2 \leq \epsilon^2, \tag{14}$$

for any $k > N$. Therefore, we can obtain

$$\|u - u^k\|_{B^7_{2,T}} = \sum_{i=1}^{\infty}\left(\left(\mu_i^7\|u_i - u_i^k\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq \epsilon, \quad k > N, \tag{15}$$

which leads to the fact that $u^k(x,t)$ converges to $u(x,t)$. Finally, since $\|u\|^2_{B^7_{2,T}} \leq 2\|u - u^k\|^2_{B^7_{2,T}} + 2\|u^k\|^2_{B^7_{2,T}}$, we can get $u \in B^7_{2,T}$. Thus, $B^7_{2,T}$ is a Banach space. □

Lemma 2.2. *For any* $\mathbf{z} \in E^7_{2,T}$*, there is* $\Phi(\mathbf{z}) \in E^7_{2,T}$ *if the following conditions are satisfied:*
(1) $\phi \in C^7[0,1]$, $\phi(0) = \phi''(0) = \phi^{(4)}(0) = \phi^{(6)}(0) = \phi(1) = \phi''(1) = \phi^{(4)}(1) = \phi^{(6)}(1) = 0$.
(2) $\psi \in C^4[0,1]$, $\psi(0) = \psi''(0) = \psi(1) = \psi''(1) = 0$.
(3) $f(x,t) \in C(\overline{D_T})$, $f(\cdot,t) \in C^4[0,1]$ *for any* $t \in [0,T]$, $f(0,t) = f(1,t) = f_{xx}(0,t) = f_{xx}(1,t) = 0$.
(4) $h(t) \in C^2[0,T]$, $h(t) \neq 0$, $\forall t \in [0,T]$.

Remark 2.1. *Lemma 2.1 and 2.2 imply that* $\mathbf{\Phi}$ *is an operator from the Banach space* $E^7_{2,T}$ *to itself.*

Remark 2.2. *The conditions in Lemma 2.2 are more restrictive than the consistency conditions* (3)*. For example, as we will show later,* $\phi \in C^7[0,1]$ *and* $\phi^{(6)}(0) = \phi^{(6)}(1) = 0$ *are needed to estimate the magnitude of* $\phi_n = \sqrt{2}\int_0^1 \phi(x)\sin(\mu_n x)dx$.

Proof. We prove of Lemma 2.2 in two steps.
Step 1. We show that $\Phi_2(\mathbf{z}) \in C[0,T]$, for any $\mathbf{z} = [u(x,t), a(t)]^T \in E^7_{2,T}$.
From (10) and the fact $h(t) \in C^2[0,T]$, $h \neq 0$, $f \in C(\overline{D_T})$, it is easy to see that we only need to show that $\sum_{n=1}^{\infty}(\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6)u_n(t)\sin(\mu_n x_0) \in C[0,T]$, where $u_n(t)$ is given in (6). We apply integration-by-parts and condition (1) in Lemma 2.2 to get

$$\phi_n = \sqrt{2}\int_0^1 \phi(x)\sin(\mu_n x)dx = -\frac{\sqrt{2}}{\mu_n^7}\int_0^1 \phi^{(7)}(x)\cos(\mu_n x)dx. \tag{16}$$

Similarly, we can derive

$$\psi_n = \frac{\sqrt{2}}{\mu_n^4}\int_0^1 \psi_{xxxx}(x)\sin(\mu_n x)dx, \tag{17}$$

and

$$f_n(t) = \frac{\sqrt{2}}{\mu_n^4}\int_0^1 f_{xxxx}(x,t)\sin(\mu_n x)dx. \tag{18}$$

We combine the equalities about ϕ_n, ψ_n, $f_n(t)$ to get

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty} b_n^2 \left(\phi_n \cos(b_n t) + \frac{\psi_n}{b_n}\sin(b_n t) + \frac{1}{b_n}\int_0^t F_n(s;a,u)\sin(b_n(t-s))ds\right)\sin(\mu_n x_0)\right| \\
\leq\ & (1+\beta_1+\beta_2)\sum_{n=1}^{\infty}\mu_n^6\left(|\phi_n| + \frac{|\psi_n|}{b_n} + \frac{1}{b_n}\int_0^t |F_n(s;a,u)|ds\right) \\
\leq\ & (1+\beta_1+\beta_2)\sum_{n=1}^{\infty}\left(\frac{1}{\mu_n}\left|\sqrt{2}\int_0^1 \phi^{(7)}\cos(\mu_n x)dx\right| + \frac{1}{\sqrt{\beta_2}\mu_n}\left|\sqrt{2}\int_0^1 \psi^{(4)}\sin(\mu_n x)dx\right|\right. \\
&\left. + \frac{\mu_n^3}{\sqrt{\beta_2}}\int_0^T |a(t)||u_n(t)|dt + \frac{T}{\mu_n}\max_{t\in[0,T]}\left|\sqrt{2}\int_0^1 f_{xxxx}(x,t)\sin(\mu_n x)dx\right|\right). \qquad (19)
\end{aligned}$$

Here we have used the fact that $b_n = \sqrt{\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6} \geq \sqrt{\beta_2}\mu_n^3$ in the second inequality above. In addition, using the boundary conditions of $u(x,t)$ and integration-by-parts, it is easy to show that

$$u_n(t) = \sqrt{2}\int_0^1 u(x,t)\sin(\mu_n x)dx = \frac{\sqrt{2}}{\mu_n^4}\int_0^1 u_{xxxx}(x,t)\sin(\mu_n x)dx. \qquad (20)$$

Therefore, (19) and (20) lead to the following estimates

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}(\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6)u_n(t)\sin(\mu_n x_0)\right| \qquad (21)\\
\leq\ & (1+\beta_1+\beta_2)\sum_{n=1}^{\infty}\left(\frac{1}{\mu_n}\left|\sqrt{2}\int_0^1 \phi^{(7)}\cos(\mu_n x)dx\right| + \frac{1}{\sqrt{\beta_2}\mu_n}\left|\sqrt{2}\int_0^1 \psi^{(4)}\sin(\mu_n x)dx\right|\right. \\
+\ & \left.\frac{T\|a\|_{C[0,T]}}{\sqrt{\beta_2}\mu_n}\max_t\left|\sqrt{2}\int_0^1 u_{xxxx}\sin(\mu_n x)dx\right| + \frac{T}{\mu_n}\max_{t\in[0,T]}\left|\sqrt{2}\int_0^1 f_{xxxx}\sin(\mu_n x)dx\right|\right) \\
\leq\ & (1+\beta_1+\beta_2)\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}\right)^{\frac{1}{2}}\left(\|\phi^{(7)}\|_{L^2[0,1]} + \frac{1}{\sqrt{\beta_2}}\|\psi^{(4)}\|_{L^2[0,1]}\right. \\
&\left. + \frac{T\|a\|_{C[0,T]}}{\sqrt{\beta_2}}\max_{t\in[0,T]}\|u_{xxxx}(\cdot,t)\|_{L^2[0,1]} + T\max_{t\in[0,T]}\|f_{xxxx}(\cdot,t)\|_{L^2[0,1]}\right),
\end{aligned}$$

where Cauchy-Schwartz inequality and Bessel's inequality have been applied in the second inequality above. Furthermore, we have

$$|u_{xxxx}| = \sqrt{2}\left|\sum_{n=1}^{\infty}\mu_n^4 u_n(t)\sin(\mu_n x)\right| \leq \sqrt{2}\left(\sum_{n=1}^{\infty}(\mu_n^7\|u_n(\cdot)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n^6}\right)^{\frac{1}{2}}.$$

for any $t \in [0,T]$, which implies that $\max_{t\in[0,T]}\|u_{xxxx}(\cdot,t)\|_{L^2[0,1]}$ is bounded. Combining inequality (21), $\sum_{n=1}^{\infty}(1/\mu_n^2) = 1/6$ and Weierstrass M-test, we can see that $\sum_{n=1}^{\infty}(\mu_n^2 + \beta_1\mu_n^4 + \beta_2\mu_n^6)u_n(t)\sin(\mu_n x_0)$ converges absolutely and uniformly. Thus the series is continuous on $t \in [0,T]$, and $\Phi_2 \in C[0,T]$.

Step 2. we then show that $\Phi_1(\mathbf{z}) \in B_{2,T}^7$.

Since

$$\begin{aligned}
&\mu_n^7\left|\phi_n\cos(b_n t) + \frac{\psi_n}{b_n}\sin(b_n t) + \frac{1}{b_n}\int_0^t F_n(s;a,u)\sin(b_n(t-s))ds\right| \qquad (22)\\
\leq\ & \left|\sqrt{2}\int_0^1 \phi^{(7)}\cos(\mu_n x)dx\right| + \sqrt{\frac{2}{\beta_2}}\left|\int_0^1 \psi^{(4)}\sin(\mu_n x)dx\right| \\
&+\sqrt{\frac{2}{\beta_2}}\int_0^T\left(|a(t)|\left|\int_0^1 u_{xxxx}\sin(\mu_n x)dx\right| + \left|\int_0^1 f_{xxxx}\sin(\mu_n x)dx\right|\right)dt,
\end{aligned}$$

for any $t \in [0,T]$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\mu_n^7\max_t\left|\phi_n\cos(b_n t) + \frac{\psi_n}{b_n}\sin(b_n t) + \frac{1}{b_n}\int_0^t F_n(s;a,u)\sin(b_n(t-s))ds\right|\right)^2 \\
\leq\ & 4\left(\|\phi^{(7)}\|_{L^2[0,1]}^2 + \frac{1}{\beta_2}\|\psi^{(4)}\|_{L^2[0,1]}^2 + \frac{T^2}{\beta_2}(\|a\|_{C[0,T]}^2\cdot\max_t\|u_{xxxx}\|^2 + \max_t\|f_{xxxx}\|^2)\right).
\end{aligned}$$

Applying the conditions of the lemma to the equation above, we can show that the right side of the inequality is bounded. Therefore, we have proved $\Phi_1(\mathbf{z}) \in B_{2,T}^7$ which concludes the lemma. □

From the proof of Lemma 2.2, we can show that for any $\mathbf{z} = [u(x,t), a(t)]^T \in E^7_{2,T}$, the following estimates for $\|\Phi_1(\mathbf{z})\|$ and $\|\Phi_2(\mathbf{z})\|$ hold:

$$\|\Phi_1(\mathbf{z})\| \le 2\left(\|\phi^{(7)}\|_{L[0,1]} + \tfrac{1}{\sqrt{\beta_2}}\|\psi^{(4)}\|_{L^2[0,1]} + \tfrac{T}{\sqrt{\beta_2}}(\|f_{xxxx}\|_{C(\overline{D}_T)} + \|a\|_{C[0,T]}\|u\|_{B^7_{2,T}})\right),$$

$$\|\Phi_2(\mathbf{z})\| \le \|\tfrac{1}{h}\|_{C[0,T]}\Big[\|f\|_{C(\overline{D}_T)} + \|h''\|_{C[0,T]} + (1+\beta_1+\beta_2)(\textstyle\sum_{n=1}^{\infty}\frac{1}{\mu_n^2})^{\frac12}\times$$
$$\left(\|\phi^{(7)}\|_{L^2[0,1]} + \tfrac{1}{\sqrt{\beta_2}}\|\psi^{(4)}\|_{L^2[0,1]} + \tfrac{2T}{\sqrt{\beta_2}}\|a\|_{C[0,T]}\|u\|_{B^7_{2,T}} + T\|f_{xxxx}\|_{C(\overline{D}_T)}\right)\Big].$$

Let $A(T)$ and $B(T)$ be two functions of T, defined by

$$A(T) = 2\|\phi^{(7)}\|_{L[0,1]} + \tfrac{2}{\sqrt{\beta_2}}\|\psi^{(4)}\|_{L^2[0,1]} + \|\tfrac{1}{h}\|_{C[0,T]}\left(\|f\|_{C(\overline{D}_T)} + \|h''\|_{C[0,T]}\right)$$
$$+\tfrac{2T}{\sqrt{\beta_2}}\|f_{xxxx}\|_{C(\overline{D}_T)} + T\|\tfrac{1}{h}\|_{C[0,T]}(1+\beta_1+\beta_2)(\textstyle\sum_{n=1}^{\infty}\frac{1}{\mu_n^2})^{\frac12}\|f_{xxxx}\|_{C(\overline{D}_T)}$$
$$B(T) = \tfrac{2T}{\sqrt{\beta_2}} + \tfrac{2T}{\sqrt{\beta_2}}(1+\beta_1+\beta_2)\|\tfrac{1}{h}\|_{C[0,T]}(\textstyle\sum_{n=1}^{\infty}\frac{1}{\mu_n^2})^{\frac12}.$$

Then we can obtain

$$\|\mathbf{\Phi}(\mathbf{z})\|_{E^7_{2,T}} \le A(T) + B(T)\|a\|_{C[0,T]}\|u\|_{B^7_{2,T}}. \tag{23}$$

Due to the fact that $B(T) \to 0$ as $T \to 0$, and $A(T)$ is a continuous function of T, there exists a sufficiently small $T > 0$ such that

$$(A(T)+2)^2 B(T) < 1. \tag{24}$$

For the fixed T, we define a ball $K := \{z \in E^7_{2,T} : \|z\|_{E^7_{2,T}} \le A(T)+2\}$. Then, for every z that belongs to K, one has

$$\begin{aligned}\|\mathbf{\Phi}(\mathbf{z})\|_{E^7_{2,T}} &\le A(T) + B(T)\|a\|_{C[0,T]}\|u\|_{B^7_{2,T}} \le A(T) + B(T)(A(T)+2)^2\\ &< A(T)+1 \le A(T)+2.\end{aligned}$$

This implies that $\mathbf{\Phi}(\mathbf{z}) \in K$ for any $\mathbf{z} \in K$. That is, $\mathbf{\Phi}$ is an operator from K to itself. The next lemma gives another important property of $\mathbf{\Phi}$.

Lemma 2.3. *For the constant T that satisfies* (24)*, the operator $\mathbf{\Phi} : E^7_{2,T} \to E^7_{2,T}$ is a contraction mapping on the ball $K = \{z \in E^7_{2,T} : \|z\|_{E^7_{2,T}} \le A(T)+2\}$.*

Proof. For any $\mathbf{z}^{(1)}, \mathbf{z}^{(2)} \in E^7_{2,T}$, let $\mathbf{z}^{(i)} = [u^{(i)}(x,t), a^i(t)]^T$ for $i = 1,2$. Then

$$\|\mathbf{\Phi}(\mathbf{z}^{(1)}) - \mathbf{\Phi}(\mathbf{z}^{(2)})\|_{E^7_{2,T}} = \|\Phi_1(\mathbf{z}^{(1)}) - \Phi_1(\mathbf{z}^{(2)})\|_{B^7_{2,T}} + \|\Phi_2(\mathbf{z}^{(1)}) - \Phi_2(\mathbf{z}^{(2)})\|_{C[0,T]} \tag{25}$$
$$= \left\|\sqrt{2}\textstyle\sum_{n=1}^{\infty}\left[\frac{1}{b_n}\int_0^t(a^{(1)}(t)u_n^{(1)} - a^{(2)}(t)u_n^{(2)})\sin(b_n(t-s))ds\right]\sin(\mu_n x)\right\|_{B^7_{2,T}}$$
$$+ \left\|\tfrac{1}{h(t)}\textstyle\sum_{n=1}^{\infty}\left[\frac{\mu_n^2+\beta_1\mu_n^4+\beta_2\mu_n^6}{b_n}\int_0^t(a^{(1)}(t)u_n^{(1)} - a^{(2)}(t)u_n^{(2)})\sin(b_n(t-s))ds\right]\sin(\mu x_0)\right\|_{C[0,T]}.$$

Note that we have used the fact $F_n(t; a^{(i)}, u^{(i)}) = a^{(i)}(t)u_n^{(i)}(t) + f_n(t)$, and $u_n^{(i)} = \phi_n\cos(b_n t) + \frac{\psi_n}{b_n}\sin(b_n t) + \frac{1}{b_n}\int_0^t F_n(s; a^{(i)}, u^{(i)})\sin(b_n(t-s))ds$ for $i = 1,2$ when we derive the equation above. We now estimate each term of the right side of equation (25). By rewriting $a^{(1)}(t)u_n^{(1)} - a^{(2)}(t)u_n^{(2)}$ as $a^{(1)}(t)(u_n^{(1)} - u_n^{(2)}) + u_n^{(2)}(a^{(1)}(t) - a^{(2)}(t))$, we have

$$\|\Phi_1(\mathbf{z}^{(1)}) - \Phi_1(\mathbf{z}^{(2)})\|_{B^7_{2,T}} \tag{26}$$
$$= \left(\textstyle\sum_{n=1}^{\infty}\left(\mu_n^7\left\|\frac{1}{b_n}\int_0^t\left(a^{(1)}(u_n^{(1)} - u_n^{(2)}) + u_n^{(2)}(a^{(1)} - a^{(2)})\right)\sin(b_n(t-s))ds\right\|_{C[0,T]}\right)^2\right)^{1/2}$$
$$\le \left(\textstyle\sum_{n=1}^{\infty}\left(\frac{T\mu_n^4}{\sqrt{\beta_2}}\left(\|a^{(1)}\|_{C[0,T]}\cdot\|u_n^{(1)} - u_n^{(2)}\|_{C[0,T]} + \|u_n^{(2)}\|_{C[0,T]}\cdot\|a^{(1)} - a^{(2)}\|_{C[0,T]}\right)\right)^2\right)^{1/2}$$
$$\le \tfrac{T}{\sqrt{\beta_2}}\left(\|a^{(1)}\|_{C[0,T]}\|u^{(1)} - u^{(2)}\|_{B^7_{2,T}} + \|a^{(1)} - a^{(2)}\|_{C[0,T]}\|u^{(2)}\|_{B^7_{2,T}}\right).$$

Similarly, we can estimate $\|\Phi_2(\mathbf{z}^{(1)}) - \Phi_2(\mathbf{z}^{(2)})\|_{C[0,T]}$ as follows,

$$\begin{aligned} & \|\Phi_2(\mathbf{z}^{(1)}) - \Phi_2(\mathbf{z}^{(2)})\|_{C[0,T]} \\ \leq\ & \max_{t\in[0,T]} \left|\frac{1}{h(t)}\right| \cdot \sum_{n=1}^{\infty} \frac{1+\beta_1+\beta_2}{\sqrt{\beta_2}} \mu_n^3 \int_0^T |a^{(1)}u_n^{(1)} - a^{(2)}u_n^{(2)}|dt \\ \leq\ & \frac{(1+\beta_1+\beta_2)T}{\sqrt{\beta_2}} \cdot \|\tfrac{1}{h}\|_{C[0,T]} \cdot \left(\sum_{n=1}^{\infty} \frac{1}{\mu_n^2}\right)^{\frac{1}{2}} \cdot \left(\|a^{(1)}\|_{C[0,T]}\|u^{(1)} - u^{(2)}\|_{B_{2,T}^7} \right. \\ & \left. + \|u^{(2)}\|_{B_{2,T}^7}\|a^{(1)} - a^{(2)}\|_{C[0,T]}\right). \end{aligned} \tag{27}$$

Note that we have used Cauchy-Schwartz inequality in the last inequality. We then combine the estimates of $\|\Phi_1(\mathbf{z}^{(1)}) - \Phi_1(\mathbf{z}^{(2)})\|_{B_{2,T}^7}$ and $\|\Phi_2(\mathbf{z}^{(1)}) - \Phi_2(\mathbf{z}^{(2)})\|_{C[0,T]}$ above to obtain that

$$\begin{aligned} & \|\mathbf{\Phi}(\mathbf{z}^{(1)}) - \mathbf{\Phi}(\mathbf{z}^{(2)})\|_{E_{2,T}^7} \\ \leq\ & \tfrac{1}{2}B(T)\left(\|a^{(1)}\|_{C[0,T]}\|u^{(1)} - u^{(2)}\|_{B_{2,T}^7} + \|u^{(2)}\|_{B_{2,T}^7}\|a^{(1)} - a^{(2)}\|_{C[0,T]}\right) \\ \leq\ & \tfrac{1}{2}B(T)(A(T)+2)2\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\|_{E_{2,T}^7} \leq \tfrac{1}{A(T)+2}\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\|_{E_{2,T}^7} < \|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\|_{E_{2,T}^7}. \end{aligned}$$

Therefore, $\mathbf{\Phi}$ is a contraction mapping on K. □

Lemma 2.1-2.3 lead to the conclusion that the inverse problem (1)-(2) has a unique solution in the ball K.

Theorem 2.1. *Given $f(x,t)$, $\phi(x)$, $\psi(x)$ and $h(t)$ that satisfy the following conditions for sufficiently small $T > 0$ with $(A(T)+2)^2 B(T) < 1$:*
(1) $\phi \in C^7[0,1]$, $\phi(0) = \phi''(0) = \phi^{(4)}(0) = \phi^{(6)}(0) = \phi(1) = \phi''(1) = \phi^{(4)}(1) = \phi^{(6)}(1) = 0$.
(2) $\psi \in C^4[0,1]$, $\psi(0) = \psi''(0) = \psi(1) = \psi''(1) = 0$.
(3) $f(x,t) \in C(\overline{D_T})$, $f(\cdot,t) \in C^4[0,1]$ for any $t \in [0,T]$, $f(0,t) = f(1,t) = f_{xx}(0,t) = f_{xx}(1,t) = 0$.
(4) $h(t) \in C^2[0,T]$, $h(t) \neq 0$, $\forall t \in [0,T]$,
the inverse problem (1)-(2) has a unique solution in the ball $K = \{z \in E_{2,T}^7 : \|z\|_{E_{2,T}^7} \leq A(T)+2\}$.

2.2. Well-Posedness

We then consider the well-posedness of the problem. Suppose we take two arbitrary sets of data, denoted by $\{f(x,t), \phi(x), \psi(x), h(t)\}$ and $\{\tilde{f}(x,t), \tilde{\phi}(x), \tilde{\psi}(x), \tilde{h}(t)\}$ that satisfy the conditions in Theorem 2.1, and the corresponding solutions to the inverse problem are $\{u(x,t), a(t)\}$ and $\{\tilde{u}(x,t), \tilde{a}(t)\}$, respectively. Moreover, suppose $\|f\|_{C(\overline{D_T})}$ and $\|\tilde{f}\|_{C(\overline{D_T})} \leq C_{f0}$; $\|f_{xxxx}\|_{C(\overline{D_T})}$ and $\|\tilde{f}_{xxxx}\|_{C(\overline{D_T})} \leq C_f$; $\|\phi^{(7)}\|_{C[0,1]}$, $\|\tilde{\phi}^{(7)}\|_{C[0,1]} \leq C_\phi$; $\|\psi^{(4)}\|_{C[0,1]}$, $\|\tilde{\psi}^{(4)}\|_{C[0,1]} \leq C_\psi$; $\|h\|_{C^2[0,T]}$, $\|\tilde{h}\|_{C^2[0,T]} \leq C_h$; $\|a\|_{C[0,T]}$, $\|\tilde{a}\|_{C[0,T]} \leq C_a$ and $\min_{t\in[0,T]}|h(t)| \geq h_0$ for positive constants C_{f0}, C_f, C_ϕ, C_ψ, C_h, C_a and h_0. Therefore, we can show the estimates of $\|u\|_{B_{2,T}^7}$ as follows:

$$\begin{aligned} \|u\|_{B_{2,T}^7}^2 &\leq \sum_{n=1}^{\infty}\left(\mu_n^7|\phi_n| + \frac{\mu_n^4}{\sqrt{\beta_2}}|\psi_n| + \frac{T\mu_n^4}{\sqrt{\beta_2}}\left(\|a\|_{C[0,T]}\|u_n\|_{C[0,T]} + \|f_n\|_{C[0,T]}\right)\right)^2 \\ &\leq 4\|\phi^{(7)}\|_{C[0,1]}^2 + \frac{4}{\beta_2}\|\psi^{(4)}\|_{C[0,1]}^2 + 4\frac{T^2}{\beta_2}\|a\|_{C[0,T]} \cdot \|u\|_{B_{2,T}^7}^2 + 4\frac{T^2}{\beta_2}\|f_{xxxx}\|_{C[0,T]}^2 \\ &\leq 4C_\phi^2 + \frac{4}{\beta_2}C_\psi^2 + \frac{4T^2C_a}{\beta_2}\|u\|_{B_{2,T}^7}^2 + \frac{4T^2C_f^2}{\beta_2}, \end{aligned} \tag{28}$$

where we have used the fact $b_n \geq \sqrt{\beta_2}\mu_n^3$ in the first inequality; and (16)-(18), (20), Cauchy-Schwartz inequality as well as Bessel's inequality in the second inequality above. For sufficiently small $T > 0$ such that $5T^2C_a < \beta_2$, it follows that $\|u\|_{B_{2,T}^7} \leq C_u$, where $C_u = \sqrt{(4\beta_2C_\phi^2 + 4C_\psi^2 + 4T^2C_f^2)/(\beta_2 - 4T^2C_a)}$. Similarly, we can also get $\|\tilde{u}\|_{B_{2,T}^7} \leq C_u$.

Now we can estimate $\|u-\tilde{u}\|_{B^7_{2,T}}$ and $\|a-\tilde{a}\|_{C[0,T]}$. (7) leads to

$$\begin{aligned}\|u-\tilde{u}\|^2_{B^7_{2,T}} \leq\ & 5\|\phi^{(7)}-\tilde{\phi}^{(7)}\|^2_{C[0,1]}+\tfrac{5}{\beta_2}\|\psi^{(4)}-\tilde{\psi}^{(4)}\|^2_{C[0,1]}+\tfrac{5T^2}{\beta_2}\|a\|^2_{C[0,T]}\cdot\|u-\tilde{u}\|^2_{B^7_{2,T}}\\ &+\tfrac{5T^2}{\beta_2}\|a-\tilde{a}\|^2_{C[0,T]}\cdot\|\tilde{u}\|^2_{B^7_{2,T}}+\tfrac{5T^2}{\beta_2}\|f_{xxxx}-\tilde{f}_{xxxx}\|^2_{C(\overline{D}_T)}.\end{aligned}$$

Since $\|a\|_{C[0,T]}\leq C_a$, $\|\tilde{u}\|_{B^7_{2,T}}\leq C_u$ and $5T^2C_a<\beta_2$, the inequality above leads to

$$\begin{aligned}\|u-\tilde{u}\|_{B^7_{2,T}} \leq\ & M_1\|\phi^{(7)}-\tilde{\phi}^{(7)}\|_{C[0,1]}+M_2\|\psi^{(4)}-\tilde{\psi}^{(4)}\|_{C[0,1]}+TM_3\|a-\tilde{a}\|_{C[0,T]}\\ &+TM_2\|f_{xxxx}-\tilde{f}_{xxxx}\|_{C(\overline{D}_T)},\end{aligned}\tag{29}$$

where $M_1=\sqrt{5\beta_2/(\beta_2-5T^2C_a)}$, $M_2=\sqrt{5/(\beta_2-5T^2C_a)}$ and $M_3=\sqrt{5C_u/(\beta_2-5T^2C_a)}$.

Next, we estimate $\|a-\tilde{a}\|_{C[0,T]}$. From (8), we can show that

$$\begin{aligned}&\|a-\tilde{a}\|_{C[0,T]}\\ \leq\ & \tfrac{1}{h_0^2}\left(\sum_{n=1}^{\infty}b_n^2\left\|u_n\tilde{h}-\tilde{u}_nh\right\|_{C[0,T]}+\left\|h''\tilde{h}-\tilde{h}''h-f(x_0,\cdot)\tilde{h}+\tilde{f}(x_0,\cdot)h\right\|_{C[0,T]}\right)\\ \leq\ & \tfrac{1+\beta_1+\beta_2}{\sqrt{6}h_0^2}\left(C_h\|u-\tilde{u}\|_{B^7_{2,T}}+C_u\|\tilde{h}-h\|_{C[0,T]}\right)+\tfrac{2C_h+C_{f0}}{h_0^2}\|\tilde{h}-h\|_{C^2[0,T]}+\tfrac{C_h}{h_0^2}\|\tilde{f}-f\|_{C(\overline{D_T})}\\ \leq\ & M_4\|u-\tilde{u}\|_{B^7_{2,T}}+M_5\|h-\tilde{h}\|_{C^2[0,T]}+M_6\|f-\tilde{f}\|_{C(\overline{D}_T)},\end{aligned}\tag{30}$$

where $M_4=(1+\beta_1+\beta_2)C_h/(\sqrt{6}h_0^2)$, $M_5=[(1+\beta_1+\beta_2)C_h+(2C_h+C_{f0})\sqrt{6}]/(\sqrt{6}h_0^2)$ and $M_6=C_h/h_0^2$.

Finally, we combine (29) and (30) and eventually obtain

$$\begin{aligned}\|a-\tilde{a}\|_{C[0,T]} \leq\ & \tfrac{D_1}{D_2}\Big(\|\phi^{(7)}-\tilde{\phi}^{(7)}\|_{C[0,1]}+\|\psi^{(4)}-\tilde{\psi}^{(4)}\|_{C[0,1]}+\|f_{xxxx}-\tilde{f}_{xxxx}\|_{C(\overline{D}_T)}\\ &+\|h-\tilde{h}\|_{C^2[0,1]}+\|f-\tilde{f}\|_{C(\overline{D}_T)}\Big),\\ \|u-\tilde{u}\|_{B^7_{2,T}} \leq\ & \tfrac{D_3}{D_2}\Big(\|\phi^{(7)}-\tilde{\phi}^{(7)}\|_{C[0,1]}+\|\psi^{(4)}-\tilde{\psi}^{(4)}\|_{C[0,1]}+\|f_{xxxx}-\tilde{f}_{xxxx}\|_{C(\overline{D}_T)}\\ &+\|h-\tilde{h}\|_{C^2[0,1]}+\|f-\tilde{f}\|_{C(\overline{D}_T)}\Big),\end{aligned}\tag{31}$$

where we denoted by $D_1=\max\{M_1M_4,M_2M_4,TM_2M_4,M_5,M_6\}$, $D_2=1-TM_3M_4$, $D_3=\max\{M_1,M_2,TM_2,TM_3M_5,TM_3M_6\}$ for sufficiently small $T>0$. Inequality (31) and (31) imply that the solution of the inverse problem depends on the given data continuously.

3. Numerical Experiments

In this section, we introduce the numerical methods to solve the inverse problem (1)-(2) and present some numerical results.

For a given final time T, we divide the time domain $[0,T]$ into N_t steps with uniform step size Δt. We first compute $a^0:=a(0)$ using

$$a^0=\tfrac{h''(0)-\phi''(x_0)+\beta_1\phi^{(4)}(x_0)+\beta_2\phi^{(6)}(x_0)-f(x_0,0)}{h(0)},\tag{32}$$

and initialize $u_N^0:=u_N(0)$ for $N=1,2,\ldots,N_{mode}$, where u_N^0 can be computed using discrete sine transformation. We then compute $u_N^1:=u_N(\Delta t)$ for $N=1,2,\ldots,N_{mode}$ using the initial conditions and the original PDE. In particular, using Taylor expansion of u at $t=\Delta t$, we get

$$u(x,\Delta t)\approx\phi(x)+\psi(x)\Delta t+\tfrac{(\Delta t)^2}{2}\left(\phi''(x)-\beta_1\phi^{(4)}(x)+\beta_2\phi^{(6)}(x)+a(0)\phi(x)+f(x,0)\right).$$

Thus, we compute u_N^1 using

$$u_N^1=\phi_N+\psi_N\Delta t+\tfrac{(\Delta t)^2}{2}\left((\phi''-\beta_1\phi^{(4)}+\beta_2\phi^{(6)})_N+a^0\phi_N+f_N^0\right),\quad N=1,2,\ldots,N_{mode},$$

where $(\phi''-\beta_1\phi^{(4)}+\beta_2\phi^{(6)})_N$ presents the N^{th} mode of the sine transformation of $\phi''-\beta_1\phi^{(4)}+\beta_2\phi^{(6)}$, and $f_N^0=f^N(0)$. Next, $a^1=\frac{1}{h(\Delta t)}\left(h''(\Delta t)-f(x_0,\Delta t)-u_{xx}(x_0,\Delta t)\right.$

$+\ \beta_1 u_{xxxx}(x_0,\Delta t) - \beta_2 u_{xxxxxx}(x_0,\Delta t))$, where $u_{xx}(x_0,\Delta t) = -\sqrt{2}\sum_{N=1}^{N_{mode}} u_N^1 \mu_n^2 \sin(\mu_n x_0)$. $u_{xxxx}(x_0,\Delta t)$ and $u_{xxxxxx}(x_0,\Delta t)$ can be computed similarly.

For the rest of the simulations, we compute $u_N^i := u_N(i\Delta t)$ and $a^i := a(i\Delta t)$ for $N = 1, 2, \ldots, N_{mode}$ in alternating order. That is, for $i = 2, 3, \ldots N_t$, we update u_N^i and a^i as follows:

$$u_N^i = 2u_N^{i-1} - u_N^{i-2} + (\Delta t)^2\left(-b_N^2 u_N^{i-1} + a^{i-1}u_N^{i-1} + f_N^{i-1}\right), \quad \forall N, \tag{33}$$

$$a^i = \frac{1}{h(i\Delta t)}\left[h''(i\Delta t) - f(x_0, i\Delta t) - u_{xx}(x_0, i\Delta t) + \beta_1 u_{xxxx}(x_0, i\Delta t) - \beta_2 u_{xxxxxx}(x_0, i\Delta t)\right]. \tag{34}$$

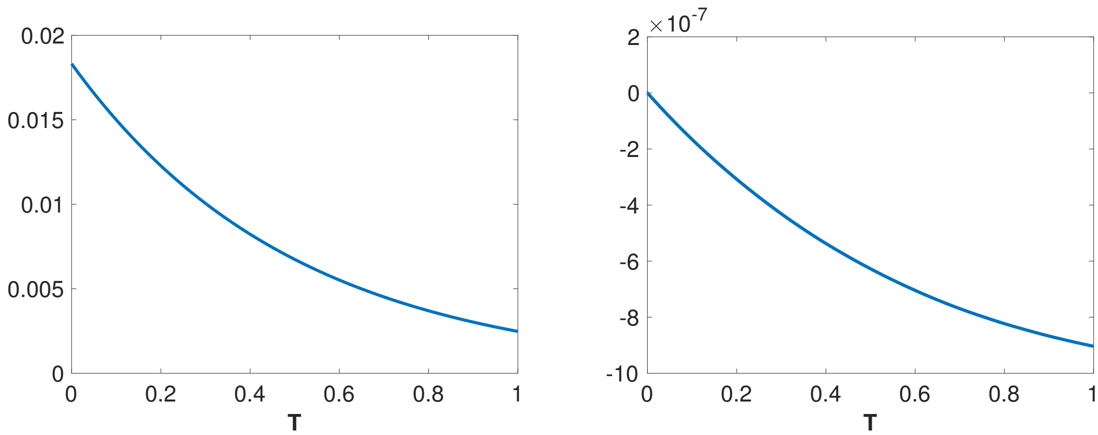
Finally, we compute the numerical solution of $u(x,T)$ using the inverse sine transformation of $u_N^{N_t}$ in (33).

Example 1

In this numerical example, we take the parameters in equation (1)-(2) as $\beta_1 = \beta_2 = 1$ and $x_0 = 1/2$. We choose the following data

$$\begin{cases} \phi(x) = \psi(x) = \exp(2)\sin(\pi x), \quad h(t) = \exp(t+2), \\ f(x,t) = \left((1+\pi^2+\pi^4+\pi^6)\exp(t+2) - \exp(-t-2)\right)\sin(\pi x), \end{cases} \tag{35}$$

for $x \in [0,1]$ and $t \in [0,T]$. The exact solution to the inverse problem is given by $u(x,t) = \exp(t+2)\sin(\pi x)$ and $a(t) = \exp(-2t-4)$. We take $N_{mode} = 7$, $\Delta t = 10^{-4}$, $T = 1$. Our numerical results show that the absolute error of $a(t)$ for $t \in [0,1]$ is 9.0372×10^{-7} (see Figure 1), and the absolute maximum error of $u(x, t = 1)$ is 1.6987×10^{-8} (see Figure 2).

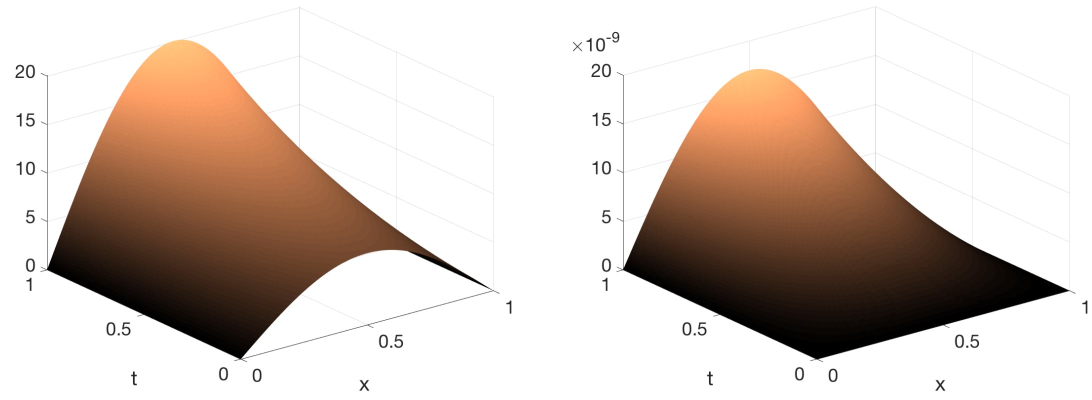


(A) Numerical solution of $a(t)$ (B) Error of $a(t)$

FIGURE 1. Numerical solution and error of $a(t)$ in example 1.

Example 2

We then consider the next numerical example, where we take parameters to be $\beta_1 = 1$, $\beta_2 = 0.01$, $x_0 = 1/4$ and $T = 0.1$. The given data is as follows



(A) Numerical solution of $u(x,t)$ (B) Error of $u(x,t)$

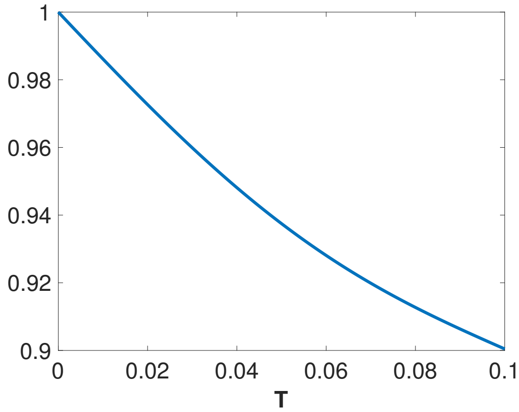
Figure 2. Numerical solution and error of $u(x,t)$ for $(x,t) \in [0,1] \times [0,1]$ in example 1.

$$\begin{cases} \phi(x) = \sin(\pi x) + \sin(2\pi x), \quad \psi(x) = \sin(\pi x) - \sin(2\pi x), \\ h(t) = \sin(\pi/4) + \exp(-t), \\ f(x,t) = (\pi^2 + \pi^4 + 0.01\pi^6)\sin(\pi x) + (1 + 4\pi^2 + 16\pi^4 + 0.64\pi^6)\times \\ \qquad \exp(-t)\sin(2\pi x) - \exp(-t)\sin(\pi x) - \exp(-2t)\sin(2\pi x). \end{cases} \tag{36}$$

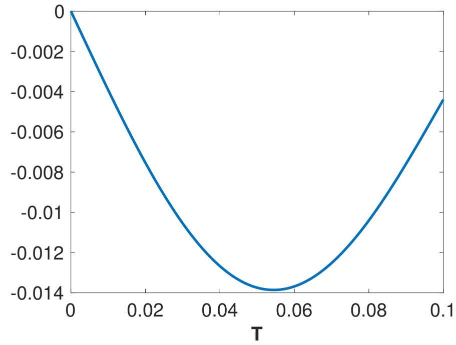
The exact solution to the inverse problem is $u(x,t) = \sin(\pi x) + \sin(2\pi x)\exp(-t)$ and $a(t) = \exp(-t)$. Our numerical simulations show that the error of $a(t)$ is sensitive to the accuracy of u_N^1. For $\Delta t = 10^{-4}$, the absolute error of $u(x,t)$ at $t = 0.1$ is 4.0667×10^{-3} and the absolute error of $a(t)$ for $t \in [0, 0.1]$ is 1.385. To obtain more accurate results, we can take $\Delta t = 10^{-6}$ which leads to the absolute error of $u(x,T)$ to be 4.0717×10^{-5} and the absolute error of $a(t)$ to be 1.3857×10^{-2}. Figure 3 shows the numerical solution and error of $a(t)$ for $t \in [0, 0.1]$. As t increases, the absolute error of $a(t)$ increases and then decreases. The numerical solution and error of $u(x,t)$ are given in Figure 4. The magnitude in the error in this example is much larger compared to the results in the previous example. This is due to the fact that the exact solution has more nonzero modes, which leads to a larger error when we compute $u_{xx}(x_0,t)$, $u_{xxxx}(x_0,t)$ and $u_{xxxxxx}(x_0,t)$, and it gives a larger error in a.

4. **Conclusions**

In this paper, we study and analyze an inverse problem for the sixth-order linear Boussinesq-type equation. Under certain conditions of the given data, we prove that the solution of the inverse problem exists and it is unique in a ball of the Banach space. Moreover, the solution depends continuously on the given data. The numerical methods for this problem is to update a and u in alternating order. Numerical results show that our numerical methods lead to accurate solutions for sufficiently small Δt.

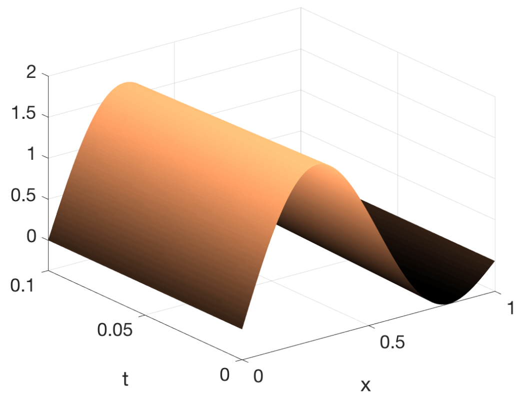


(A) Numerical solution of $a(t)$

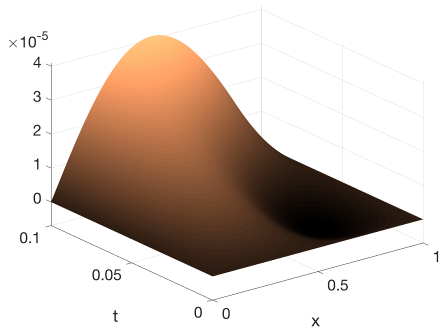


(B) Error of $a(t)$

FIGURE 3. Numerical solution and error of $a(t)$ for $t \in [0, 1]$ in example 2.



(A) Numerical solution of $u(x, t)$



(B) Error of $u(x, t)$

FIGURE 4. Numerical solution and error of $u(x, t)$ for $(x, t) \in [0, 1] \times [0, 0.1]$ in example 2.

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