STUDY ON OPERATOR REPRESENTATION OF FRAMES IN HILBERT SPACES

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The purpose of this paper is to give an overview of the operator structure of frames $\{f_k\}_{k=1}^{\infty}$ as $\{T^k\phi\}_{k=0}^{\infty}$, where the operator $T : \mathcal{H} \to \mathcal{H}$ belongs to certain classes of linear operators and the element ϕ belongs to $\mathcal{H}$. We discuss the size of the set of such elements. Also, for a given frame $\{f_k\}_{k=1}^{\infty}$ and any $n \in \mathbb{N}$, some results are obtained for $T^n(\{f_k\}_{k=1}^{\infty}) = \{T^n f_1, T^n f_2, \cdots\}$. Finally, we conclude this note by raising several questions connecting frame theory and operator theory.

Keywords: frames, operator representation of frames, spectrum, spectral radius, hypercyclic operators.

MSC2020: 42C15.

1. Introduction

The system of iterations $\{T^k\phi\}_{k=0}^{\infty}$, where T is a bounded linear operator on a separable Hilbert space $\mathcal{H}$ and $\phi \in \mathcal{H}$, the so-called dynamical sampling problem is a relatively new research topic in Harmonic analysis. This topic has been studied since the work of Aldroubi and Petrosyan [1] for some new results concerning frames and Bessel systems, see for example Aldroubi et al. [2] to study dynamical sampling problem in finite dimensional spaces. Christensen et al. [9, 11, 12] studied some properties of systems arising via iterated actions of operators and frame properties of operator orbits. For more details, we refer the interested reader to the [1, 4, 7, 8].

Let $F := \{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ which spans an infinite dimensional subspace. A natural question to ask is whether there exists a linear operator T such that $f_{k+1} = Tf_k$, for all $k \in \mathbb{N}$?

In [12], it was proved that such an operator exists if and only if F is linearly independent; also, T is bounded if and only if the kernel of the synthesis operator of F is invariant under the right shift operator on $\ell^2(\mathbb{N})$, in the affirmative case, $F = \{T^k f_1\}_{k=0}^{\infty}$.

In this note, for given frame F, some results are obtained for $T^n(F) = \{T^n f_1, T^n f_2, \cdots\}$. Note that the problem considered in this note is of some interest from other points of view. Indeed, assume that the system $\{T^k\phi\}_{k=0}^{\infty}$ is a frame for $\mathcal{H}$. It is natural to ask for a characterization of $\mathcal{V}(T)$: the set of all $\phi \in \mathcal{H}$ such that $\{T^k\phi\}_{k=0}^{\infty}$ is a frame for $\mathcal{H}$. In [11], $\mathcal{V}(T)$ is obtained by applying all invertible operators from the set of commutant T' of T to ϕ. The chief aim of this paper is to understand the size of the set $\mathcal{V}(T)$ with a restriction on T. Also, we show that the set of all invertible operators $T \in B(\mathcal{H})$, for which $\{T^k\phi\}_{k=0}^{\infty}$ is a frame for $\mathcal{H}$ for some $\phi \in \mathcal{H}$, is open. Finally, we close this work by raising some questions connecting frame theory and operator theory.

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In the following, we denote by $\mathcal{H}$ a separable Hilbert space. We use $B(\mathcal{H})$ for the set of all bounded linear operators on $\mathcal{H}$, $\mathbb{N}$ for the natural numbers as the index set and we take $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The spectrum of an operator $T \in B(\mathcal{H})$ is denoted by $\sigma(T)$, which is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

and the spectral radius of T is denoted by $r(T)$, which is defined as

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Given an operator Λ, we denote its domain by $dom(\Lambda)$ and its range by $ran(\Lambda)$.

Definition 1.1. [8] *A sequence of vectors F in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ so that*

$$A \parallel f \parallel^2 \leq \sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 \leq B \parallel f \parallel^2 , \tag{1}$$

for all $f \in \mathcal{H}$.

It follows from the definition that if F is a frame for $\mathcal{H}$, then

$$\overline{span} F = \mathcal{H} . \tag{2}$$

If $A = B$, then F is called a *tight frame* and if $A = B = 1$, then F is called a *Parseval frame or a normalized tight frame.* Moreover, F is called a *Bessel sequence* if at least the upper condition in (1) holds.

For any sequence F in $\mathcal{H}$, the associated *synthesis operator* is defined by

$$U_F : D_1(F) \to \mathcal{H}; \quad U_F(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k f_k ,$$

where

$$D_1(F) := \left\{ \{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) : \sum_{k=1}^{\infty} c_k f_k \text{ converges in } \mathcal{H} \right\}.$$

The *analysis operator* is defined by

$$U_F^* : D_2(F) \to \ell^2(\mathbb{N}); \quad U_F^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty} ,$$

where

$$D_2(F) := \left\{ f \in \mathcal{H} : \{\langle f, f_k \rangle\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \right\}.$$

Furthermore, the *frame operator* is defined by

$$S_F : D_3(F) \to \mathcal{H}; \quad S_F f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k ,$$

where

$$D_3(F) := \left\{ f \in \mathcal{H} : \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \text{ converges in } \mathcal{H} \right\}.$$

Remark 1.1. *It is known that if F is at least a Bessel sequence, then*

(i) *the operator U_F is well-defined and $D_1(F) = \ell^2(\mathbb{N})$,*
(ii) *the operators S_F and U_F^* are well-defined and the domain of both is $\mathcal{H}$.*

Definition 1.2. [8] *A sequence of vectors F in $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ with bounds $0 < A \le B < \infty$, if F satisfies the relation (2) and for every scalar sequence $(c_k)_k \in \ell^2$, one has*

$$A \sum_k |c_k|^2 \le \| \sum_k c_k f_k \|^2 \le B \sum_k |c_k|^2 \ . \tag{3}$$

We have the following result which is well-known in frame theory in general case, (see for example [3]), we include a sketch of the proof.

Proposition 1.1. *Suppose that F is a frame (not tight frame) for a Hilbert space $\mathcal{H}$ with bounds A and B, respectively, and with frame operator S_F. If $f_j \in F$ such that $||S_F^{-1/2} f_j|| \le \sqrt{\frac{A}{B}}$, then $F \setminus \{f_j\}$ is a frame for $\mathcal{H}$.*

Proof. Let $f_j \in F$ such that $||S_F^{-1/2} f_j|| \le \sqrt{\frac{A}{B}}$. It is known that $\{S_F^{-1/2} f_k\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$, (see [14, Corollary 6.3.5] and [6, Theorem III.2]), so for any $h \in \mathcal{H}$ we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle h,\ S_F^{-1/2} f_k \rangle|^2 - |\langle h,\ S_F^{-1/2} f_j \rangle|^2 \quad &\ge \quad ||h||^2 - ||S_F^{-1/2} f_j||^2 ||h||^2 \\ &\ge \quad (1 - \frac{A}{B}) ||h||^2. \end{aligned}$$

That is, the sequence $S_F^{-1/2}(F \setminus \{f_j\})$ is a frame for $\mathcal{H}$ with lower frame bound $1 - \frac{A}{B}$. Since $S_F^{-1/2}$ is an invertible operator on $\mathcal{H}$, thus $F \setminus \{f_j\}$ is a frame for $\mathcal{H}$ as well. □

The availability of the representation $F = \{T^k f_1\}_{k=0}^{\infty}$ is characterized in [12]:

Proposition 1.2. *Consider any sequence F in $\mathcal{H}$ for which $spanF$ is infinite-dimensional. Then the following are equivalent:*

(i) *F is linearly independent.*
(ii) *There exists a linear operator $T : spanF \to \mathcal{H}$ such that $F = \{T^k f_1\}_{k=0}^{\infty}$.*

2. **Main results**

We begin with the following notations. Note that the sequence F being represented by T means that

$$F := \{f_k\}_{k=1}^{\infty} = \{f_1, f_2, f_3 \cdots\} = \{f_1, Tf_1, T^2 f_1 \cdots\} = \{T^k f_1\}_{k=0}^{\infty}.$$

Therefore, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} T_n(F) := T^n(F) = \{T^n f_k\}_{k=1}^{\infty} \quad &= \quad \{T^n f_1, T^n f_2, \cdots\} \\ &= \quad \{f_{n+1}, f_{n+2}, \cdots\} = F \backslash \{f_1, f_2, \cdots, f_n\}. \end{aligned}$$

Throughout this section, for any $n \in \mathbb{N}$, we simply take $T_n := T^n$. Suppose that F is a sequence in $\mathcal{H}$ of the form $\{T^k f_1\}_{k=0}^{\infty}$ for some operator T. For any $n \in \mathbb{N}$, the associated synthesis, analysis, and frame operators of $T_n(F)$ are given by the following proposition:

Proposition 2.1. *The synthesis, the analysis, and the frame operators for $T_n(F)$ are given by $T_n U_F$, $U_F^* T_n^*$, and $T_n S_F T_n^*$, respectively.*

Proof. For each $n \in \mathbb{N}$, we have

$$U_{T_n(F)}(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k T_n f_k = \sum_{k=1}^{\infty} T_n(c_k f_k) = T_n U_F(\{c_k\}_{k=1}^{\infty}),$$

it follows that $U_{T_n(F)} = T_n U_F$.
For the analysis operator, we have

$$U^*_{T_n(F)} f = \{\langle f,\ T_n f_k\rangle\}_{k=1}^{\infty} = \{\langle T_n^* f,\ f_k\rangle\}_{k=1}^{\infty} = U_F^* T_n^* f,$$

it follows that $U^*_{T_n(F)} = U_F^* T_n^*$. In addition, we have

$$S_{T_n(F)} f = \sum_{k=1}^{\infty} \langle f,\ T_n f_k\rangle T_n f_k = T_n \sum_{k=1}^{\infty} \langle T_n^* f,\ f_k\rangle f_k = T_n S_F T_n^* f,$$

it follows that $S_{T_n(F)} = T_n S_F T_n^*$. □

It is well-known that F is a Riesz basis for $\mathcal{H}$ if and only if the analysis operator U_F^* is bijective (because by [8, Theorem (5.4.1)] and [13, Proposition (5.1.5)], F is a Riesz basis if and only if F is a frame and U_F^* is surjective and also, by [8, Corollary (5.5.3)] F is a frame if and only if U_F^* is an injective operator with closed range). Now we have the following result:

Proposition 2.2. *Let F be a Riesz basis for $\mathcal{H}$ of the form $\{T^k f_1\}_{k=0}^{\infty}$, for some operator T. Then, for any $n \in \mathbb{N}$, $T_n(F)$ is a Riesz basis for $\mathcal{H}$ if and only if T_n is an invertible operator on $\mathcal{H}$.*

Proof. Since F is a Riesz basis for $\mathcal{H}$, then the analysis operator U_F^* is invertible. For any $n \in \mathbb{N}$, if T_n is an invertible operator on $\mathcal{H}$, then $U_F^* T_n^*$ (the analysis operator of $T_n(F)$) is invertible. Therefore, for any $n \in \mathbb{N}$, $T_n(F)$ is a Riesz basis for $\mathcal{H}$. Conversely, for any $n \in \mathbb{N}$, if $T_n(F)$ is a Riesz basis for $\mathcal{H}$, then the analysis operator $U_F^* T_n^*$ is invertible. Also, since the analysis operator U_F^* is an invertible operator on $\mathcal{H}$, it follows that T_n^* and then T_n is an invertible operator on $\mathcal{H}$. □

We give some relations associated to the domain and range of the synthesis, analysis, and frame operators for $T_n(F)$.

Proposition 2.3. *For any $n \in \mathbb{N}$, the following statements hold:*

(i) $dom(T_n S_F T_n^*) = dom(S_{T_n(F)}) \subseteq dom(U^*_{T_n(F)}) = dom(U_F^* T_n^*)$,
(ii) $ran(T_n S_F T_n^*) = ran(S_{T_n(F)}) \subseteq ran(U_{T_n(F)}) = ran(T_n U_F)$,
(iii) $dom(S_{T_n(F)}) = dom(U^*_{T_n(F)})$ *iff* $ran(U^*_{T_n(F)}) \subseteq dom(U_{T_n(F)})$.

Proof. For any $n \in \mathbb{N}$, it is known that

$$dom(T_n S_F T_n^*) = dom(S_{T_n(F)}) = \{f \in \mathcal{H} :\ \sum_{k=1}^{\infty} \langle f,\ T_n f_k\rangle T_n f_k \text{ converges in } \mathcal{H}\},$$

$$dom(U_F^* T_n^*) = dom(U^*_{T_n(F)}) = \{f \in \mathcal{H} :\ \{\langle f,\ T_n f_k\rangle\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})\},$$

$$\begin{aligned} dom(T_n U_F) &= dom(U_{T_n(F)}) \\ &= \{(c_k)_{k=1}^{\infty} \in \ell^2(\mathbb{N}) :\ \sum_{k=1}^{\infty} c_k T_n f_k = \sum_{k=1}^{\infty} T_n(c_k f_k) \text{ converges in } \mathcal{H}\}. \end{aligned}$$

For (i), let $f \in dom(S_{T_n(F)})$. It follows that

$$\sum_{k=1}^{\infty} \langle f,\ T_n f_k \rangle T_n f_k \ \text{ converges in } \mathcal{H}.$$

Then, we obtain that

$$\left\langle \sum_{k=1}^{N} \langle f,\ T_n f_k \rangle T_n f_k,\ f \right\rangle \longrightarrow \langle S_{T_n(F)} f,\ f \rangle, \ \text{ as } N \longrightarrow \infty,$$

which implies that

$$\sum_{k=1}^{N} |\langle f,\ T_n f_k \rangle|^2 = \sum_{k=1}^{N} |\langle T_n^* f,\ f_k \rangle|^2 \ \text{ converges as } N \longrightarrow \infty.$$

Therefore, $f \in dom(U^*_{T_n(F)}) = dom(U^*_F T^*_n)$.

(ii) follows from the fact that

$$S_{T_n(F)} = T_n S_F T_n^* = T_n U_F U_F^* T_n^* = \underbrace{T_n U_F}\, U^*_{T_n(F)}.$$

For (iii), let $dom(S_{T_n(F)}) = dom(U^*_{T_n(F)})$ and take $U^*_{T_n(F)} f \in ran(U^*_{T_n(F)})$. Then, $f \in dom(S_{T_n(F)})$ which implies that $U^*_{T_n(F)} f \in dom(U_{T_n(F)})$. Now, if $ran(U^*_{T_n(F)}) \subseteq dom(U_{T_n(F)})$, it is clear that $dom(U^*_{T_n(F)}) \subseteq dom(S_{T_n(F)})$ and the other inclusion is given in (i). Therefore,
$dom(S_{T_n(F)}) = dom(U^*_{T_n(F)})$, as desired. □

For a given operator $T \in B(\mathcal{H})$, let

$$\mathcal{V}(T) := \left\{ \phi \in \mathcal{H} : \{T^k \phi\}_{k \in \mathbb{N}_0} \text{ is a frame for } \mathcal{H} \right\}.$$

Also, let

$$\mathcal{E}(\mathcal{H}) := \left\{ T \in B(\mathcal{H}) : \{T^k \phi\}_{k \in \mathbb{N}_0} \text{ is a frame for } \mathcal{H}, \text{ for some } \phi \in \mathcal{H} \right\}.$$

Let $T \in B(\mathcal{H})$ be an operator for which there exists some $f \in \mathcal{H}$, such that $\{T^k f\}_{k=0}^{\infty}$ is a frame for $\mathcal{H}$. A natural question to ask is whether there exist other vectors $\phi \in \mathcal{H}$ for which $\{T^k \phi\}_{k=0}^{\infty}$ is also a frame for $\mathcal{H}$. In the following theorem, we consider a viewpoint of this query; we discuss the size of the set of vectors $\phi \in \mathcal{H}$ for which $\{T^k \phi\}_{k=0}^{\infty}$ is a frame for $\mathcal{H}$, with a restriction on T.

Recall that an operator $T \in B(\mathcal{H})$ is said to be *hypercyclic* if there is some vector $\phi \in \mathcal{H}$, such that its T-orbit

$$O(\phi, T) := \{T^k \phi\}_{k \in \mathbb{N}_0}$$

is dense in $\mathcal{H}$. Such a vector ϕ is said to be hypercyclic for T.

Proposition 2.4. *Let $T \in B(\mathcal{H})$ be invertible. Then*

$$\bigcap_{f \in \mathcal{V}(T),\ k \in \mathbb{N}} \mathbf{B}(f, k) \subseteq \mathcal{V}(T), \tag{4}$$

where,

$$\mathbf{B}(f, k) = \bigcup_{n \in \mathbb{N}_0} \{\phi \in \mathcal{H} : \|T^n \phi - f\| < \frac{1}{k}\}.$$

Moreover, if T admits $\mathcal{V}(T)$ as hypercyclic vectors, then

$$\mathcal{V}(T) = \bigcap_{f \in \mathcal{V}(T),\ k \in \mathbb{N}} \mathbf{B}(f,k). \tag{5}$$

Proof. For any $f \in \mathcal{V}(T)$, it is obvious that $T \in \mathcal{E}(\mathcal{H})$. Now, let $B_{f,k}$ be an open ball centered at $f \in \mathcal{V}(T)$ and with radius $\frac{1}{k}$. Then by continuity, for any $n \in \mathbb{N}_0$

$$(T^n)^{-1} B_{f,k} = \{\phi \in \mathcal{H} : \|T^n \phi - f\| < \frac{1}{k}\},$$

is open in $\mathcal{H}$. Therefore, for any $k \in \mathbb{N}$

$$\mathbf{B}(f,k) := \bigcup_{n \in \mathbb{N}_0} (T^n)^{-1} B_{f,k}, \tag{6}$$

is open in $\mathcal{H}$.

Claim:

$$\bigcap_{f \in \mathcal{V}(T),\ k \in \mathbb{N}} \mathbf{B}(f,k) \subseteq \mathcal{V}(T).$$

Let $\phi \in \bigcap_{f \in \mathcal{V}(T), k \in \mathbb{N}} \mathbf{B}(f,k)$, then for any $f \in \mathcal{V}(T)$ and any $k \in \mathbb{N}$,

$$\phi \in \mathbf{B}(f,k). \tag{7}$$

Hence, for any $f \in \mathcal{V}(T)$ and any $k \in \mathbb{N}$, from (6) and (7) we conclude that there exists $m \in \mathbb{N}_0$ such that $\phi \in (T^m)^{-1} B_{f,k}$, it follows that

$$T^m \phi \in B_{f,k}. \tag{8}$$

Now we need the following fact in the sequel.

Fact. For any $f \in \mathcal{V}(T)$ and for $k \in \mathbb{N}$ sufficiently large, $B_{f,k} \subset \mathcal{V}(T)$.

Proof of the fact. We have to show that for any $f \in \mathcal{V}(T)$, if $\|f - \phi\|$ is small enough, then $\phi \in \mathcal{V}(T)$, i.e., $\{\phi_i\}_{i \in \mathbb{N}_0} := \{T^i \phi\}_{i \in \mathbb{N}_0}$ is a frame for $\mathcal{H}$. For this purpose, it is enough that $\{\phi_i\}_{i \in \mathbb{N}_0}$ satisfies the assumptions of Theorem (2) in [5]. Let A be a lower frame bound for the frame $\{f_i\}_{i \in \mathbb{N}_0} := \{T^i f\}_{i \in \mathbb{N}_0}$. We have

$$\begin{aligned} \|\sum_{i=0}^{n} c_i (f_i - \phi_i)\| &= \sup_{\|g\|=1} |\langle \sum_{i=0}^{n} c_i (f_i - \phi_i),\ g\rangle| \\ &\leq \sup_{\|g\|=1} \sum_{i=0}^{n} |c_i \langle (f_i - \phi_i),\ g\rangle| \\ \text{(Cauchy-Schwarz inequality)} &\leq (\sum_{i=0}^{n} |c_i|^2)^{1/2} \sup_{\|g\|=1} (\sum_{i=0}^{n} |\langle f_i - \phi_i,\ g\rangle|^2)^{1/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup_{\|g\|=1} (\sum_{i=0}^{n} |\langle f_i - \phi_i,\ g\rangle|^2)^{1/2} &= \sup_{\|g\|=1} (\sum_{i=0}^{n} |\langle T^i (f - \phi),\ g\rangle|^2)^{1/2} \\ &\leq \sup_{\|g\|=1} (\sum_{i=0}^{n} \|T\|^{2i} \|f - \phi\|^2 \|g\|^2)^{1/2} \\ &\leq \frac{1}{k} (\sum_{i=0}^{n} \|T\|^{2i})^{1/2}. \end{aligned}$$

Therefore, we have

$$\|\sum_{i=0}^{n} c_i(f_i - \phi_i)\| \leq \mu(\sum_{i=0}^{n} |c_i|^2)^{1/2},$$

where $\mu := \frac{1}{k}(\sum_{i=0}^{n} \|T\|^{2i})^{1/2}$. Choose $k \in \mathbb{N}$ sufficiently large such that $\mu < \sqrt{A}$. Therefore, $\{\phi_i\}_{i\in\mathbb{N}_0} := \{T^i\phi\}_{i\in\mathbb{N}_0}$ satisfies in [5, Theorem (2)] with $\lambda_1 = \lambda_2 = 0$.

Hence, by (8) and the previous fact we have $T^m\phi \in \mathcal{V}(T)$, it follows that $\{T^n(T^m\phi)\}_{n\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$. By adding finite elements of points $\{\phi, T\phi, \cdots T^{m-1}\phi\}$ to the system $\{T^n(T^m\phi)\}_{n=0}^{\infty}$, it yields that the system $\{T^n\phi\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$, that is, $\phi \in \mathcal{V}(T)$ which proves inclusion (4).

Finally, let T admit any $\phi \in \mathcal{V}(T)$ as a hypercyclic vector. Let $\phi \in \mathcal{V}(T)$. Since the T-orbit $O(\phi, T) := \{T^n\phi\}_{n\in\mathbb{N}_0}$ is dense in $\mathcal{V}(T)$ with respect to the relative topology, then for any $f \in \mathcal{V}(T)$ and any $k \in \mathbb{N}$, there exists $n \in \mathbb{N}_0$ such that $T^n\phi \in B_{f,k}$, it follows that $\phi \in (T^n)^{-1}B_{f,k}$. Therefore,

$$\phi \in \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}_0} (T^n)^{-1}B_{f,k}.$$

It follows that $\phi \in \bigcap_{k\in\mathbb{N}} \mathbf{B}(f, k)$, for any $f \in \mathcal{V}(T)$. Therefore,

$$\mathcal{V}(T) \subseteq \bigcap_{f\in\mathcal{V}(T),\ k\in\mathbb{N}} \mathbf{B}(f, k).$$

The other inclusion is given in (4). Therefore,

$$\mathcal{V}(T) = \bigcap_{f\in\mathcal{V}(T),\ k\in\mathbb{N}} \mathbf{B}(f, k),$$

as desired. □

It is well-known that for any $T \in B(\mathcal{H})$, $\sigma(T)$ is included in the ball $B(0, \|T\|)$. Therefore, $r(T) \leq \|T\|$ for any $T \in B(\mathcal{H})$. In [10], it is proved that $\mathcal{E}(\mathcal{H})$ does not form an open set in $B(\mathcal{H})$. The next proposition yields that the set of invertible elements in $\mathcal{E}(\mathcal{H})$ is relatively open in $\mathcal{E}(\mathcal{H})$.

Proposition 2.5. *The set of invertible operators in $\mathcal{E}(\mathcal{H})$ is relatively open in $\mathcal{E}(\mathcal{H})$.*

Proof. Suppose that $\widetilde{\mathcal{E}}(\mathcal{H})$ is a set of invertible operators in $\mathcal{E}(\mathcal{H})$ and take $T \in \widetilde{\mathcal{E}}(\mathcal{H})$. For each $U \in \mathcal{E}(\mathcal{H})$, we have

$$U = T(I - T^{-1}(T - U)). \tag{9}$$

Assume that $\mathcal{N}_r(T)$ is an open neighborhood centered at T and with radius $r := \|T^{-1}\|^{-1}$, i.e., for every $U \in \mathcal{N}_r(T)$, we have

$$\|T - U\| < \|T^{-1}\|^{-1}.$$

Also, we have

$$\begin{aligned} r(T^{-1}(T-U)) &\leq \|T^{-1}(T-U)\| \\ &\leq \|T^{-1}\|\|(T-U)\| \\ &< \|T^{-1}\|\|T^{-1}\|^{-1} \\ &= 1. \end{aligned}$$

Hence, the value 1 is not in $\sigma(T^{-1}(T-U))$, that is, the operator $I - T^{-1}(T-U)$ is invertible, and therefore, by (9) U is invertible and belongs to $\widetilde{E}(\mathcal{H})$. We have proved that there exists an open ball around T made of bounded invertible operators in $\mathcal{E}(\mathcal{H})$, that is, $\widetilde{\mathcal{E}}(\mathcal{H})$ is open. □

Remark 2.1. *It is clear that $\mathcal{E}(\mathcal{H})$ cannot be dense in $B(\mathcal{H})$ with respect to the norm topology. Indeed, by Prop. 2.2. in* [9], *every operator $T \in \mathcal{E}(\mathcal{H})$ has norm greater than or equal to* 1. *Therefore, the norm topology is not always the most natural topology on $B(\mathcal{H})$. It is often more useful to consider the weakest topology on $B(\mathcal{H})$, the so-called strong operator topology, which is defined by the family of seminorms $\{p_h : h \in \mathcal{H}\}$, where $p_h(T) = \|Th\|$.*

We conclude this note by raising the following questions:

Q1. For $T \in B(\mathcal{H})$, take $c = \|T\| + \alpha$, $(0 < \alpha < 1)$ and replace the norm topology by the strong operator topology. What can we say for the size of the set of all operators in $\mathcal{E}(\mathcal{H})$ with the norm of at most c?

Q2. Does there exist $T \in \mathcal{E}(\mathcal{H})$ such that $T^{-1} \in \mathcal{E}(\mathcal{H})$?

Q3. Let $T \in B(\mathcal{H})$ be invertible and that there exists a dense subset $D \subset \mathcal{H}$ such that T^k and $(T^{-1})^k$ tend to zero, as $k \to \infty$ on D. Does there exist $\varphi \in \mathcal{H}$ such that $\{T^k\phi\}_{k=0}^{\infty}$ and $\{(T^{-1})^k\phi\}_{k=0}^{\infty}$ are frames in $\mathcal{H}$?

Acknowledgment
The authors would like to thank the anonymous reviewers for important comments and suggestions that improved the quality of the paper.

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