SOLITARY WAVE AND SHOCK WAVE SOLUTIONS OF THE VARIANTS OF BOUSSINESQ EQUATIONS

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This paper obtains the solitary wave as well as the shock wave solutions of the variants of the Boussinesq equations in both (1+1) and (1+2) dimensions. The domain restrictions are also identified in the process.

Keywords: Evolution equations; solitons; integrability;

1. Introduction

The theory of nonlinear evolution equations (NLEEs) is a very important area of research in the fields of Applied Mathematics and Theoretical Physics [1-15]. There are various issues of NLEEs that need to be addressed. These include the integrability aspect, conservation laws, wave interactions and many more. In this paper, the first aspect is going to be studied in detail for a few generalized versions of familiar NLEEs.

There are various tools that have been developed in the past couple of decades that enables the issue of integrability of NLEEs to be addressed with ease. Some of these familiar tools of integrability are variational iteration method, semi-inverse variational principle, $G'/G-$expansion method, exponential function method, Fan's F-expansion method, simple equation method, $tanh$-$coth$ method, just to name a few of these techniques. These techniques lead to several kinds of solutions that the Theoretical Physicists and Applied Mathematicians need to carry out further studies in these areas.

Once these solutions are available, it is not a difficult task to carry out further studies related to NLEEs, including the computation of conserved quantities, wave-wave interactions, quasi-stationary solutions in presence of perturbations and many other aspects of NLEEs. Therefore it is of prime importance to first focus on the integrability aspects of NLEEs to retrieve various solutions of NLEEs, for example the cnoidal and snoidal waves, solitary waves, peakons, cuspons and others.

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2. Governing Equations

This paper is going to focus on the integration of a generalized version of Boussinesq equation (BE) that comes with three variants. These generalized variants will be studied in both (1+1) and (1+2) dimensions. These three variants will be respectively labeled as Variants I, II and III. The focus, in this paper, will be on solving these variants of the BE for solitary wave solutions and shock wave solutions.

These equations arise in the area of Applied Mathematics as a generalized version of the regular BE. Rosenau generalized the KdV equation to formulate the $K(m,n)$ equation. Later this equation was further generalized to formulate the $K(m,n)$ with generalized evolution. Similarly the BE was generalized to formulate the $B(m,n)$ equation, that was solved in 2009 [1]. Another kind of generalization of the BE was given by Wazwaz in 2005 [10, 11]. In this paper, the three variants are a further study of the three variants given by Wazwaz in 2005, with generalized evolution. It is well known that both KdV equation and BE study the shallow water waves.

3. Variant-I

We first consider the Variant I of the improved Boussinesq-type equation:

$$u_{tt} - k_1^2 u_{xx} - k_2^2 u_{yy} - a\left(u^{2n}\right)_{xx} - b\left[u^n \left(u^n\right)_{xx}\right]_{xx} = 0, \tag{1}$$

where $u(x,y,t)$ represents the wave profile, depending on the space coordinates x and y, and the time variable t. The subscripts x, y and t denote partial derivatives with respect to these variables, and $a, b \in \mathbb{R}$ are constants.

In (1), the first term is the evolution term, the second and the third terms represent the dispersion terms in the x- and y-directions respectively, while the last two terms are the nonlinear dispersion terms.

3.1. **Solitary Waves.** We start the analysis by assuming a solitary wave ansatz of the form [1-4]

$$u(x,y,t) = \frac{A}{\cosh^p \tau} \tag{2}$$

where

$$\tau = B_1 x + B_2 y - vt \tag{3}$$

and

$$p > 0 \tag{4}$$

for solitons to exist. Here, in (2) and (3), A represents the amplitude of the soliton, v is the velocity of the soliton while B_1 and B_2 are the inverse widths in the x and y directions, respectively. The exponent p is unknown at this point and its value will be evaluated in the process of deriving the solution of this equation. Substituting

(2) into (1) yields

$$\frac{p^2A\left(v^2-k_1^2B_1^2-k_2^2B_2^2\right)}{\cosh^p\tau}-\frac{p(p+1)A\left(v^2-k_1^2B_1^2-k_2^2B_2^2\right)}{\cosh^{p+2}\tau} \tag{5}$$
$$+\frac{2npA^{2n}B_1^2\left\{a+2anp+bB_1^2\left(4n^3p^3+7n^2p^2+6np+2\right)\right\}}{\cosh^{2np+2}\tau}$$
$$-\frac{4n^2p^2B_1^2A^{2n}\left(a+bn^2p^2B_1^2\right)}{\cosh^{2np}\tau}-\frac{2A^{2n}B_1^4bnp\left(np+1\right)^2\left(2np+3\right)}{\cosh^{2np+4}\tau}=0.$$

From (5), equating the exponents p and $(2pn+2)$ gives

$$p=2pn+2 \tag{6}$$

that leads to

$$p=2/(1-2n) \tag{7}$$

which is also obtained by equating the exponents pair $(p+2)$ and $(2pn+4)$.

Now, the functions, $1/\cosh^{np+j}\tau$ for $j=0,2,4$ in (5) are linearly independent. Thus, their respective coefficients must vanish. Setting their coefficients to zero gives the system of algebraic equations:

$$p^2A\left(v^2-k_1^2B_1^2-k_2^2B_2^2\right)+2npA^{2n}B_1^2\Big\{a(1+2np)$$
$$+bB_1^2\left(4n^3p^3+7n^2p^2+6np+2\right)\Big\}=0 \tag{8}$$

$$p(p+1)A\left(v^2-k_1^2B_1^2-k_2^2B_2^2\right)+2A^{2n}B_1^4bnp\left(np+1\right)^2\left(2np+3\right)=0 \tag{9}$$

$$4n^2p^2B_1^2A^{2n}\left(a+bn^2p^2B_1^2\right)=0. \tag{10}$$

Solving the above system yields:

$$v=\left\{k_1^2B_1^2+k_2^2B_2^2-nA^{2n-1}B_1^2\left[a+2an+\frac{2bB_1^2\left(4n^3+2n^2+1\right)}{(1-2n)^2}\right]\right\}^{\frac{1}{2}} \tag{11}$$

$$v=\left\{k_1^2B_1^2+k_2^2B_2^2-\frac{2bnA^{2n-1}B_1^4}{(1-2n)^2}\right\}^{\frac{1}{2}} \tag{12}$$

and

$$B_1=\frac{(1-2n)}{2n}\sqrt{-\frac{a}{b}}. \tag{13}$$

Now, equating the two values of the velocity v from (11) and (12) yields the expression of the inverse width B_1 that is given by (13). This shows the consistency of the used method.

Further, the expression (13) implies that the soliton will exist for

$$ab<0. \tag{14}$$

Therefore the bright soliton solution of Boussinesq equation (1) is given by

$$u(x,y,t)=\frac{A}{\cosh^{\frac{2}{1-2n}}\left(B_1x+B_2y-vt\right)} \tag{15}$$

where the velocity of the soliton is given by (11) or (12) and the inverse width B_1 of the soliton is given by (13).

Finally, it is necessary to have $n < 1/2$ as seen from (4) and (7). Notice that the bright soliton solution (15) exists provided that $ab < 0$.

3.2. **Shock Waves.** In order to look for the shock waves solution to (1), the starting assumption is [8]

$$u(x,y,t) = A\tanh^p(B_1x + B_2y - vt) \tag{16}$$

where

$$\tau = B_1x + B_2y - vt \tag{17}$$

and

$$p > 0 \tag{18}$$

for solitons to exist. Here A, B_1 and B_2 are free parameters, while v represents the velocity of the soliton. The value of the unknown exponent p will be determined during the course of derivation of the soliton solution of (1).

Substituting (16)-(17) into (1) yields

$$\begin{aligned}
&pv^2A\Big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\Big\}\\
&-k_1^2pAB_1^2\Big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\Big\}\\
&-k_2^2pAB_2^2\Big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\Big\}\\
&-2apnA^{2n}B_1^2\Big\{(2pn-1)\tanh^{2pn-2}\tau - 4pn\tanh^{2pn}\tau + (2pn+1)\tanh^{2pn+2}\tau\Big\}\\
&-2bpnA^{2n}B_1^4\{(pn-1)^2(2pn-3)\tanh^{2pn-4}\tau + (pn+1)^2(2pn+3)\tanh^{2pn+4}\tau\\
&-\Big\{2p^2n^2(2pn-1) + 4(pn-1)^3\Big\}\tanh^{2pn-2}\tau\\
&-\Big\{2p^2n^2(2pn+1) + 4(pn+1)^3\Big\}\tanh^{2pn+2}\tau\\
&+\big\{8p^3n^3 + (pn-1)^2(2pn-1) + (pn+1)^2(2pn+1)\big\}\tanh^{2pn}\tau\big\} = 0
\end{aligned} \tag{19}$$

From (19), equating the exponents p and $2pn+2$ gives $p = 2pn+2$ so that $p = 2/(1-2n)$. It needs to be noted that the same value of p is yielded when the exponents pair $p+2$ and $2pn+4$, and the exponents $p-2$ and $2pn$, respectively, are equated with each other.

Thus, the linearly independent functions in (19) are $\tanh^{pn+j}\tau$, where $j = -4,-2,0,2,4$. So, from (19), each of the coefficients of these linearly independent functions must be zero. Setting their respective coefficients to zero yields the following parametric equations:

$$\begin{aligned}
&pv^2A(p-1) - k_1^2pAB_1^2(p-1) - k_2^2pAB_2^2(p-1) + 8ap^2n^2A^{2n}B_1^2\\
&-2bpnA^{2n}B_1^4\left\{8p^3n^3 + (pn-1)^2(2pn-1) + (pn+1)^2(2pn+1)\right\} = 0
\end{aligned} \tag{20}$$

$$-2p^2v^2A+2k_1^2p^2AB_1^2+2k_2^2p^2AB_2^2-2apnA^{2n}B_1^2(2pn+1)$$
$$+2bpnA^{2n}B_1^4\left\{2p^2n^2(2pn+1)+4(pn+1)^3\right\}=0 \tag{21}$$

$$\left\{pv^2A-k_1^2pAB_1^2-k_2^2pAB_2^2\right\}(p+1)-2bpnA^{2n}B_1^4(pn+1)^2(2pn+3)=0 \tag{22}$$

$$-2apnA^{2n}B_1^2(2pn-1)+2bpnA^{2n}B_1^4\left\{2(pn)^2(2pn-1)+4(pn-1)^3\right\}=0 \tag{23}$$

$$-2bpnA^{2n}B_1^4(pn-1)^2(2pn-3)=0. \tag{24}$$

To solve (24), we have considered firstly the case $pn-1=0$. This yields

$$p=\frac{1}{n}. \tag{25}$$

Substituting (25) into (23) gives

$$B_1=\sqrt{\frac{a}{2b}} \tag{26}$$

which forces the constraint relation

$$ab>0. \tag{27}$$

Substituting (25) into (20), (21) and (22), respectively, gives

$$v=\left\{k_1^2B_1^2+k_2^2B_2^2-\frac{\left(2aB_1^2-10bB_1^4\right)A^{2n-1}}{3}\right\}^{\frac{1}{2}} \tag{28}$$

$$v=\left\{k_1^2B_1^2+k_2^2B_2^2+\frac{\left(38bB_1^4-3aB_1^2\right)A^{2n-1}}{16}\right\}^{\frac{1}{2}} \tag{29}$$

and

$$v=\left\{k_1^2B_1^2+k_2^2B_2^2+2bA^{2n-1}B_1^4\right\}^{\frac{1}{2}}. \tag{30}$$

Equating any two values of v from (28), (29) and (30) gives the the same value of B_1 given in (26). Notice that the second case $2pn-3=0$ is not considered here as it does not give a unique value of B_1.

Thus, finally, the shock waves solution to the Boussinesq equation (1) is given by

$$u(x,y,t)=A\tanh^{\frac{2}{1-2n}}\left(B_1x+B_2y-vt\right) \tag{31}$$

where the free parameter B_1 is given by (26) and the velocity by (28) or (29) or (30). Notice that this solution exists provided that $n<1/2$ as seen from (18) and $p=2/(1-2n)$ and $ab>0$.

4. Variant-II

In this section, we consider the variant II of the improved Boussinesq equation:

$$\left(u^l\right)_{tt} - k^2\left(u^m\right)_{xx} - a\left(u^{2n}\right)_{xx} - b\left[u^n\left(u^n\right)_{xx}\right]_{tt} = 0. \tag{32}$$

The focus will be on searching the bright and dark soliton solutions to (32).

4.1. **Solitary Waves.** The starting hypothesis is the following [1-4]

$$u(x,t) = \frac{A}{\cosh^p \tau} \tag{33}$$

where

$$\tau = B(x - vt). \tag{34}$$

Here, in (33)-(34), A is the amplitude of the soliton while v is the velocity of the soliton and B is the inverse width. The exponents p is unknown at this point and their values will fall out in the process of deriving the solution of this equation.

Substituting (33)-(34) into (32) yields

$$\begin{aligned}
&\frac{p^2l^2v^2A^lB^2}{\cosh^{pl}\tau} - \frac{pl(pl+1)v^2A^lB^2}{\cosh^{pl+2}\tau} - \frac{k^2p^2m^2A^mB^2}{\cosh^{pm}\tau} + \frac{k^2pm(pm+1)A^mB^2}{\cosh^{pm+2}\tau} \\
&\quad - \frac{4n^2p^2A^{2n}B^2\left(a+bn^2p^2v^2B^2\right)}{\cosh^{2np}\tau} - \frac{2bnp\left(np+1\right)^2\left(2np+3\right)A^{2n}v^2B^4}{\cosh^{2np+4}\tau} \\
&\quad + \frac{2npA^{2n}B^2\left\{a(2np+1)+b\left(4n^3p^3+7n^2p^2+6np+2\right)v^2B^2\right\}}{\cosh^{2np+2}\tau} = 0.
\end{aligned} \tag{35}$$

Now, from (35), matching the exponents of $\cosh^{pm+2}\tau$ and $\cosh^{2np+4}\tau$ gives

$$pm + 2 = 2np + 4 \tag{36}$$

so that

$$p = \frac{2}{m-2n} \tag{37}$$

which is also obtained by equating the exponents of $\cosh^{pm}\tau$ and $\cosh^{2np+2}\tau$ functions.

Also, from (35), equating the exponent of $\cosh^{pl}\tau$ and $\cosh^{2np}\tau$ yields

$$pl = 2np \tag{38}$$

and therefore

$$l = 2n. \tag{39}$$

Now, from (35), setting the coefficients of the linearly independent functions $1/\cosh^{2pn+j}\tau$ to zero, where $j = 0, 2, 4$ gives the following system of algebraic equations,

$$p^2l^2v^2A^lB^2 - 4an^2p^2A^{2n}B^2 - 4bn^4p^4A^{2n}v^2B^4 = 0 \tag{40}$$

$$\begin{aligned}
&2bnp\left(4n^3p^3+7n^2p^2+6np+2\right)A^{2n}v^2B^4 - k^2p^2m^2A^mB^2 \\
&- pl(pl+1)v^2A^lB^2 + 2anp(2np+1)A^{2n}B^2 = 0
\end{aligned} \tag{41}$$

$$2bnp\,(np+1)^2\,(2np+3)A^{2n}v^2B^4-k^2pm(pm+1)A^mB^2=0. \tag{42}$$

Solving this system, one obtains

$$A=\left\{\frac{\left(v^2-a\right)m}{2nk^2}\right\}^{\frac{1}{m-2n}}, \tag{43}$$

$$B=\frac{(m-2n)}{2nv}\sqrt{\frac{(v^2-a)}{b}}. \tag{44}$$

Thus (44) introduces the constraint conditions:

$$b\left(v^2-a\right)>0,\quad v\neq 0,\quad v\neq\pm\sqrt{a} \tag{45}$$

Hence, finally, the 1-soliton solution of the Boussinesq equation (32) is given by

$$u(x,t)=\frac{A}{\cosh^{\frac{2}{m-2n}}[B(x-vt)]} \tag{46}$$

where the amplitude A and the velocity v are connected by (43) and the width of the soliton is given by (44).

4.2. **Shock Waves.** In this subsection the search is going to be for shock wave solution to the Boussinesq-type equation given by (32). To start off, the hypothesis is given by [8]

$$u(x,t)=A\tanh^p\tau \tag{47}$$

where

$$\tau=B(x-vt) \tag{48}$$

where in (47) and (48), A and B are free parameters and v is the velocity of the wave. Also, the unknown exponent p will be determined during the course of the derivation of the soliton solution to (32). By inserting (47)-(48) into (32), we obtain

$$\begin{aligned}
&plA^lB^2v^2\left\{(pl-1)\tanh^{pl-2}\tau-2pl\tanh^{pl}\tau+(pl+1)\tanh^{pl+2}\tau\right\}\\
&-k^2mpA^mB^2\left\{(mp-1)\tanh^{mp-2}\tau-2mp\tanh^{mp}\tau+(mp+1)\tanh^{mp+2}\tau\right\}\\
&-2apnA^{2n}B^2\left\{(2pn-1)\tanh^{2pn-2}\tau-4pn\tanh^{2pn}\tau+(2pn+1)\tanh^{2pn+2}\tau\right\}\\
&-2bpnA^{2n}B^4v^2\left\{(pn-1)^2(2pn-3)\tanh^{2pn-4}\tau+(pn+1)^2(2pn+3)\tanh^{2pn+4}\tau\right.\\
&-\left\{p^2n^2(4pn-2)+4(pn-1)^3\right\}\tanh^{2pn-2}\tau\\
&-\left\{p^2n^2(4pn+2)+4(pn+1)^3\right\}\tanh^{2pn+2}\tau\\
&\left.+\left\{8(pn)^3+(pn-1)^2(2pn-1)+(pn+1)^2(2pn+1)\right\}\tanh^{2pn}\tau\right\}=0.
\end{aligned} \tag{49}$$

By equating the exponents $(pm+2)$ and $(2pn+4)$ in (49) gives

$$pm+2=2pn+4 \tag{50}$$

so that

$$p = \frac{2}{m-2n}. \tag{51}$$

It needs to be noted that the same value of p is yielded when the exponents pair pm and $2pn+2$, and the exponents $pm-2$ and $2pn$, respectively, are equated with each other.

From (49), equating the exponents pl and $2pn$ gives

$$pl = 2pn \tag{52}$$

so that

$$l = 2n. \tag{53}$$

Finally, setting the coefficients of the linearly independent functions $\tanh^{2pn+j}\tau$, for $j = -2, 0, 2$ in (49), to zero yields

$$\begin{aligned} &plA^l B^2 v^2 (pl-1) - 2apnA^{2n}B^2(2pn-1) \\ &+2bpnA^{2n}B^4v^2\left\{2p^2n^2(2pn-1)+4(pn-1)^3\right\} = 0 \end{aligned} \tag{54}$$

$$\begin{aligned} &-2p^2l^2A^lB^2v^2 + 8ap^2n^2A^{2n}B^2 - k^2mpA^mB^2(mp-1) - 2bpnA^{2n}B^4v^2 \\ &\times\left\{8(pn)^3+(pn-1)^2(2pn-1)+(pn+1)^2(2pn+1)\right\} = 0 \end{aligned} \tag{55}$$

$$\begin{aligned} &plA^lB^2v^2(pl+1) - 2apnA^{2n}B^2(2pn+1) + 2p^2m^2k^2A^mB^2 \\ &+2bpnA^{2n}B^4v^2\left\{2(pn)^2(2pn+1)+4(pn+1)^3\right\} = 0 \end{aligned} \tag{56}$$

$$-k^2mpA^mB^2(mp+1) - 2bpnA^{2n}B^4v^2(pn+1)^2(2pn+3) = 0 \tag{57}$$

$$2bpnA^{2n}B^4v^2(pn-1)^2(2pn-3) = 0. \tag{58}$$

To solve (58), we have considered firstly the case $pn-1=0$. This yields

$$p = \frac{1}{n}. \tag{59}$$

Substituting (59) into the above system gives

$$B = \frac{1}{v}\sqrt{\frac{v^2-a}{2b}} \tag{60}$$

and

$$A = \left(-\frac{2bB^2v^2}{k^2}\right)^{\frac{1}{m-2n}} \tag{61}$$

which shows that solitons will exist for

$$b < 0 \tag{62}$$

if $m-2n$ is an even integer. However, if $m-2n$ is an odd integer there is no such restriction but the soliton will be pointing downwards. Also from (60) the following restriction is obtained

$$b\left(v^2-a\right)>0. \tag{63}$$

Notice that the second case $2pn-3=0$ in (58) is not considered here as it does not give unique values of A and B.

Equating the two values of p from (51) and (59) gives the condition:

$$4n=m. \tag{64}$$

Also from the necessary condition $p>0$ for the existence of the dark soliton solution (47) and (51) the following restrictions are obtained.

$$m>2n. \tag{65}$$

Thus, finally, the shock waves solution to the Boussinesq equation (32) is given by

$$u(x,t)=A\tanh^{\frac{2}{m-2n}}\left[B(x-vt)\right] \tag{66}$$

where the the free parameters A and B are given by (60) and (61).

5. Variant-III

Now, we consider the variant III of the generalized (2+1)–dimensional of the improved Boussinesq equation:

$$\left(u^l\right)_{tt}-k^2\left(u^m\right)_{xx}-a_1\left(u^{2n}\right)_{xx}-a_2\left(u^{2n}\right)_{yy}-b_1\left[u^n\left(u^n\right)_{xx}\right]_{tt}-b_2\left[u^n\left(u^n\right)_{yy}\right]_{tt}=0 \tag{67}$$

The focus will be on searching the bright and dark soliton solutions to (67).

5.1. **Solitary Waves.** The starting hypothesis for the solution to (67) is the same as in the Variant I that is given by (2) and (3).

Substituting (2)-(3) into (67), we get

$$\begin{aligned}
&\frac{p^2l^2v^2A^l}{\cosh^{pl}\tau}-\frac{pl(pl+1)v^2A^l}{\cosh^{pl+2}\tau}-\frac{k^2p^2m^2A^mB_1^2}{\cosh^{pm}\tau}+\frac{k^2pm(pm+1)A^mB_1^2}{\cosh^{pm+2}\tau}\\
&-\frac{4n^2p^2A^{2n}\left(a_1B_1^2+a_2B_2^2+b_1n^2p^2v^2B_1^2+b_2n^2p^2v^2B_2^2\right)}{\cosh^{2np}\tau}\\
&+\frac{2npA^{2n}\left\{(2np+1)\left(a_1B_1^2+a_2B_2^2\right)+v^2\left(4n^3p^3+7n^2p^2+6np+2\right)\left(b_1B_1^2+b_2B_2^2\right)\right\}}{\cosh^{2np+2}\tau}\\
&-\frac{2np\left(np+1\right)^2\left(2np+3\right)v^2A^{2n}\left(b_1B_1^2+b_2B_2^2\right)}{\cosh^{2np+4}\tau}=0
\end{aligned} \tag{68}$$

From (68), equating the exponents pm and $2np+2$ gives $pm=2np+2$, so that $p=2/(m-2n)$.

It needs to be noted that the same value of p is yielded when the exponents $pm+2$ and $2np+4$ are equated with each other. Again equating the exponents $2np$ and pl gives $2np=pl$, that yields $l=2n$.

Again this same value of p is obtained on equating the exponents $2np+2$ and $pl+2$.

Now from (68), setting the coefficients of the linearly independent functions $1/\cosh^{2np+j}\tau$ to zero, where $j=0,2,4$ gives

$$p^2l^2v^2A^l-4n^2p^2A^{2n}\left(a_1B_1^2+a_2B_2^2+b_1n^2p^2v^2B_1^2+b_2n^2p^2v^2B_2^2\right)=0. \tag{69}$$

$$\begin{aligned}&-pl(pl+1)v^2A^l-k^2p^2m^2A^mB_1^2+2npA^{2n}(2np+1)\left(a_1B_1^2+a_2B_2^2\right)\\&+2npA^{2n}v^2\left(4n^3p^3+7n^2p^2+6np+2\right)\left(b_1B_1^2+b_2B_2^2\right)=0.\end{aligned} \tag{70}$$

$$k^2pm(pm+1)A^mB_1^2-2np\left(np+1\right)^2(2np+3)v^2A^{2n}\left(b_1B_1^2+b_2B_2^2\right)=0. \tag{71}$$

Solving the above system gives the following unique value of the soliton amplitude A:

$$A=\left\{\frac{m\left(v^2-a_1B_1^2-a_2B_2^2\right)}{2nk^2B_1^2}\right\}^{\frac{1}{m-2n}} \tag{72}$$

which forces the constraint relation

$$v\neq\pm\sqrt{a_1B_1^2+a_2B_2^2} \tag{73}$$

in order to obtain nontrivial solutions.

Thus, the bright soliton solution to the generalized two-dimensional Boussinesq equation (67) is given by

$$u(x,y,t)=\frac{A}{\cosh^{\frac{2}{m-2n}}\left(B_1x+B_2y-vt\right)} \tag{74}$$

where the amplitude A as function of the soliton velocity v and the inverse widths B_1 and B_2 of the soliton is given by (72).

Finally, it is necessary to have $m>2n$ for the soliton solution (74) to exist.

5.2. **Shock Waves.** Now, we are interested by finding the dark soliton solution for the considered Boussinesq equation (67). To do this, we use an ansatz solution of the form (16) and (17) from [8]. Substituting (16)-(17) into (67), we have

$$\begin{aligned}&plv^2A^l\left\{(pl-1)\tanh^{pl-2}\tau-2pl\tanh^{pl}\tau+(pl+1)\tanh^{pl+2}\tau\right\}\\&-k^2pmA^mB_1^2\left\{(pm-1)\tanh^{pm-2}\tau-2pm\tanh^{pm}\tau+(pm+1)\tanh^{pm+2}\tau\right\}\\&-2pnA^{2n}\left(a_1B_1^2+a_2B_2^2\right)\left\{(2pn-1)\tanh^{2pn-2}\tau-4pn\tanh^{2pn}\tau+(2pn+1)\tanh^{2pn+2}\tau\right\}\\&-2pnA^{2n}v^2\left(b_1B_1^2+b_2B_2^2\right)\left\{(pn-1)^2(2pn-3)\tanh^{2pn-4}\tau+(pn+1)^2(2pn+3)\tanh^{2pn+4}\tau\right.\\&-\left\{p^2n^2(4pn-2)+4(pn-1)^3\right\}\tanh^{2pn-2}\tau-\left\{p^2n^2(4pn+2)+4(pn+1)^3\right\}\tanh^{2pn+2}\tau\\&\left.+\left\{8(pn)^3+(pn-1)^2(2pn-1)+(pn+1)^2(2pn+1)\right\}\tanh^{2pn}\tau\right\}=0.\end{aligned} \tag{75}$$

From (75), equating the exponents pm and $2pn+2$ gives $pm = 2pn+2$, so that $p = 2/(m-2n)$.

It needs to be noted that the same value of p is yielded when the exponents pair $pm+2$ and $2pn+4$, and the exponents $pm-2$ and $2pn$, respectively, are equated with each other. Again from (75), equating the exponents $2np$ and pl gives $2np = pl$ that yields $l = 2n$. Note that this same value of p is obtained on equating the exponents pairs $2np+2$ and $pl+2$, $2np-2$ and $pl-2$.

Thus, the linearly independent functions in (75) are $\tanh^{2pn+j}\tau$, where $j = \pm 4, \pm 2, 0$. So, from (75), each of the coefficients of these linearly independent functions must be zero. Setting their respective coefficients to zero yields

$$plv^2A^l(pl-1) - 2pnA^{2n}\left(a_1B_1^2 + a_2B_2^2\right)(2pn-1)$$
$$+2pnA^{2n}v^2\left(b_1B_1^2 + b_2B_2^2\right)\left\{2(pn)^2(2pn-1) + 4(pn-1)^3\right\} = 0 \tag{76}$$

$$-2p^2l^2v^2A^l + 8p^2n^2A^{2n}\left(a_1B_1^2 + a_2B_2^2\right) - k^2pmA^mB_1^2(pm-1) - 2pnA^{2n}v^2$$
$$\times\left(b_1B_1^2 + b_2B_2^2\right)\left\{8p^3n^3 + (pn-1)^2(2pn-1) + (pn+1)^2(2pn+1)\right\} = 0 \tag{77}$$

$$plv^2A^l(pl+1) - 2pnA^{2n}\left(a_1B_1^2 + a_2B_2^2\right)(2pn+1) + 2k^2p^2m^2A^mB_1^2$$
$$+2pnA^{2n}v^2\left(b_1B_1^2 + b_2B_2^2\right)\left\{2(pn)^2(2pn+1) + 4(pn+1)^3\right\} = 0 \tag{78}$$

$$-k^2pmA^mB_1^2(pm+1) - 2pnA^{2n}v^2\left(b_1B_1^2 + b_2B_2^2\right)(pn+1)^2(2pn+3) = 0 \tag{79}$$

$$-2pnA^{2n}v^2\left(b_1B_1^2 + b_2B_2^2\right)(pn-1)^2(2pn-3) = 0. \tag{80}$$

To solve (80), we have considered firstly the case $pn-1 = 0$. This yields $p = 1/n$.

Substituting $p = 1/n$. into (76)-(79) reduces the above system to:

$$v^2 - \left(a_1B_1^2 + a_2B_2^2\right) + 2v^2\left(b_1B_1^2 + b_2B_2^2\right) = 0 \tag{81}$$

$$2v^2 - 2\left(a_1B_1^2 + a_2B_2^2\right) + 3k^2A^{m-2n}B_1^2 + 10v^2\left(b_1B_1^2 + b_2B_2^2\right) = 0 \tag{82}$$

$$3v^2 - 3\left(a_1B_1^2 + a_2B_2^2\right) + 16k^2A^{m-2n}B_1^2 + 38v^2\left(b_1B_1^2 + b_2B_2^2\right) = 0 \tag{83}$$

$$k^2A^{m-2n}B_1^2 + 2v^2\left(b_1B_1^2 + b_2B_2^2\right) = 0. \tag{84}$$

From solving the above equations, one gets a unique value of the free parameter A such that

$$A = \left(\frac{v^2 - \left(a_1B_1^2 + a_2B_2^2\right)}{k^2B_1^2}\right)^{\frac{1}{m-2n}} \tag{85}$$

which proves again the consistency of the used method. Notice that the case $2pn-3 = 0$ in (80) is not considered here as it does not give a unique value of A. Now, equating the two values of p from $p = 2/(m-2n)$ and $p = 1/n$ gives the condition $4n = m$. Lastly, we can determine the shock waves solution to Boussinesq equation (67) with generalized evolution term as

$$u(x,y,t) = A\tanh^{\frac{2}{m-2n}}(B_1x + B_2y - vt) \tag{86}$$

where the free parameter A is given by (85). It is important to note that this solution (86) exists only when $m > 2n$ which is in agrement with the equality $4n = m$.

6. Stability Analysis and Numerics

The stability analysis for the parameter regimes in order for the solitary waves to exist will be summarized in this section. The modified versions of the three variants that are conformable for the existence of the solitary waves are respectively given by

$$u_{tt} - k_1^2 u_{xx} - k_2^2 u_{yy} - a\left(u^{2n}\right)_{xx} - b\left[u^n \left(u^n\right)_{xx}\right]_{xx} = 0,$$

$$\left(u^{2n}\right)_{tt} - k^2 \left(u^m\right)_{xx} - a\left(u^{2n}\right)_{xx} - b\left[u^n \left(u^n\right)_{xx}\right]_{tt} = 0,$$

$$\left(u^{2n}\right)_{tt} - k^2 \left(u^m\right)_{xx} - a_1\left(u^{2n}\right)_{xx} - a_2\left(u^{2n}\right)_{yy} - b_1\left[u^n \left(u^n\right)_{xx}\right]_{tt} - b_2\left[u^n \left(u^n\right)_{yy}\right]_{tt} = 0.$$

Thus, Variants-II and -III are the only ones that changed. In each case, $l = 2n$. Hence, the stability criteria for the solitary waves to exist is given in the following table.

Stability Criteria for the Existence of Solitary Waves

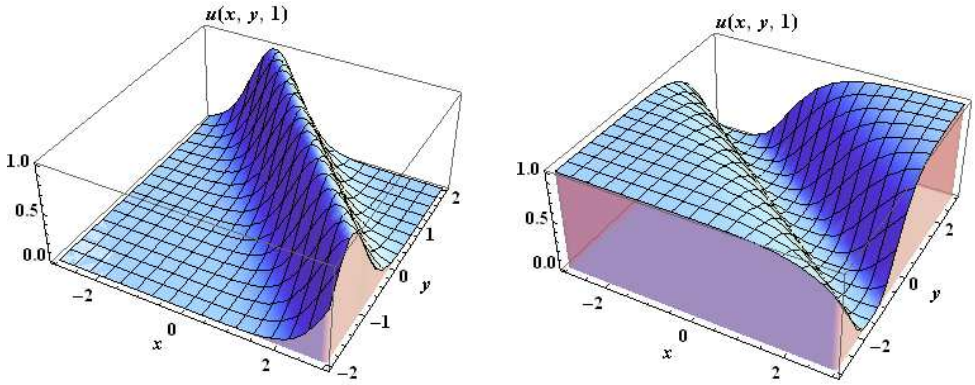
Variants	Nonlinear Wave	Stability Criteria
Variant−I	Shock Wave	$n < \frac{1}{2}$
	Solitary Wave	$n < \frac{1}{2}$
Variant−II	Shock Wave	$n < \frac{m}{2}$
	Solitary Wave	$n < \frac{m}{2}$
Variant−III	Shock Wave	$n < \frac{m}{2}$
	Solitary Wave	$n < \frac{m}{2}$

The numerical simulations are carried out for BE in three variants. The shock wave as well as solitary wave solutions are numerically obtained for all three variants.

In Figure 1, the soliton profiles for the BE with Variant-I are shown. The parameter values that are chosen are $n = 0.25$, $a = -2$, $b = 3$, $k_1 = 1$, $k_2 = 1$, $t = 1$.

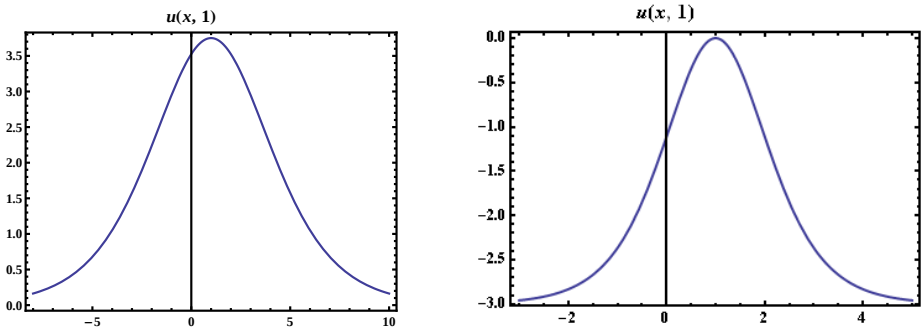
In Figure 2, the non-shock wave and shock wave solitons for BE with Variant-II are shown. The parameter values are $n = 2$, $m = 5$, $a = -2$, $b = 3$, $k = 1$, $t = 1$. In this case, for non-shock wave soliton, $A > 0$, while for shock wave soliton $A < 0$.

In Figure 3, the non-shock wave and shock wave solitons for BE with Variant-III are shown. The parameter values are chosen to be $n = 2$, $m = 5$, $a_1 = 1$, $a_2 = 1$, $b_1 = 0.5$, $b_2 = 1$, $k = 1$, $t = 1$. In this case, parameter $A < 0$ for both types of solitons.



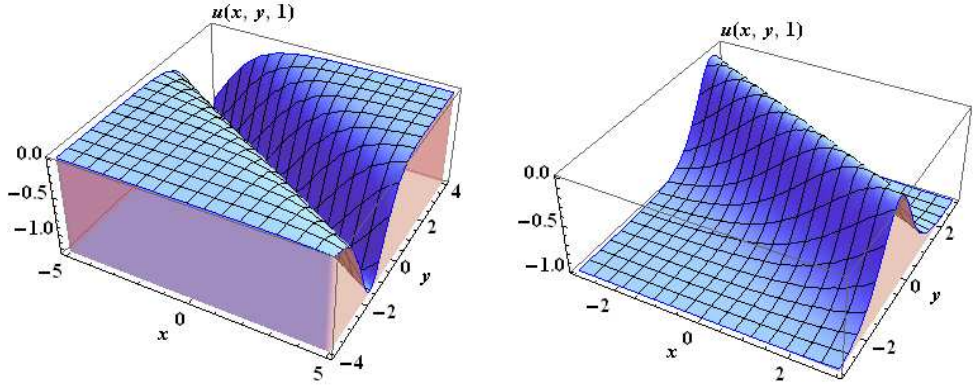
(a) Solitary Wave (b) Shock Wave

FIGURE 1. Boussinesq Equation with Variant-I



(a) Solitary Wave (b) Shock Wave

FIGURE 2. Boussinesq Equation with Variant-II



(a) Solitary wave (b) Shock Wave

FIGURE 3. Boussinesq Equation with Variant-III

7. Conclusions

This paper obtains the solitary wave and the shock wave solutions of the three variants of the Boussinesq equation. For each of the variants the solitary wave

solution as well as the shock wave solution is obtained. The parameter domains and restrictions also fall out of the analysis. The numerical simulations are also given for all the variants. These results are very important and new in the context of nonlinear evolution equations.

These results will be definitely used to carry out the further analysis of these equations. For example, one can possibly study these variants with time-dependent coefficients as opposed to the constant coefficients as studied in this paper [14]. The Lie symmetry approach can also be used to compute the conservation laws of these variants. The soliton perturbation theory can also be studied to obtain the adiabatic variation of the conserved quantities as well as the slow change in the velocity of the soliton. These results will be reported in future publications.

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