COUPLED FREE VIBRATIONS OF FLUID-STRUCTURE INTERACTION SYSTEM

Abdelghani SEGHIR[1]

Using pressure as field variable for the discretization of the fluid domain in finite element modeling of fluid-structure coupled systems, leads to a non-symmetric algebraic matrices. Classical modal superposition methods are not directly applicable and eigen-modes extraction requires special algorithms and high cost numerical treatment. Several symmetrization techniques are developed and different symmetric formulations are proposed, but practically, they all use either matrix inversion or some changes of variables. The present paper discuses the commonly used techniques and introduces two very efficient ones which are based on the mass matrix lumping. They avoid the heavy matrix inversion process, and greatly saves computer's memory allocations and CPU execution time. An example of application concerning the coupled eigenfrequencies of a partially filled tank is worked out through which the performances of the proposed techniques are demonstrated.

Keywords: fluid, structure, interaction, FEM, free vibrations, symmetrization

1. Introduction

The choice of the basic variables to be retained in numerical modeling of dynamic fluid-structure interaction, has a great incidence on the numerical properties of the resulting algebraic matrices. Sometimes, numerically undesirable properties such as symmetry loss, zeros diagonal terms, ill-conditioned or non-positive definite matrices are to be expected and special techniques are required in order to overcome these difficulties [1–9].

Structural displacements and fluid hydrodynamic pressures seems to be naturally adequate basic variables, they are often used in the finite element modeling of fluid-structure systems. However, in the case of compressible fluid hypothesis, or when dealing with free surface waves, coupled displacement-pressure based approach, called $(\mathbf{u}, p)$ formulation, leads to a non-symmetric matrix system which can not be handled with standard dynamic modal methods [10–13].

A symmetric $(\mathbf{u}, \mathbf{v})$ approach, based on the description of both the fluid and the structure behaviors using the displacement field, can be obtained due to coupling two variables of the same nature. However, it is commonly reported that this approach causes some numerical problems due to zero-energy modes, or spurious modes, and it lacks numerical stability. For this reason, it is not used in most computers codes even if some improved fluid elements are proposed and special methods are developed by some researchers such as [14–17]

[1] PhD. Research Laboratory of Applied Hydraulics and Environment, Faculty of Technology, University of Bejaia, Targua Ouzemour, 06000, Algeria. E-mail: a.seghir@univ-bejaia.dz

The most popular symmetric approaches are derived by making use of two variables to describe the fluid behavior. The mixed $(\mathbf{u}, p, \phi)$ formulation is based on pressure and fluid displacement potential coupled to the structural displacements. It makes modal methods available for dynamic analysis and it is implemented in various finite element codes. An other mixed formulation $(\mathbf{u}, p, \psi)$ is based on the the pressure and velocity potential variables to describe the fluid. This approach is considered as the most favored for representing the irrotational fluid motion as reported in [17, 18].

Nevertheless, using two variables for the fluid domain requires, even for small fluid-structure systems, large computer's memory resources and high processing performances due to the additional nodal variables. Besides increasing the size of the resulting algebraic system, zero diagonal submatrices may be introduced and special care must be taken when coupling with the structural displacements. In addition, a static condensation of one of the fluid variables is often required inducing matrix inversion and additional computations.

Other alternatives are based on combining separate solutions of each one of the two subdomains so that the equilibrium condition at the fluid-structure interface is satisfied by iterations. Example of such solution is the staggered coupling algorithm ([19, 20]) where a stage-loop is performed for each time step. In the stages of the loop, each subsystem uses the previously computed state of the other subsystem until convergence. An other idea is based on the decoupled modal analysis ([21, 22]) where a symmetric coupled system is obtained for a vector of participation factors by combining the mode shapes of the two subsystems which are evaluated as two separate eigenproblems.

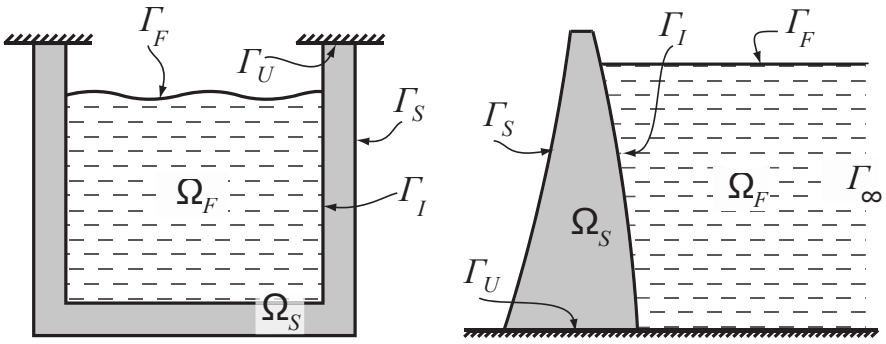
On the other hand, numerous researchers used the boundary element method for the fluid domain. This method is particularly attractive when the fluid is unbounded. It also reduces by one the problem dimension leading to large computer's memory saving especially for three dimensional cases, see for example [23]. Moreover, in addition to complex numerics induced by singular integrals of this method, the non symmetric property of the resulting algebraic system considered above against the $(\mathbf{u}, p)$ based finite element formulation is here usual. Various symmetric boundary element formulations are also developed mainly in order to derive numerical models coupling finite and boundary elements.

Finally, it can be summarized that in numerical modeling of fluid-structure interaction, the displacement is invariably used as the unknown field variable for the structure and the numerical difficulties come from the fluid part. Practically, each one of the available formulations has its own contribution in the improvement of the solution quality and its own requirements in exceptional numerical and special mathematical skills. Nowadays, coupling of Computational Fluid Dynamics (CFD) codes and Computational Structural Dynamics (CSD) codes through Fluid Structure Interaction Algorithms (FSI) is considered as a general solution for a wide range of engineering applications. Such procedures can even deal with very complex problems involving nonlinear interaction, large material deformation or failure resulting from short-duration and severe loading from impact or high pressures [24]. Unfortunately, this way of solving FSI problems requires advanced numerical techniques and very expensive computing tools. Thus, searching for fast and simplified methods still remain of great interest. In the present paper, the $(\mathbf{u}, p)$ formulation is considered, the work is focused on the symmetrization of the vibration problem arising from the discretization of coupled fluid-structure interaction systems.

Some commonly used techniques are first reported and two others whose performances come essentially from the mass matrix lumping concept, are introduced. Then, a numerical example is treated in order to examine the accuracy and the features of these techniques.

2. The displacement-pressure based formulation

We consider a fluid-structure system constituted by a linear elastic structure in contact with an inviscid fluid with free surface. The fluid can be contained within the structure like in the case of a filled reservoir, or retained as in the case of a dam's reservoir. In the later, the fluid domain is unbounded and a fictitious truncation boundary is necessary. As shown in Figure 1, the fluid-structure interface is denoted by Γ_I the structure domain is referred to by Ω_S, the fixed part of its boundary is Γ_U and the rest is Γ_S. The fluid domain is Ω_F, and its free surface is Γ_F. The truncation boundary, if any, is noted Γ_∞.



(a) case of contained fluid (b) case of unbounded fluid

Fig. 1. Examples of fluid-structure systems

The continuum field equations in terms of displacements $\mathbf{u}$ for a flexible structure subjected to dynamic motion of accelerations $\ddot{\mathbf{u}}$ are:

$$\sigma_{ij,j} + f_i - \rho_s \ddot{u}_i = 0 \qquad \text{in } \Omega_S \tag{1}$$

The subscripts i and j refer to the spatial directions and $(\,,j\,)$ denotes the derivative with respect to the coordinate x_j. f_i is the body force acting in the i direction, ρ_s is the structure's material density and σ is the stress tensor which is related to the linearized strains ε by the generalized Hooke's laws:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \mathbf{e}\delta_{ij} \tag{2}$$

where $\mathbf{e} = \text{trace}(\varepsilon)$, λ and μ are the Lamé's coefficients defining the structure's material. Under small deformations hypothesis, the strain tensor is related to the displacement by:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

The appropriate boundary conditions are as follows:

$$\sigma_{ij} n_j = p n_i \qquad \text{on } \Gamma_I \tag{4a}$$

$$\sigma_{ij} n_j = 0 \qquad \text{on } \Gamma_S \tag{4b}$$

$$u_i = 0 \qquad \text{on } \Gamma_U \tag{4c}$$

where $\mathbf{n}$ is the unit outward normal vector along the boundary and p is the hydrodynamic pressure (in excess to hydrostatic) acting on the structure along Γ_I.

Using the classical assumptions of linearly compressible and inviscid fluid, and assuming small amplitude motions of the fluid-structure system, it can be shown that the continuous hydrodynamic pressure field satisfies the wave equation:

$$\nabla^2 p - \frac{1}{c^2}\ddot{p} = 0 \qquad \text{in } \Omega_F \tag{5}$$

with the following boundary conditions:

$$\frac{\partial p}{\partial \mathbf{n}} = -\rho_F \ddot{\mathbf{u}} \cdot \mathbf{n} \qquad \text{on } \Gamma_I \tag{6a}$$

$$\frac{\partial p}{\partial \mathbf{n}} = \frac{-1}{g}\ddot{p} \qquad \text{on } \Gamma_F \tag{6b}$$

On the remaining fluid boundary like a reservoir bottom in Figure 1b, we can take $\partial p/\partial \mathbf{n} = 0$ when only horizontal motion is considered. In general, this condition applies on boundaries that are not subjected to prescribed accelerations.

For the case of unbounded fluid domain, some boundary conditions on Γ_∞ are suggested and several numerical techniques are developed to take into account the infinite part of the fluid and the radiations conditions [18]

In equations (5) and (6), the symbols ∇^2 and $(\ddot{})$ denote respectively the Laplacian operator and the second derivative with respect to time. The physical parameters c, ρ_F and g designate respectively the velocity of sound in fluid, the fluid density and the gravitational constant. Equation (6a) links the outward normal component of the solid particles accelerations to the normal pressure gradient at the fluid-structure interface. Equation (6b) is an approximate condition for the free surface fluctuations, it is a good evaluation of the free surface waves and of the fluid mass sloshing effects.

3. **The finite element discretization**

In order to derive the discrete equations corresponding to the coupled $(\mathbf{u}, p)$ based finite element formulation of the fluid-structure interaction problem, it is first necessary to write a variational formulation for the governing equilibrium equations (1) for the structure and the governing wave equation (5) for the hydrodynamic pressure field in the fluid domain. With the use of regular test functions $\delta\mathbf{u}$ for the displacement and δp for the pressure fields, it may be shown that the following integral forms hold:

$$-\int_{\Omega_S} \mathbf{S}\delta\mathbf{u}\,\mathbf{D}\,\mathbf{S}\mathbf{u}\,\mathrm{d}\Omega - \int_{\Omega_S} \delta\mathbf{u}\rho_S\ddot{\mathbf{u}}\,\mathrm{d}\Omega + \int_{\Omega_S} \delta\mathbf{u}\mathbf{f}\,\mathrm{d}\Omega + \int_{\Gamma_I} \delta\mathbf{u}p\mathbf{n}\,\mathrm{d}\Gamma = 0 \tag{7}$$

$$\int_{\Omega_F} \nabla\delta p \nabla p\,\mathrm{d}\Omega + \int_{\Omega_F} \delta p \frac{1}{c^2}\ddot{p}\,\mathrm{d}\Omega + \int_{\Gamma_I} \delta p \rho_F \ddot{\mathbf{u}}\cdot\mathbf{n}\,\mathrm{d}\Gamma + \int_{\Gamma_F} \delta p \frac{1}{g}\ddot{p}\,\mathrm{d}\Gamma = 0 \tag{8}$$

Matrices $\mathbf{D}$ and $\mathbf{S}$ represent the elasticity matrix and the derivative operator. They arises, respectively, from equations (2) and (3) when the tensors σ and ε are written in vectorial form.

The discrete form of these two equations are obtained by approximating the continuous displacement field in the structure and the continuous pressure field in the fluid using standard finite element shape functions.

$$\mathbf{u} \approx \mathbf{N_u}\,\mathbf{U} \quad ; \quad p \approx \mathbf{N}_p\,\mathbf{P} \tag{9}$$

where $\mathbf{U}$ and $\mathbf{P}$ are vectors of nodal displacements and nodal pressures, respectively, and $\mathbf{N_u}$ and $\mathbf{N}_p$ are the corresponding appropriate shape functions. Substituting (9) in (7) and (8),we get the following two algebraic systems:

$$\mathbf{M}_S\ddot{\mathbf{U}} + \mathbf{K}_S\mathbf{U} - \mathbf{Q}\,\mathbf{P} = \mathbf{F} \tag{10}$$

$$\mathbf{M}_F\ddot{\mathbf{P}} + \mathbf{K}_F\mathbf{P} + \rho_{_F}\mathbf{Q}^T\ddot{\mathbf{U}} = \mathbf{0} \tag{11}$$

where $\mathbf{M}_S$ and $\mathbf{K}_S$ are the classical assembled mass and stiffness matrices of the structure. The corresponding element matrices are given by:

$$\mathbf{M}_S = \sum_{e=1}^{N_{\Omega_S^{(e)}}} \int_{\Omega_S^{(e)}} \mathbf{N}_\mathbf{u}^T \rho_{_S} \mathbf{N_u}\ \mathrm{d}\Omega \tag{12}$$

$$\mathbf{K}_S = \sum_{e=1}^{N_{\Omega_S^{(e)}}} \int_{\Omega_S^{(e)}} [\mathbf{S}\mathbf{N_u}]^T\,\mathbf{D}\,[\mathbf{S}\mathbf{N_u}]\ \mathrm{d}\Omega \tag{13}$$

The superscript (e) in the above expressions refers to the element of area $\Omega_S^{(e)}$ and the sum sign is regarded as the assembling operator over the whole $N_{\Omega_S^{(e)}}$ elements constituting the structure's finite element mesh.

The vector $\mathbf{F}$ contains external body forces such as driving force components generated by prescribed base accelerations.

It is to notice that the coherent mass matrix given by (12) can be lumped and instead of this expression one can use, as indicated in [25], the following diagonal matrix which allows substantial computer's memory and CPU time savings.

$$\mathbf{M}_S = \sum_{e=1}^{N_{\Omega_S^{(e)}}} \int_{\Omega_S^{(e)}} \rho_{_S} \mathbf{N}_\mathbf{u}^T\ \mathrm{d}\Omega \tag{14}$$

The global fluid matrices, called "mass matrix" and "stiffness matrix" by analogy to the structure's ones, are expressed as follows:

$$\mathbf{M}_F = \sum_{e=1}^{N_{\Omega_F^{(e)}}} \int_{\Omega_F^{(e)}} \mathbf{N}_p^T \frac{1}{c^2} \mathbf{N}_p\ \mathrm{d}\Omega + \sum_{e=1}^{N_{\Gamma_F^{(e)}}} \int_{\Gamma_F^{(e)}} \mathbf{N}_p^T \frac{1}{g} \mathbf{N}_p\ \mathrm{d}\Gamma \tag{15}$$

$$\mathbf{K}_F = \sum_{e=1}^{N_{\Omega_F^{(e)}}} \int_{\Omega_F^{(e)}} \nabla\mathbf{N}_p^T\ \nabla\mathbf{N}_p\ \mathrm{d}\Omega \tag{16}$$

Note that the fluid mass matrix account for compressibility effects over the $N_{\Omega_F^{(e)}}$ fluid elements and for free surface waves effects by the second integral over the $N_{\Gamma_F^{(e)}}$ free surface elements. The second term is sometimes neglected especially in problems where the free surface effects are small like in the case of dam-reservoir interaction.

In the same way as done for the structure's mass matrix, lumping can also be applied to fluid mass. We see from expression (15) that the fluid mass operator is exactly the same as the structure's one, only the scalar factors differ. Thus instead of expression (15), the following can be used:

$$\mathbf{M}_F = \sum_{e=1}^{N_{\Omega_F^{(e)}}} \int_{\Omega_F^{(e)}} \frac{1}{c^2} \mathbf{N}_p^T \, \mathrm{d}\Omega + \sum_{e=1}^{N_{\Gamma_F^{(e)}}} \int_{\Gamma_F^{(e)}} \frac{1}{g} \mathbf{N}_p^T \, \mathrm{d}\Gamma \tag{17}$$

The interaction matrix $\mathbf{Q}$ appearing in the two equations (10) and (11) due to discretization of the two boundary conditions (4)a and (6)a is given by assembling integrations over $N_{\Gamma_I^{(e)}}$ interface elements

$$\mathbf{Q} = \sum_{e=1}^{N_{\Gamma_I^{(e)}}} \int_{\Gamma_I^{(e)}} \mathbf{N}_\mathbf{u}^T \mathbf{n} \mathbf{N}_p \, \mathrm{d}\Gamma \tag{18}$$

Consider now the free vibration problem of a fluid-structure system, the corresponding coupled algebraic system is obtained from equations (10) and (11) by omitting the forcing vector. It is written as follows:

$$\left(\begin{bmatrix} \mathbf{K}_S & -\mathbf{Q} \\ \mathbf{0} & \mathbf{K}_F \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_S & \mathbf{0} \\ \rho_F \mathbf{Q}^T & \mathbf{M}_F \end{bmatrix} \right) \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \mathbf{0} \tag{19}$$

where ω represent the set of coupled eigenfrequencies of the system.

It is to notice that, unfortunately, this system is not symmetric, so classical mode superposition methods are not directly applicable. Consequently, various symmetrization techniques are proposed, some of them are reported in the following section where two are currently proposed.

4. Symmetrization of the free vibration problem

Even if the algebraic system in expression (19) is unsymmetric and non positive definite, but physically it is known that free vibration modes of the fluid-structure system exist. Therefore, equation (19) admits real solutions for eigenvalues and eigenvectors, and it can be symmetrized. It is possible to arrive at a symmetric form by various matrix manipulations. The most famous symmetrization techniques are reported here and are categorized in two classes depending on whether the coupling is localized in the mass matrix or in the stiffness matrix. Three techniques leading to the mass-coupling are reported and two techniques leading to stiffness-coupling are proposed.

4.1. Technique using inversion of the two fluid matrices

This technique due to [25], is based on a change of variable such that:

$$\mathbf{V} = \frac{1}{c^2} \mathbf{M}_F \mathbf{P} \tag{20}$$

then, the second equation of the system (19) is rewritten as:

$$\mathbf{K}_F \mathbf{P} - \omega^2 \rho_F \mathbf{Q}^T \mathbf{U} - \omega^2 c^2 \mathbf{V} = \mathbf{0} \tag{21}$$

from which $\mathbf{P}$ can be eliminated and replaced in the first equation of the unsymmetric system to get:

$$\mathbf{K}_S\,\mathbf{U} - \omega^2\left(\mathbf{M}_S + \rho_F\,\mathbf{Q}\mathbf{K}_F^{-1}\mathbf{Q}^T\right)\mathbf{U} - \omega^2 c^2\mathbf{Q}\mathbf{K}_F^{-1}\,\mathbf{V} = \mathbf{0} \tag{22}$$

Now, the symmetry can be achieved by dividing by c^2, and then, left multiplying equation (21) by $\mathbf{K}_F^{-1}$, dividing by ρ_F and replacing $\mathbf{P}$ by $c^2\mathbf{M}_F^{-1}\mathbf{V}$. This gives the final symmetric system:

$$\left(\begin{bmatrix} \dfrac{c^2}{\rho_F}\mathbf{M}_F^{-1} & \mathbf{0} \\ \mathbf{0} & \dfrac{1}{c^2}\mathbf{K}_S \end{bmatrix} - \omega^2 \begin{bmatrix} \dfrac{c^2}{\rho_F}\mathbf{K}_F^{-1} & \mathbf{K}_F^{-1}\mathbf{Q}^T \\ \mathbf{Q}\mathbf{K}_F^{-1} & \dfrac{1}{c^2}\left(\mathbf{M}_S + \rho_F\,\mathbf{Q}\mathbf{K}_F^{-1}\mathbf{Q}^T\right) \end{bmatrix}\right)\begin{Bmatrix}\mathbf{V}\\ \mathbf{U}\end{Bmatrix} = \mathbf{0} \tag{23}$$

We see that this system requires inversion of both the two fluid matrices. This makes essential boundary condition necessary for the pressure field to make the matrices positive definite. Hence, free surface effects cannot be handled by this technique.

4.2. Technique using inversion of the fluid stiffness matrix

This second symmetrization method is suggested by [26]. It introduces an additional variable such that $\mathbf{P} = \omega^2\mathbf{V}$ which adds a third equation to the system (19)

$$\mathbf{M}_F\,\mathbf{P} - \omega^2\mathbf{M}_F\,\mathbf{V} = 0 \tag{24}$$

Then the system is rewritten as follows using both $\mathbf{P}$ and $\mathbf{V}$ after multiplying its first equation by ρ_F

$$\left(\begin{bmatrix} \rho_F\mathbf{K}_S & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} - \omega^2 \begin{bmatrix} \rho_F\mathbf{M}_S & \mathbf{0} & \rho_F\mathbf{Q} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_F \\ \rho_F\mathbf{Q}^T & \mathbf{M}_F & -\mathbf{K}_F \end{bmatrix}\right)\begin{Bmatrix}\mathbf{U}\\ \mathbf{P}\\ \mathbf{V}\end{Bmatrix} = \mathbf{0} \tag{25}$$

This algebraic system is of course symmetric due to the symmetry of the submatrices but it contains zero diagonal terms. Eigenmodes extraction may pose serious numerical difficulties. However static condensation of the introduced variable $\mathbf{V}$ eliminates the zeros and reduces the size of the system which still remain symmetric but contain only the basic variables.

From the third equation of the matrix system (25), $\mathbf{V}$ is expressed as follows:

$$\mathbf{V} = \rho_F\mathbf{K}_F^{-1}\mathbf{Q}^T\,\mathbf{U} + \mathbf{K}_F^{-1}\mathbf{M}_F\,\mathbf{P} \tag{26}$$

and replaced in the first two equations to get the final symmetric form

$$\left(\begin{bmatrix} \mathbf{K}_S & \mathbf{0} \\ \mathbf{0} & \dfrac{1}{\rho_F}\mathbf{M}_F \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_S + \rho_F\,\mathbf{Q}\mathbf{K}_F^{-1}\mathbf{Q}^T & \mathbf{Q}\mathbf{K}_F^{-1}\mathbf{M}_F \\ \mathbf{M}_F\mathbf{K}_F^{-1}\mathbf{Q}^T & \dfrac{1}{\rho_F}\mathbf{M}_F\mathbf{K}_F^{-1}\mathbf{M}_F \end{bmatrix}\right)\begin{Bmatrix}\mathbf{U}\\ \mathbf{P}\end{Bmatrix} = \mathbf{0} \tag{27}$$

As noticed in the previous technique, the inversion of $\mathbf{K}_F$ requires natural boundary condition for the pressure field, so this technique can not handle free surface effects. The condition $\mathbf{P} = 0$ on Γ_F must be applied.

4.3. Technique using inversion of the structure stiffness matrix

This technique is proposed by [27]. It uses accelerations vector rather than the displacements one, this vector is computed from the first equation of the system (19) as follows

$$\ddot{\mathbf{U}} = \omega^2 \mathbf{K}_S^{-1} \mathbf{M}_S \ddot{\mathbf{U}} - \omega^2 \mathbf{K}_S^{-1} \mathbf{Q}\, \mathbf{P} \tag{28}$$

then a left multiplication by $\mathbf{M}_S$ yield an equation of the desired symmetric system:

$$\mathbf{M}_S \ddot{\mathbf{U}} - \omega^2 \mathbf{M}_S \mathbf{K}_S^{-1} \mathbf{M}_S \ddot{\mathbf{U}} + \omega^2 \mathbf{M}_S \mathbf{K}_S^{-1} \mathbf{Q}\mathbf{P} = \mathbf{0} \tag{29}$$

Now the displacement vector will be eliminated from the system. It is first expressed by:

$$\mathbf{U} = -\mathbf{K}_S^{-1} \mathbf{M}_S \ddot{\mathbf{U}} + \mathbf{K}_S^{-1} \mathbf{Q}\mathbf{P} \tag{30}$$

then replaced in the second equation of (19) to give

$$\mathbf{K}_F \mathbf{P} - \omega^2 \left(\mathbf{M}_F + \rho_F \mathbf{Q}^T \mathbf{K}_S^{-1} \mathbf{Q} \right) \mathbf{P} + \omega^2 \rho_F \mathbf{Q}^T \mathbf{K}_S^{-1} \mathbf{M}_S \ddot{\mathbf{U}} = \mathbf{0} \tag{31}$$

which is divided by ρ_F to yield the other equation of the symmetric system. The final expression is:

$$\left(\begin{bmatrix} \dfrac{1}{\rho_F} \mathbf{K}_F & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_S \end{bmatrix} - \omega^2 \begin{bmatrix} \dfrac{1}{\rho_F} \mathbf{M}_F + \mathbf{Q}^T \mathbf{K}_S^{-1} \mathbf{Q} & -\mathbf{Q}^T \mathbf{K}_S^{-1} \mathbf{M}_S \\ -\mathbf{M}_S \mathbf{K}_S^{-1} \mathbf{Q} & \mathbf{M}_S \mathbf{K}_S^{-1} \mathbf{M}_S \end{bmatrix} \right) \begin{Bmatrix} \mathbf{P} \\ \ddot{\mathbf{U}} \end{Bmatrix} = \mathbf{0} \tag{32}$$

The most advantage of this technique compared to the two above ones is that it avoids inversion of the fluid matrices. No natural boundary condition is necessary for the pressure, the free surface effects can thus be counted for in free vibration modes extraction. In addition, for a fixed structure, the stiffness matrix $\mathbf{K}_F$ is always positive definite, its inversion should not pose any special numerical difficulties.

The inconvenient retained here concerns the change of variable, using accelerations as basic unknown variables for the structure requires additional computations when interest is focused on the structure response. Consequently this technique is preferred for problems dealing mostly with the fluid part.

4.4. Technique using inversion of the fluid mass matrix

It is proposed here to make the same change of variable as in Ohayon's technique ($\mathbf{P} = \rho_F \omega^2 \mathbf{V}$) but this time $\mathbf{P}$ is eliminated from the system. The second equation of (19) is rewritten by introducing $\mathbf{V}$ as follows

$$\rho_F \omega^2 \mathbf{K}_F \mathbf{V} - \omega^2 \mathbf{Q}^T \mathbf{U} - \omega^2 \mathbf{M}_F \mathbf{P} = \mathbf{0} \tag{33}$$

then $\mathbf{P}$ is expressed as a function of both $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{P} = \rho_F \mathbf{M}_F^{-1} \left(\mathbf{K}_F \mathbf{V} - \mathbf{Q}^T \mathbf{U} \right) \tag{34}$$

and replaced in the first equation of (19) to yield

$$\left(\mathbf{K}_S + \rho_F \mathbf{Q} \mathbf{M}_F^{-1} \mathbf{Q}^T \right) \mathbf{U} - \omega^2 \mathbf{M}_S \mathbf{U} - \rho_F \mathbf{Q} \mathbf{M}_F^{-1} \mathbf{K}_F \mathbf{V} = \mathbf{0} \tag{35}$$

This is the first equation of the target system. Now to achieve the symmetry, the second equation is obtained by left-multiplying (34) by $\mathbf{K}_F^T$ and replacing $\mathbf{P}$ by $\rho_F \omega^2 \mathbf{V}$. The final symmetric shape is

$$\left(\begin{bmatrix} \mathbf{K}_S + \rho_F \mathbf{Q} \mathbf{M}_F^{-1} \mathbf{Q}^T & -\rho_F \mathbf{Q} \mathbf{M}_F^{-1} \mathbf{K}_F \\ -\rho_F \mathbf{K}_F^T \mathbf{M}_F^{-1} \mathbf{Q}^T & \rho_F \mathbf{K}_F^T \mathbf{M}_F^{-1} \mathbf{K}_F \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_S & \mathbf{0} \\ \mathbf{0} & \rho_F \mathbf{K}_F^T \end{bmatrix} \right) \begin{Bmatrix} \mathbf{U} \\ \mathbf{V} \end{Bmatrix} = \mathbf{0} \tag{36}$$

We see that we use here inversion of the fluid mass matrix which is positive definite independently on whether essential or natural boundary condition is used for the pressure. Thus free surface fluctuations can be included in the free vibrations modes. This is one advantage of this technique, another advantage concern numerical aspect. If lumping concept is applied when evaluating the fluid mass, $\mathbf{M}_F$ becomes a diagonal matrix and its inversion is simply computed by inverting its diagonal components. In addition, the number of computation operations in triple matrix products can considerably be reduced when $\mathbf{M}_F$ is diagonal.

4.5. Technique using inversion of the structure mass matrix

In this technique, any change of variable is required, only matrix manipulations conduct to a symmetric form. From the first equation of (19) the accelerations vector is expressed as:

$$\omega^2 \mathbf{U} = \mathbf{M}_S^{-1} \mathbf{K}_S \mathbf{U} - \mathbf{M}_S^{-1} \mathbf{Q} \mathbf{P} \tag{37}$$

and replaced in the second equation to get:

$$\left(\mathbf{K}_F + \rho_F \mathbf{Q}^T \mathbf{M}_S^{-1} \mathbf{Q} \right) \mathbf{P} - \rho_F \mathbf{Q}^T \mathbf{M}_S^{-1} \mathbf{K}_S \mathbf{U} - \omega^2 \mathbf{M}_F \mathbf{P} = \mathbf{0} \tag{38}$$

which is an equation of the symmetric system then left multiplying equation (37) by $\mathbf{K}_S^T$ gives the other equation.

The final symmetric system is written, after dividing (38) by ρ_F, as follows

$$\left(\begin{bmatrix} \mathbf{K}_S^T \mathbf{M}_S^{-1} \mathbf{K}_S & -\mathbf{K}_S^T \mathbf{M}_S^{-1} \mathbf{Q} \\ -\mathbf{Q}^T \mathbf{M}_S^{-1} \mathbf{K}_S & \dfrac{1}{\rho_F} \mathbf{K}_F + \mathbf{Q}^T \mathbf{M}_S^{-1} \mathbf{Q} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{K}_S^T & \mathbf{0} \\ \mathbf{0} & \dfrac{1}{\rho_F} \mathbf{M}_F \end{bmatrix} \right) \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \mathbf{0} \tag{39}$$

By inverting the structure's mass matrix, this technique combines the advantages of the previous ones, and at the same time it avoids the inconvenient of making change of variable.

For finite element models utilizing rotational degrees of freedom, such rods and shells, the components of $\mathbf{M}_S$ corresponding to the rotations are null, and then, the matrix cannot be inverted unless static condensation of the rotations is first achieved.

5. Application and results

The presented symmetrization techniques are implemented in an existing finite element program of our own which is based on the "ARPACK" library for eigenvalues and eigenvectors computation.

The program is validated through an example of a 2D steel vessel partially filled with water that has been given in [14]. This problem is solved in the reference by using two different methods for the fluid: a finite element added mass and Raviart-Thomas discretization.

5.1. Problem description and FE modeling

The geometry of the coupled system is given in figure 2 with the finite element mesh used in the numerical model.

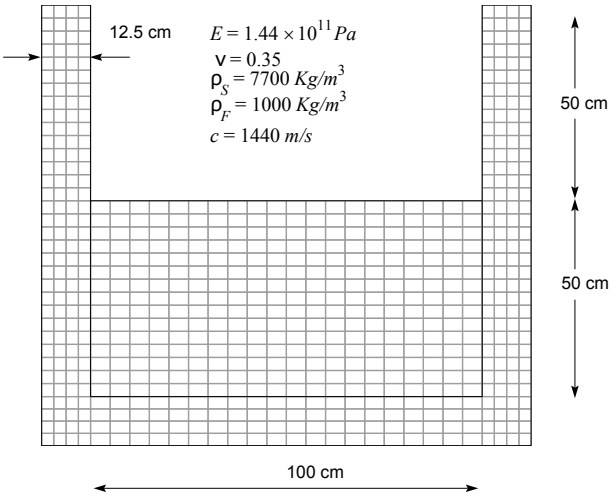


Fig. 2. Finite element model

The container of $1m$ inner dimensions and $12.5cm$ of thickness is fixed at its top and half filled of water (up to $50cm$). It is made of steel of density $\rho_S = 7700Kg/m^3$. Its elastic characteristics are given by the Young modulus $E = 1.44 \times 10^{11}Pa$ and the Poison's ratio $\nu = 0.35$.

The finite element model used for the structure consist on 352 linear plain strain quadrilateral elements, which correspond to a total number of 870 degrees of freedom.

The contained water of density $\rho_F = 1000Kg/m^3$ is considered linearly compressible with bulk modulus $k = 2.074 \times 10^9 Pa$ which gives $c = 1440m/s$). It is discretized by 300 quadrilateral acoustic elements, the element matrices are given by equations (15) to (17).The mesh correspond to a total number of 336 degrees of freedom.

It is to notice that the same element size is fixed for the two matched meshes in order to make easy the computation of the interaction matrix $\mathbf{Q}$.

5.2. Uncoupled modes

Firstly, in order to show that the mass matrix lumping does not affect the quality of the obtained results, we present in tables 1 and 2 the computed eigenfrequencies of the two subsystems separately with no interaction. The first five frequencies of vibration modes of the elastic vessel without fluid are reported in table 1. This modes are computed by solving equation (10) without external forces: $(\mathbf{K}_S - \omega_S^2 \mathbf{M}_S)\mathbf{U} = \mathbf{0}$. The mass matrix is taken once distributed and once lumped (equations 12 and 14 respectively).

As expected, this table shows that distributing or lumping the structure mass matrix leads to very small differences in the computed eigenvalues. This is well known in structural dynamics, the idea here is to apply the lumping to the fluid.

Table 1

Eigenfrequencies (rad/sec) of the empty structure

Mode #	1	2	3	4	5
Distributed mass	480.57	1717.07	2945.01	3126.03	4968.98
Lumped mass	480.47	1713.98	2937.86	3120.68	4954.42

As the fluid mass matrix contains two terms, we obtain two kinds of modes by solving eigenvalue problem associated to equation (11): $(\mathbf{K}_F - \omega_F^2 \mathbf{M}_F)\mathbf{P} = \mathbf{0}$. The first ones called sloshing modes correspond to the gravity waves due to free surface fluctuations at low frequencies, and the second ones, called hydrodynamic modes, correspond to volumetric contractions due to pressure fluctuations at high frequencies within the fluid mass. The first five modes of each kind are reported in table 2 for the two cases of distributed and lumped fluid mass matrix.

Table 2

Eigenfrequencies (rad/sec) of the fluid without structure

Mode #	1	2	3	4	5
			Sloshing modes		
Distributed mass	5.37	7.93	9.77	11.34	12.77
Lumped mass	5.36	7.87	9.59	10.97	12.13
			Hydrodynamic modes		
Distributed mass	4525.96	6402.50	10149.98	13627.56	14360.29
Lumped mass	4521.83	6383.51	10057.52	13515.94	14148.53

It is noticed from this table that practically, the same results are obtained for all range of the computed eigenfrequencies independently from lumping or distributing the mass. The relative difference between the values do not excess a maximum of 5%, it can thus be concluded that lumped mass matrix can be used not only for the structure as commonly done but also for the fluid.

5.3. Coupled modes

Now, we consider the fluid-structure interaction effects, tables 3 and 4 show the computed eigenfrequencies of the coupled system constituted by the steel vessel partially filled with water. The first four frequencies of the free vibration modes of the elastic vessel modified by the interaction effects, called hydroelastic modes, are reported in table 3, while the first four frequencies of the fluid sloshing modes are reported in table 4. The two kinds of modes are evaluated by using distributed and lumped mass matrices for both the structure and the fluid. Also, values computed by means of an added mass found in [14], are reported for comparison. It is to precise that in the reference, the fluid was considered incompressible and triangular elements were used for the finite element discretization.

We see from these results that the same values are obtained for the coupled frequencies by the present model using even distributed or lumped matrices.

The computed hydroelastic frequencies are slightly different (8% to 10%) from those of [14] which are evaluated by an added mass method. This can be explained at a part by the differences in types and sizes of the used meshes, close results are obtained with

Table 3

Eigenfrequencies (rad/sec) of the first hydroelastic modes

Mode #	1	2	3	4
Distributed mass	459.62	1524.80	2679.82	2865.23
Lumped mass	459.53	1522.63	2676.56	2858.47
Added mass (from [14])	503.49	1656.80	2981.84	3107.46

Table 4

Eigenfrequencies (rad/sec) of the first sloshing modes

Mode #	1	2	3	4
Distributed mass	5.37	7.94	9.77	11.84
Lumped mass	5.36	7.87	9.59	11.68
Added mass (from [14])	5.32	7.87	9.71	–

less number of quadrilateral elements. In addition, the converged values can be affected by the fluid compressibility. Meanwhile, the computed sloshing frequencies are very close to those given by [14], and moreover, if we compare them to those of table 2 computed without interaction, we see that neither the structure flexibility nor the fluid compressibility affect the sloshing modes.

5.4. Performances of the symmetrization techniques

Finally, the performances of the two proposed symmetrization techniques, obtained essentially by exploiting mass lumping concept, are shown through examining computer resources mobilized by the computer program in order to construct the coupled system.

Figure 3 plot CPU times (t/t_r) and storage memory capacities (m/m_r) required by each one of the symmetrization techniques exposed in section 4. The values are normalized to the time (t_r) and memory capacity (m_r) demanded by the simple construction of the non symmetric coupled system (equation 19). The first bar (a) in the plots correspond to unity (for the non symmetric system $t/t_r = 1$) and the subsequent bars (b) to (f) correspond respectively to the symmetric systems as they are exposed in section 4 (equations: 23, 27, 32 36 and 39, respectively).

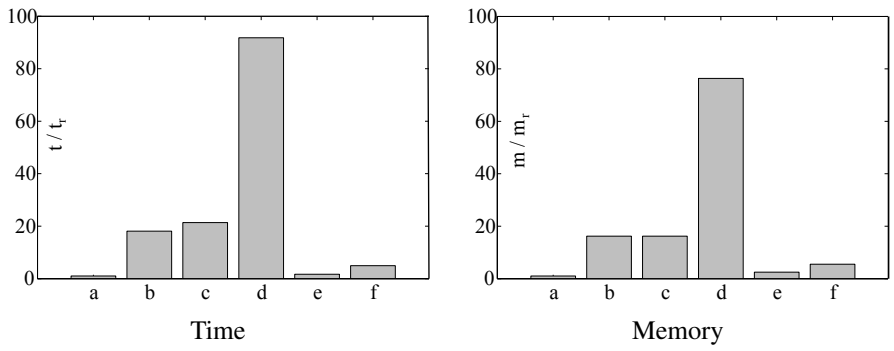


Fig. 3. CPU time and memory capacity requirements

It can easily seen from this figure that the height of the two bars (e) and (f) are close to unity (height of bar a). The CPU-time and the storage memory capacity required by the two proposed symmetrization techniques are slightly greater than those required by a simple assembling of the matrices constituting the coupled non symmetric system.

Bars (b) and (c) are lesser that bar (d) due to the size of the inverted matrices. The fluid matrices in this example are smaller than those of the structure. This situation can not be generalized, it can be inverted in problems requiring large fluid mesh such in the case of unbounded fluid domain.

An ultimate comparison of the symmetrization techniques, concern how they affect the banded shape of the original unsymmetric matrices. In figure 4, points represent nonzero matrix component, only this numbers are stored in computer's memory with the use of sparse storage algorihm. This figure show clearly that the proposed techniques preserve the sparsity pattern and the banded shapes of the matrix system. With the other symmetrization techniques, blocs of nonzeros components appear in the symmetric coupled matrices because of banded matrix inversion gives a full populated matrix.

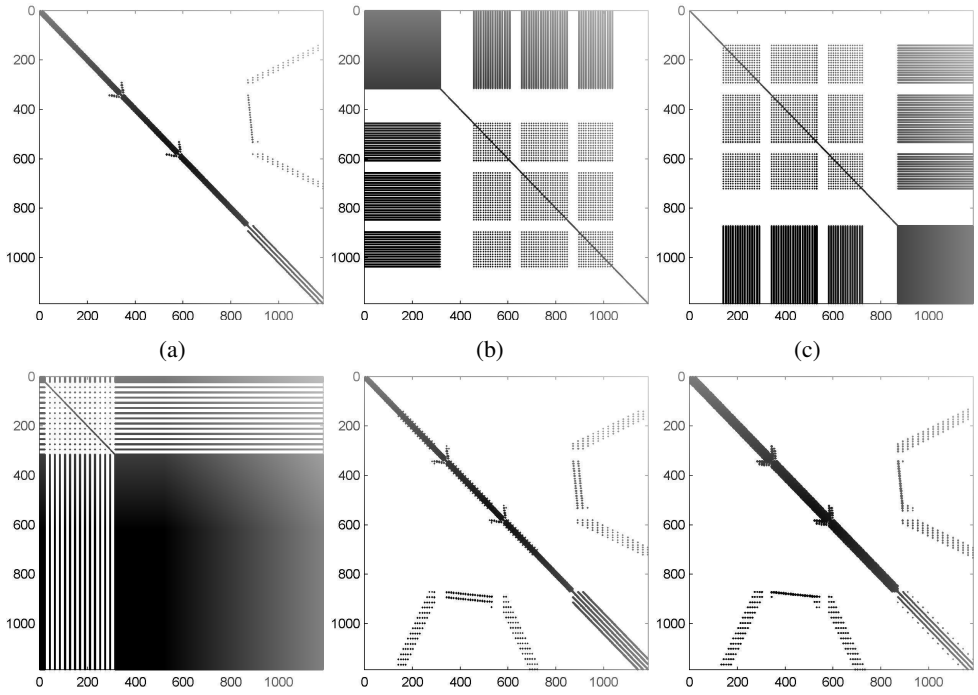


Fig. 4. Sparsity pattern of the coupled matrices

6. **Case of incompressible fluid**

In the case of incompressible fluid ($c = \infty$ in eq. 15), the fluid mass matrix $\mathbf{M}_F$ reduces to the sum of the contribution of only the surface integral over Γ_F. We note this contribution $\mathbf{M}_F^{(s)}$, where the subscript "s" refers to the surface nodes. By denoting the other fluid nodes by "v" for volume, the matrices in equation (11) may be partitioned as

follows:

$$\mathbf{M}_F = \begin{bmatrix} \mathbf{M}_F^{(\mathrm{s})} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad ; \quad \mathbf{K}_F = \begin{bmatrix} \mathbf{K}_F^{(\mathrm{s})} & \mathbf{K}_F^{(\mathrm{sv})} \\ \mathbf{K}_F^{(\mathrm{vs})} & \mathbf{K}_F^{(\mathrm{v})} \end{bmatrix} \quad ; \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}^{(\mathrm{s})} & \mathbf{Q}^{(\mathrm{v})} \end{bmatrix}$$

The corresponding eigenvalue problem for fluid splits then into the following two equations :

$$\mathbf{K}_F^{(\mathrm{s})}\mathbf{P}^{(\mathrm{s})} + \mathbf{K}_F^{(\mathrm{sv})}\mathbf{P}^{(v)} - \omega_{\mathrm{s}}^2\mathbf{M}_F^{(\mathrm{s})}\mathbf{P}^{(\mathrm{s})} + \rho_F\,\omega_{\mathrm{u}}^2\mathbf{Q}^{(\mathrm{s})^T}\mathbf{U} = \mathbf{0} \tag{40}$$

and

$$\mathbf{K}_F^{(\mathrm{vs})}\mathbf{P}^{(\mathrm{s})} + \mathbf{K}_F^{(\mathrm{v})}\mathbf{P}^{(\mathrm{v})} + \rho_F\,\omega_{\mathrm{u}}^2\mathbf{Q}^{(\mathrm{v})^T}\mathbf{U} = \mathbf{0} \tag{41}$$

where the pressure vector $\mathbf{P}$ is also split into the surface part $\mathbf{P}^{(s)}$ and the volume part $\mathbf{P}^{(v)}$. This last one can be evaluated from (41) and replaced in (40) to give:

$$\left(\mathbf{K}_F^{(\mathrm{s})*} - \omega_s^2\mathbf{M}_F^{(\mathrm{s})}\right)\mathbf{P}^{(\mathrm{s})} + \rho_F\,\omega_u^2\mathbf{Q}^{(\mathrm{s})*^T}\mathbf{U} = \mathbf{0} \tag{42}$$

where $\mathbf{K}_F^{(\mathrm{s})*} = \mathbf{K}_F^{(\mathrm{s})} - \mathbf{K}_F^{(\mathrm{sv})}{\mathbf{K}_F^{(\mathrm{v})}}^{-1}\mathbf{K}_F^{(\mathrm{vs})}$ and $\mathbf{Q}^{(\mathrm{s})*^T} = \mathbf{Q}^{(\mathrm{s})^T} - \mathbf{K}_F^{(\mathrm{sv})}{\mathbf{K}_F^{(\mathrm{v})}}^{-1}\mathbf{Q}^{(\mathrm{v})^T}$

Equation (42) without its second part ($\mathbf{U} = \mathbf{0}$) is a reduced form of the sloshing modes eigenproblem of a confined fluid. The submatrix $\mathbf{Q}^{(\mathrm{s})}$ contains terms coupling the pressure at fluid surface nodes $\mathbf{P}^{(\mathrm{s})}$ to the accelerations of the solid particles $\ddot{\mathbf{U}}$, this concerns, in 2D problems, only the two nodes belonging to the fluid surface and in contact with the solid walls. This submatrix can be neglected and the sloshing modes are, in fact, decoupled from the fluid-structure interaction problem.

Considering now the structure's eigensystem given by the first equation of the coupled system (19). By splitting the pressure vector and consequently the interaction matrix $\mathbf{Q}$, this equation is rewritten as follows after replacing $\mathbf{P}^{(\mathrm{v})}$ from (41) as done for equation (40)

$$\left(\mathbf{K}_S - \omega_{\mathrm{u}}^2\mathbf{M}_S^*\right)\mathbf{U} - \mathbf{Q}^{(\mathrm{s})*}\mathbf{P}^{(\mathrm{s})} = \mathbf{0} \tag{43}$$

with $\mathbf{M}_S^{(\mathrm{s})*} = \mathbf{M}_S + \rho_F\mathbf{Q}^{(\mathrm{v})}{\mathbf{K}_F^{(\mathrm{v})}}^{-1}\mathbf{Q}^{(\mathrm{v})^T}$ and $\mathbf{Q}^{(\mathrm{s})*} = \mathbf{Q}^{(\mathrm{s})} - \mathbf{Q}^{(\mathrm{v})}{\mathbf{K}_F^{(\mathrm{v})}}^{-1}\mathbf{K}_F^{(\mathrm{vs})}$

It is to note that equation (43) an added mass term due to fluid and it is independent from the fluid sloshing eigen frequencies (ω_{s}) while equation (42) include those of the solid displacements

The fluid-structure interaction problem coupling both sloshing pressures and structural displacements arises from gathering these two equations:

$$\left(\begin{bmatrix} \mathbf{K}_S & \mathbf{Q}^{(s)*} \\ \mathbf{0} & \mathbf{K}_F^{(\mathrm{s})*} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_S^* & \mathbf{0} \\ \rho_F\mathbf{Q}^{(\mathrm{s})*^T} & \mathbf{M}_F^{(\mathrm{s})} \end{bmatrix}\right)\begin{Bmatrix} \mathbf{U} \\ \mathbf{P}^{(\mathrm{s})} \end{Bmatrix} = \mathbf{0} \tag{44}$$

This system is exactly of the same form as (19) but of reduced size. All the the submatrices are symmetric, consequently all the previous symmetrization techniques presented above apply.

7. Conclusion

This work has been focused on the symmetrization of the free vibration problem of fluid-structure coupled systems arising from the finite element formulation which uses structural displacements and fluid hydrodynamic pressure fields as basic unknown variables. Three commonly used symmetrization techniques are reported in details and two others are introduced.

The performances of the proposed techniques are obtained essentially by exploiting mass lumping concept. Therefore, it has been first shown, through a numerical example that lumping mass matrix, usually used in structure finite element modeling, can also be applied to the fluid mass matrix accounting for both compressibility and free surface waves. Then, coupled eigenfrequencies of the considered fluid-structure system are computed using the five symmetrization techniques in order to show the huge economy that can be saved in terms of the computing resources.

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