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TAXATION, RISK AVERSION AND OPTIMAL INVESTMENT

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The effect of taxation on irreversible investment has often been considered expensively under risk neutral by using real option. This paper derives neutral tax system for a constant relative risk aversion entrepreneur and the normal and paradoxical effects of taxation on irreversible investment.

Keywords: Investment under uncertainty; Real option; Constant relative risk aversion; Tax neutrality

MSC2010: 91B55, 91B30

1. Introduction

Irreversible investment based on the theory of real option, which was studied deeply and expensively after the paradigm of [1], has been enriched by introducing taxation(e.g. [2] and [3]). In recent literatures, tax neutral system under uncertainty has been studied by public economists in the risk neutral scenario (e.g. [4] and [5]). Risk aversion is not considered widely yet in neutral tax systems in irreversible investment. Neutral tax systems was derived for the first time under risk aversion and irreversibility (see [6]).

This paper extends the model of [7] by introducing taxation in irreversible investment. We succeed in deriving neutral tax systems and the investment threshold as general analytical solution for investment decisions after taxes for a constant relative risk aversion entrepreneur. Moreover, we examine the effects of risk aversion and taxation on the entrepreneurs' investment decision.

2. Model and assumptions

Along the lines of [1] we consider an irreversible investment model for a

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constant relative risk aversion entrepreneur, in which he has exclusive access to a project and decides whether to invest at any time. The investment opportunity includes an option to wait. The entrepreneur can compare the costs and benefits of investing at each moment.

The project's completion takes a one-time investment cost *I*, which we assume is the entrepreneur's initial wealth. Prior to investment, the entrepreneur's initial wealth is invested in risk-free bond with risk-free pre-tax interest rate *r*.

Assuming that the project is infinitely lived and generates a instantaneous pre-tax stochastic revenue of P_t after the completion of investment that follows exogenously according to a geometric Brown motion (GBM)

$$dP_t = \mu P_t dt + \sigma P_t dZ_t \tag{1}$$

Where dZ_t is an increment to a standard Wiener process with mean zero and unit variance. The parameter μ and σ measure the trend and volatility in the price process respectively.

Profits from investment and interest from risk-free bond are both subject to a proportional profit tax with rate τ. As [5], we exclude periodical tax-deductible depreciation allowances and define the post-tax cash flow from investment $P_t^\tau = (1-\tau)P_t$ and the risk-free post-tax interest rate $r^\tau = (1-\tau)r$.

The constant relative risk aversion expected utility function

$$U(x) = \frac{x^{1-R}}{1-R} \;;\;\; R \geq 0 \,\&\, R \neq 1 \tag{2}$$

is introduced to represent the entrepreneur's preference if risk aversion is considered, which is typically increasing ($U'(x>0)$) and concave ($U''(x)<0$) R reflects the entrepreneur's relative risk aversion, as usual, the case $R=1$ corresponds to the logarithmic utility function. The entrepreneur's time-preference parameter we assume is ρ and for simplicity the post-tax discount rate is defined as $\rho^\tau = (1-\tau)\rho > \mu$.

Our goal is to determine whether and when the entrepreneur should invest in the project. In making this investment decision it is important to not only take into account the expected utility of future cash flow produced by the project, but also the real option value embedded in its irreversible investment. Once the investment has been made, it cannot be undone should prospects change for the worse. By deferring the investment, however, the investor can await new information about investment.

3. Results

By investment the project, the entrepreneur gives up a free-risk cash flow stream $(1-\tau)rI$ and gets in return a risk cash flow stream P_t. Using the law of iterated expectations and the strong Markov property of the GBM, which states that price values after investment time T are independent of the values before T and depend only on the value of the process at T, the time-zero discounted expected utility of the cash flows is:

$$\begin{aligned} V^\tau(P_0) &= \sup_{T\in S} E_{P_0}\left[\int_0^T U(r^\tau I)e^{-\rho^\tau t}dt + \int_T^\infty U(P_t)e^{-\rho^\tau t}dt\right] \\ &= \frac{U(r^\tau I)}{\rho^\tau} + \sup_{T\in S} E_{P_0}\left[\int_T^\infty [U(P_t)-U(r^\tau I)]e^{-\rho^\tau t}dt\right] \\ &= \frac{U(r^\tau I)}{\rho^\tau} + \sup_{T\in S} E_{P_0}[e^{-\rho^\tau T}]E_{P_T}\left[\int_0^\infty [U(P_s)-U(r^\tau I)]e^{-\rho^\tau s}ds\right] \\ &= \frac{U(r^\tau I)}{\rho^\tau} + F^\tau(P_0) \end{aligned} \tag{3}$$

By S, we define the set of stopping times of the filtration generated by the price process.

Using theorem 9.18 of [8] for the constant relative risk aversion utility function, we derive that

$$E_{P_T}\left[\int_0^\infty U(P_s)e^{-\rho^\tau s}ds\right] = \frac{\beta_1^\tau\beta_2^\tau}{\rho^\tau(1-\beta_1^\tau-R)(1-\beta_2^\tau-R)}U(P_0) \tag{4}$$

where β_1^τ and β_2^τ are respectively the positive and negative roots of the quadratic equation:

$$\frac{\sigma^2}{2}\beta^\tau(\beta^\tau - 1) + \mu\beta^\tau - (1-\tau)\rho = 0 \tag{5}$$

Since the expected discount factor $P_T^* = \left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}(1-\tau)rI$ (see [8]), $F^\tau(P_0)$ can be written as follows:

$$F^\tau(P_0) = \max_{P_T \geq P_0}\left(\frac{P_0}{P_T}\right)^{\beta_1^\tau}\left[\frac{\beta_1^\tau\beta_2^\tau}{\rho^\tau(1-\beta_1^\tau - R)(1-\beta_2^\tau - R)}U(P_0) - \frac{U(r^\tau I)}{\rho^\tau}\right] \tag{6}$$

For the constant relative risk aversion entrepreneur, the utility maximizing investment threshold is given by

$$P_T^* = \left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}(1-\tau)rI \tag{7}$$

4. Discussion

In order to analyze the effects of changes in the one-time investment cost, entrepreneur's attitudes toward risk, the cash flow's volatility and the profit tax rate on the entrepreneur's optimal investment decision, we introduce the following propositions.

Clearly, as $\left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau}\right)^{\frac{1}{1-R}} > 0$, this implies that an increase in I raises the optimal investment threshold for the constant relative risk aversion entrepreneur. It is not surprising that only when the optimal investment threshold reach a sufficient level would the entrepreneur start to invest.

Proposition 1: *When the entrepreneur's relative risk aversion coefficient R is less than 1, the objection function is strictly concave.*

Proof.

The objection function evaluated at the utility maximizing investment threshold, P_T^*, is

$$F^{\tau}(P_T^*) = \left(\frac{P_0}{P_T^*}\right)^{\beta_1^{\tau}} \left[\frac{\beta_1^{\tau}\beta_2^{\tau}}{\rho^{\tau}(1-\beta_1^{\tau}-R)(1-\beta_2^{\tau}-R)} U(P_T^*) - \frac{U(r^{\tau}I)}{\rho^{\tau}}\right] \tag{8}$$

Differentiating the objection function (8) with respect to P_T^* yields the following result:

$$\frac{\partial F^{\tau}(P_T^*)}{\partial P_T^*} = (1-\beta_2^{\tau}-R)\frac{\beta_1^{\tau}\beta_2^{\tau}}{\rho^{\tau}(1-\beta_2^{\tau}-R)(1-\beta_2^{\tau}-R)(1-R)} P_0^{\beta_1^{\tau}}(P_T^*)^{-\beta_1^{\tau}-R} + \beta_1^{\tau}\frac{U(r^{\tau}I)}{\rho^{\tau}} P_0^{\beta_1^{\tau}}(P_T^*)^{-\beta_1^{\tau}-1}$$

$$\begin{aligned}
\frac{\partial^2 F^{\tau}(P_T^*)}{\partial P_T^{*2}} &= (1-\beta_2^{\tau}-R)(-\beta_1^{\tau}-R)\frac{\beta_1^{\tau}\beta_2^{\tau}}{\rho^{\tau}(1-\beta_2^{\tau}-R)(1-\beta_2^{\tau}-R)(1-R)} P_0^{\beta_1^{\tau}}(P_T^*)^{-\beta_1^{\tau}-R-1} \\
&\quad + \beta_1^{\tau}(-\beta_1^{\tau}-1)\frac{U(r^{\tau}I)}{\rho^{\tau}} P_0^{\beta_1^{\tau}}(P_T^*)^{-\beta_1^{\tau}-2} \\
&= P_0^{\beta_1^{\tau}}(P_T^*)^{-\beta_1^{\tau}-2}\left[\begin{array}{l}(1-\beta_2^{\tau}-R)(-\beta_1^{\tau}-R)\dfrac{\beta_1^{\tau}\beta_2^{\tau}}{\rho^{\tau}(1-\beta_2^{\tau}-R)(1-\beta_2^{\tau}-R)(1-R)}\dfrac{\beta_2^{\tau}+R-1}{\beta_2^{\tau}}(r^{\tau}I)^{1-R} \\ -\beta_1^{\tau}(\beta_1^{\tau}+1)\dfrac{U(r^{\tau}I)}{\rho^{\tau}}\end{array}\right] \\
&= \beta_1^{\tau}(R-1)P_0^{\beta_1^{\tau}}(P_T^*)^{-\beta_1^{\tau}-2}\frac{(r^{\tau}I)^{1-R}}{\rho^{\tau}} < 0.
\end{aligned}$$

As proved above, the objection function (8) is concave only and only if $0 \le R < 1$.

Under the traditional NPV rule the investment decision is undertaken as soon as the project value exceeds the one-time investment cost, which is at the revenue level equal to $(1-\tau)rI$. This value is always lower than P_T^*, as $\left(\frac{\beta_2^{\tau}+R-1}{\beta_2^{\tau}}\right)^{\frac{1}{1-R}} > 1$. So there are states where the expected payoff is positive and the entrepreneur chooses to wait and not to invest. The option to invest captures this positive value of waiting.

Proposition 2: *The optimal investment threshold* P_T^* *increases with risk aversion* R.

Proof.

Differentiating the entrepreneur's optimal investment threshold P_T^* with respect to R yields:

$$\frac{\partial P_T^*}{\partial R} = P_T^* \frac{\partial}{\partial R}\left(\frac{1}{1-R}\ln\frac{\beta_2^\tau + R - 1}{\beta_2^\tau}\right)$$
$$= \frac{P_T^*}{(1-R)^2}\left(\frac{\beta_2^\tau}{\beta_2^\tau + R - 1} - 1 - \ln\frac{\beta_2^\tau}{\beta_2^\tau + R - 1}\right) \quad (9)$$

For $f(x) = x - 1 - \ln x, x \in (1, \infty)$, we know

$$f'(x) = \frac{x-1}{x} > 0 \text{ and } f(x) = x - 1 - \ln x > 0, \forall x \in (1, \infty)$$

Since, $\frac{\beta_2^\tau}{\beta_2^\tau + R - 1} > 1$ we can derive $P_T^* > 0$ and

$$\frac{\beta_2^\tau}{\beta_2^\tau + R - 1} - 1 - \ln\frac{\beta_2^\tau}{\beta_2^\tau + R - 1} > 0$$

So, $\frac{\partial P_T^*}{\partial R} > 0$.

Proposition 3: *The optimal investment threshold* P_T^* *increases with uncertainty.*

Proof.

The derivatives of the entrepreneur's optimal investment threshold P_T^* with respect to σ^2 is:

$$\frac{\partial P_T^*}{\partial \sigma^2} = \frac{1}{1-R}\left[1 + \frac{R-1}{\beta_2^\tau}\right]^{\frac{1}{1-R}-1}\left[\frac{1-R}{(\beta_2^\tau)^2}\frac{\partial \beta_2^\tau}{\partial \sigma^2}\right](1-\tau)rI$$
$$= \left[1 + \frac{R-1}{\beta_2^\tau}\right]^{\frac{1}{1-R}-1}\frac{(1-\tau)rI}{(\beta_2^\tau)^2}\frac{\partial \beta_2^\tau}{\partial \sigma^2} \quad (10)$$

Note that:

$$\rho^{\tau} = (1-\tau)\rho > \mu \Rightarrow \frac{\partial \beta_2^{\tau}}{\partial \sigma^2} = \frac{1}{\sigma^4}\left[\mu + \frac{(\frac{\mu}{\sigma^2}-\frac{1}{2})\mu + \rho^{\tau}}{\sqrt{(\frac{\mu}{\sigma^2}-\frac{1}{2})^2 + \frac{2\rho^{\tau}}{\sigma^2}}}\right] > 0$$

so we can conclude that $\frac{\partial P_T^*}{\partial \sigma^2} > 0$.

The effect of uncertainty on the optimal investment threshold is unambiguously positive. This is the standard real options result, which says that the value of waiting is increasing with uncertainty. This is reflected by higher optimal investment threshold, because the revenue must reach a higher level before investment decision is optimally undertaken.

In addition to factors above, the constant relative risk aversion entrepreneur also needs to examine the effect of the profit tax rate on the optimal investment threshold.

Definition 1: *A tax system is called* ***neutral*** *for a entrepreneur with the standard constant relative risk aversion utility function iff* $\frac{\partial P_T^*}{\partial \tau} = 0$. *A neutral tax rate* $\tau_{neutral} = \tau_{neutral}(\mu, \sigma^2, \rho, R)$ *is an element of a neutral tax system with*

$$\frac{\partial P_T^*}{\partial \tau} = -\left(\frac{\beta_2^{\tau} + R - 1}{\beta_2^{\tau}}\right)^{\frac{1}{1-R}} + \frac{1-\tau}{\beta_2^{\tau}(\beta_2^{\tau} + R - 1)}\left(\frac{\beta_2^{\tau} + R - 1}{\beta_2^{\tau}}\right)^{\frac{1}{1-R}} \frac{\partial \beta_2^{\tau}}{\partial \tau} = 0 \quad (11)$$

The first term tends to lower the threshold, while the second tends to raise it. We are looking for settings that balance these effects ($\frac{\partial P_T^*}{\partial \tau} = 0$, neutral tax system).

Definition 2: *A tax reaction is defined* ***normal*** *under the case of constant relative risk aversion when an increase in the tax rate increase the utility maximizing investment threshold* P_T^*, *i.e.* $\frac{\partial P_T^*}{\partial \tau} > 0$.

Definition 3: *A tax reaction is defined* ***paradoxical*** *under the case of constant relative risk aversion when an increase in the tax rate decrease the utility maximizing investment threshold* P_T^*, *i.e.* $\frac{\partial P_T^*}{\partial \tau} < 0$.

For simplicity and convenience, we set $\delta = \rho^\tau - \mu > 0$ and $\frac{1}{2} - \frac{\sigma^2}{u} > 0$, so

$$\lim_{\delta \to 0} \beta_2^\tau = -\frac{2\mu}{\sigma^2}, \lim_{\sigma \to \infty} \beta_2^\tau = 0, \lim_{\delta \to 0}\left[(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2(1-\tau)\rho}{\sigma^2} \right]^{\frac{1}{2}} = \frac{1}{2} + \frac{\mu}{\sigma^2}.$$

Along the assumptions above, we can derive the following lemma.

Lemma 1: *Let* $G = \frac{\partial P_T^*}{\partial \tau} = -\left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau} \right)^{\frac{1}{1-R}} + \frac{1-\tau}{\beta_2^\tau (\beta_2^\tau + R - 1)} \left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau} \right)^{\frac{1}{1-R}} \frac{\partial \beta_2^\tau}{\partial \tau}$,

$\delta = \rho^\tau - \mu > 0$ *and* $\frac{1}{2} - \frac{\sigma^2}{u} > 0$, *then* $\frac{\partial G}{\partial \tau} = \frac{\partial^2 P_T^*}{\partial \tau^2} > 0$ *for sufficiently small* δ.

Proof.

Differentiating the utility maximizing investment P_T^* threshold with respect to the tax rate τ, yields:

$$\begin{aligned} G = \frac{\partial P_T^*}{\partial \tau} &= -\left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau} \right)^{\frac{1}{1-R}} + \frac{1-\tau}{\beta_2^\tau (\beta_2^\tau + R - 1)} \left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau} \right)^{\frac{1}{1-R}} \frac{\partial \beta_2^\tau}{\partial \tau} \\ &= \left(\frac{\beta_2^\tau + R - 1}{\beta_2^\tau} \right)^{\frac{1}{1-R}} \left[-1 + \frac{1-\tau}{\beta_2^\tau (\beta_2^\tau + R - 1)} \frac{\partial \beta_2^\tau}{\partial \tau} \right] \end{aligned} \tag{12}$$

Since $\frac{\partial \beta_2^\tau}{\partial \tau} = \left[(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2(1-\tau)\rho}{\sigma^2} \right]^{-\frac{1}{2}} \frac{\rho}{\sigma^2} > 0$ and

$$\frac{\partial^2 \beta_2^\tau}{\partial \tau^2} = \left[(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2(1-\tau)\rho}{\sigma^2} \right]^{-\frac{1}{2}} \left(\frac{\partial \beta_2^\tau}{\partial \tau} \right)^2 > 0,$$

we can conclude that

$$\frac{\partial G}{\partial \tau}=\frac{\partial^2 P_T^*}{\partial \tau^2}=\left(\frac{\beta_2^\tau+R-1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}\frac{1}{\beta_2^\tau(\beta_2^\tau+R-1)}\frac{\partial \beta_2^\tau}{\partial \tau}\left[-1+\frac{1-\tau}{\beta_2^\tau(\beta_2^\tau+R-1)}\frac{\partial \beta_2^\tau}{\partial \tau}\right]$$

$$+\left(\frac{\beta_2^\tau+R-1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}\left[\frac{-(1-\tau)}{(\beta_2^\tau)^2(\beta_2^\tau+R-1)}\left(\frac{\partial \beta_2^\tau}{\partial \tau}\right)^2+\frac{-(1-\tau)}{(\beta_2^\tau)^2(\beta_2^\tau+R-1)^2}\left(\frac{\partial \beta_2^\tau}{\partial \tau}\right)^2+\frac{1-\tau}{\beta_2^\tau(\beta_2^\tau+R-1)}\frac{\partial^2 \beta_2^\tau}{\partial \tau^2}\right]$$

$$=\left(\frac{\beta_2^\tau+R-1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}\frac{1}{\beta_2^\tau(\beta_2^\tau+R-1)}\left[(1-\tau)\frac{\partial^2 \beta_2^\tau}{\partial \tau^2}-\frac{\partial \beta_2^\tau}{\partial \tau}+(1-\tau)\frac{2-2\beta_2^\tau-R}{\beta_2^\tau(\beta_2^\tau+R-1)}\left(\frac{\partial \beta_2^\tau}{\partial \tau}\right)^2\right]$$

$$=\left(\frac{\beta_2^\tau+R-1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}\frac{1}{\beta_2^\tau(\beta_2^\tau+R-1)}\cdot$$

$$\cdot\left\{(1-\tau)\left[\frac{2-2\beta_2^\tau-R}{\beta_2^\tau(\beta_2^\tau+R-1)}+\left[(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2(1-\tau)\rho}{\sigma^2}\right]^{-\frac{1}{2}}\right]\left(\frac{\partial \beta_2^\tau}{\partial \tau}\right)^2-\frac{\partial \beta_2^\tau}{\partial \tau}\right\}$$

$$=\left[\left(\frac{\beta_2^\tau+R-1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}\frac{1}{\beta_2^\tau(\beta_2^\tau+R-1)}\frac{\partial \beta_2^\tau}{\partial \tau}\right]\cdot$$

$$\cdot\left\{(1-\tau)\left[\frac{2-2\beta_2^\tau-R}{\beta_2^\tau(\beta_2^\tau+R-1)}+\left[(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2(1-\tau)\rho}{\sigma^2}\right]^{-\frac{1}{2}}\right]\frac{\partial \beta_2^\tau}{\partial \tau}-1\right\}.$$

As the sign of square bracket is positive, i.e., $\left(\frac{\beta_2^\tau+R-1}{\beta_2^\tau}\right)^{\frac{1}{1-R}}\frac{1}{\beta_2^\tau(\beta_2^\tau+R-1)}\frac{\partial \beta_2^\tau}{\partial \tau}>0$, we must check the sign of

$$(1-\tau)\left[\frac{2-2\beta_2^\tau-R}{\beta_2^\tau(\beta_2^\tau+R-1)}+\left[(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2(1-\tau)\rho}{\sigma^2}\right]^{-\frac{1}{2}}\right]\frac{\partial \beta_2^\tau}{\partial \tau}-1 .$$

As $\beta_2^\tau=(\frac{1}{2}-\frac{\mu}{\sigma^2})-\sqrt{(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2(1-\tau)\rho}{\sigma^2}}<0$ tends to become sufficiently close to $-2\frac{\mu}{\sigma^2}$ for sufficiently small δ, we have the following result:

$$(1-\tau)\left[\frac{2-2\beta_2^\tau-R}{\beta_2^\tau(\beta_2^\tau+R-1)}+\left[(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2(1-\tau)\rho}{\sigma^2}\right]^{-\frac{1}{2}}\right]\frac{\partial\beta_2^\tau}{\partial\tau}$$

$$=(1-\tau)\left[\frac{2+4\frac{\mu}{\sigma^2}-R}{-2\frac{\mu}{\sigma^2}(-2\frac{\mu}{\sigma^2}+R-1)}+\frac{1}{\frac{1}{2}+\frac{\mu}{\sigma^2}}\right]\frac{1}{\frac{1}{2}+\frac{\mu}{\sigma^2}}\frac{\mu}{1-\tau}\frac{1}{\sigma^2}$$

$$=\frac{\left(1+2\frac{\mu}{\sigma^2}\right)^2+(1+4\frac{\mu}{\sigma^2}+4\frac{\mu}{\sigma^2}\cdot\frac{1}{2})(1+2\frac{\mu}{\sigma^2}-R)}{(1+2\frac{\mu}{\sigma^2}-R)\left(1+2\frac{\mu}{\sigma^2}\right)^2}$$

$$>\frac{\left(1+2\frac{\mu}{\sigma^2}\right)^2+(1+4\frac{\mu}{\sigma^2}+4\frac{\mu}{\sigma^2}\cdot\frac{\mu}{\sigma^2})(1+2\frac{\mu}{\sigma^2}-R)}{(1+2\frac{\mu}{\sigma^2}-R)\left(1+2\frac{\mu}{\sigma^2}\right)^2}$$

$$=1+\frac{1}{1+2\frac{\mu}{\sigma^2}-R}>1.$$

Hence, if δ decreases there will be a sufficiently small δ so that

$$\frac{\partial G}{\partial\tau}=\frac{\partial^2 P_T^*}{\partial\tau^2}>0.$$

Proposition 4: *A tax reaction of the utility maximizing investment threshold is defined* ***normal/paradoxical*** *under the case of constant relative risk aversion when the tax rates* τ *are* ***higher/lower*** *than the rate* $\tau_{neutral}$ *of the tax system* for sufficiently small δ.

Proof.

In definition 1, the neutral tax rates $\tau_{neutral}=\tau_{neutral}(\mu,\sigma^2,\rho,R)$ of a neutral tax system can be derived from $\frac{\partial P_T^*}{\partial\tau}=0.$

From Lemma 1, we know $\frac{\partial G}{\partial \tau} = \frac{\partial^2 P_T^*}{\partial \tau^2} > 0$, hence

$$\begin{cases} \frac{\partial P_T^*}{\partial \tau} > 0 & for\ \tau > \tau_{neutral}(\mu, \sigma^2, \rho, R) \\ \frac{\partial P_T^*}{\partial \tau} < 0 & for\ \tau < \tau_{neutral}(\mu, \sigma^2, \rho, R) \end{cases}.$$

5. Conclusions

In this paper, we prove neutral tax systems in analytic formula form in irreversible investment under constant relative risk aversion and uncertainty. By proper simplicity and assumption, we also derive that taxation has opposite effect on the utility maximizing investment threshold, i.e. an increase in the tax rate increase the utility maximizing investment threshold when the tax rates are higher than the rate of the neutral tax system, decrease for otherwise. Moreover, the time of the entrepreneur's investment is delayed until the optimal out price when relative risk aversion is increasing.

Directions for future research could include the neutral tax system in different class of utility functions, which could be applied in order to obtain further insight regarding the impact of risk aversion and taxation on the optimal investment policy and allow for comparisons with the approach presented in this paper. What's more, two or more risk aversion enterprises can be considered in the competition scenario with the help of game theory as [9], which will be an interesting topic in the future.

R E F E R E N C E S

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