SOME RELATIONS ON HERMITE-HERMITE MATRIX POLYNOMIALS

Ayman SHEHATA[1] and Bayram ÇEKİM[2]

The main purpose of this paper is to define a new extension of Hermite-Hermite matrix polynomials (HHMP) and derive some properties such as recurrence relations and matrix differential equation for generalized Hermite-Hermite matrix polynomials. We derive addition and multiplication theorem, and summation formula for generalized HHMP. Furthermore, via integral transform, the new families of Chebyshev-Chebyshev, Legendre-Legendre, Chebyshev-Hermite and Legendre-Hermite matrix polynomials are introduced, from which a variety of interesting results follows as special cases.

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1. Introduction and preliminaries

Special functions, as a branch of mathematics, is of utmost importance to scientists, physics, engineering, mathematical physics and engineers in many areas of applications. In the recent papers, matrix polynomials have significant emergent. Some mathematician have obtained some properties for orthogonal matrix polynomials and special matrix functions via some properties in the theory of orthogonal polynomials and special functions, see [1, 2, 3, 4, 5, 6, 7, 9, 10, 16, 17, 18, 19, 20, 21, 22]. The Hermite, Legendre and Gegenbauer matrix polynomials have been studied in many previous papers [11, 15]. The first author has studied the Hermite-Hermite matrix polynomials [12, 13, 17]. The reason of interest for this family of Hermite polynomials is due to their main mathematical significance and to these polynomials have applications in physics.

Our main aim in this paper is to take in consideration a new matrix polynomials, namely the class of the generalized HHMP taking advantage

[1] Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt. Department of Mathematics, College of Science and Arts in Unaizah, Qassim University, Qassim 10363, Kingdom of Saudi Arabia. Corresponding author. E-mail: drshehata2006@yahoo.com

[2] Gazi University, Faculty of Science, Department of Mathematics Teknikokullar TR-06500, Ankara, Turkey, e-mail: bayramcekim@gazi.edu.tr

of the recently treated in [12]. In this paper, the generalized HHMP are defined, the four terms matrix recurrence relation is satisfied, their connections with matrix differential equation of third order is established in section 2. In section 3, we define generalized Legendre and Chebyshev matrix polynomials of first and second kind and give an integral representation. Finally, we define the Legendre-Hermite and Chebyshev-Hermite-type matrix polynomials of the first and second kind in section 4.

In this paper, its spectrum $\sigma(A)$ for A in $\mathbb{C}^{N\times N}$ will symbolize the set of all the eigenvalues of A. Its two-norm will be symbolized by $\| A\|_2$ and given by $\|A\|_2 = sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}$ where $\|y\|_2 = (y^H y)^{1/2}$ is the Euclidean norm of y for a vector $y \in \mathbb{C}^N$. If $u(z)$ and $v(z)$ are holomorphic functions are defined in an open set Ω of the complex plane and P, Q are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(P)\subset\Omega$ and $\sigma(Q)\subset\Omega$, such that $PQ = QP$, via the matrix functional calculus in [8], we have

$$u(P)v(Q) = v(Q)u(P).$$

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principle logarithm of z, then $\sqrt{z}$ represents $z^{\frac{1}{2}} = \sqrt{z} = \exp\left(\frac{1}{2}\log(z)\right)$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp\left(\frac{1}{2}\log(A)\right)$ denotes the image by $\sqrt{z}$ of matrix functional calculus acting on the matrix A.

In [6], for $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} A(k,n-mk)\ ; m\in\mathbb{N}. \tag{1.1}$$

Similarly to (1.1), we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk)\ ; m\in\mathbb{N}. \tag{1.2}$$

Let A be a matrix in $\mathbb{C}^{N\times N}$ such that

$$Re(z) > 0 \text{ for } \forall z\in\sigma(A). \tag{1.3}$$

This matrix is denoted as positive stable. Then the Hermite matrix polynomials $H_n(x,A)$ are defined via the generating matrix function [7]

$$\sum_{n=0}^{\infty} H_n(x,A)\frac{t^n}{n!} = \exp\left(xt\sqrt{2A} - t^2 I\right); (x,t)\in\mathbb{R}^2. \tag{1.4}$$

Here I is the identity matrix or unit matrix in $\mathbb{C}^{N\times N}$. Furthermore, in [7] n^{th} Hermite matrix polynomials have the property

$$H_n(x,A) = n!\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}, n \geq 0. \quad (1.5)$$

In the following, we are devoted to a more substantive effort in proofs of some known properties as well as new expansions formulae related to these generalized HHMP.

2. Generalized Hermite-Hermite matrix polynomials

If A is a positive stable in $\mathbb{C}^{N\times N}$ and $\beta \notin \mathbb{Z}^-$, the generalized HHMP is defined as the series

$$_H H_n(x,A) = n!\sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k}{k!(n-2k)!(1+\beta)_k} H_{n-2k}(x,A) \quad (2.1)$$

where the Pochhammer symbol is defined in [14]

$$(\beta-\alpha)_k = (\beta-\alpha)(\beta-\alpha+1)...(\beta-\alpha+k-1); \quad k \geq 1; \quad (\beta-\alpha)_0 = 1,$$
$$(\beta+1)_k = (\beta+1)(\beta+2)...(\beta+k); \quad k \geq 1; \quad (\beta+1)_0 = 1.$$

It is clear that

$$_H H_{-1}(x,A) = \theta, \quad _H H_0(x,A) = I, \quad _H H_1(x,A) = 2xA$$

where θ is a zero matrix or null matrix in $\mathbb{C}^{N\times N}$.

By using (1.2), (1.4) and (2.1), we get

$$\begin{aligned} F(x,t,A) = \sum_{n=0}^{\infty} {}_H H_n(x,A)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k}{k!(n-2k)!(1+\beta)_k} t^n H_{n-2k}(x,A) \\ &= \left(\sum_{k=0}^{\infty} \frac{(\beta-\alpha)_k}{k!(1+\beta)_k} t^{2k}\right)\left(\sum_{n=0}^{\infty} H_n(x,A)\frac{t^n}{n!}\right) \\ &= {}_1F_1(\beta-\alpha;1+\beta;t^2)\exp(xt\sqrt{2A}-t^2 I). \end{aligned}$$

We obtain a generating function for the generalized HHMP:

$$F(x,t,A) = {}_1F_1(\beta-\alpha;1+\beta;t^2)\exp(xt\sqrt{2A}-t^2 I). \quad (2.2)$$

Here $F(x,t,A)$ is a entire matrix function of the complex variable t. Because of this, the function has the Taylor series about $t=0$ and the series converges for all values of x and t.

Writing $(-x)$ instead of x and $(-t)$ instead of t in (2.2), we get

$$_H H_n(-x,A) = (-1)^n \, _H H_n(x,A).$$

By Kummer's first formula [14], we get

$$_1F_1(\beta-\alpha;1+\beta;t^2) = \exp(t^2)\, {}_1F_1(\alpha+1;1+\beta;-t^2). \quad (2.3)$$

Thus the generalized HHMP possess the generating function

$$F(x,t,A)=\sum_{n=0}^{\infty} {}_H H_n(x,A)\frac{t^n}{n!} \quad = {}_1F_1(\alpha+1;1+\beta;-t^2)\exp(xt\sqrt{2A}). \quad (2.4)$$

Therefore (2.4) and (1.1) yields

$$\begin{aligned} F(x,t,A)=\sum_{n=0}^{\infty} {}_H H_n(x,A)\frac{t^n}{n!} &= \left(\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha+1)_k t^{2k}}{k!(\beta+1)_k}\right)\left(\sum_{n=0}^{\infty}\frac{(x\sqrt{2A})^n t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(\alpha+1)_k(x\sqrt{2A})^{n-2k}t^n}{k!(n-2k)!(\beta+1)_k}. \end{aligned} \quad (2.5)$$

We get an explicit representation for the generalized HHMP:

$${}_H H_n(x,A) \quad = n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(\alpha+1)_k(x\sqrt{2A})^{n-2k}}{k!(n-2k)!(\beta+1)_k}. \quad (2.6)$$

Put

$$\Psi_n(x,A)=\frac{{}_H H_n(x,A)}{n!} \quad = \sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(\alpha+1)_k(x\sqrt{2A})^{n-2k}}{k!(n-2k)!(\beta+1)_k}, \quad (2.7)$$

then we have

$$\Psi_n(x,A) \quad = \sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(\alpha+1)_k(x\sqrt{2A})^{n-2k}}{k!(n-2k)!(\beta+1)_k} = \sum_{k=0}^{[\frac{1}{2}n]}\varepsilon(k,n,x,A). \quad (2.8)$$

From the series, we get

$$(x\sqrt{2A})\Psi_{n-1}(x,A) \quad = \sum_{k=0}^{[\frac{1}{2}(n-1)]}\frac{(-1)^k(\alpha+1)_k(x\sqrt{2A})^{n-2k}}{k!(n-2k-1)!(\beta+1)_k} = \sum_{k=0}^{[\frac{1}{2}n]}(n-2k)\varepsilon(k,n,x,A), \quad (2.9)$$

$$\begin{aligned} \Psi_{n-2}(x,A) \quad &= \sum_{k=0}^{[\frac{1}{2}(n-2)]}\frac{(-1)^k(\alpha+1)_k(x\sqrt{2A})^{n-2k-2}}{k!(n-2k-2)!(\beta+1)_k} \\ &= \sum_{k=1}^{[\frac{1}{2}n]}\frac{(-1)^{k-1}(\alpha+1)_{k-1}(x\sqrt{2A})^{n-2k}}{(k-1)!(n-2k)!(\beta+1)_{k-1}}, \end{aligned}$$

or

$$\Psi_{n-2}(x,A)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{-k(\beta+k)}{(\alpha+k)}\varepsilon(k,n,x,A), \quad (2.10)$$

$$(x\sqrt{2A})^2\Psi_{n-2}(x,A) \quad = \sum_{k=0}^{[\frac{1}{2}n]}(n-2k)(n-2k-1)\varepsilon(k,n,x,A) \quad (2.11)$$

and, in the same manner,

$$(x\sqrt{2A})\Psi_{n-3}(x,A) \quad = \sum_{k=0}^{[\frac{1}{2}n]} \frac{-k(n-2k)(\beta+k)}{\alpha+k}\varepsilon(k,n,x,A). \qquad (2.12)$$

It follow from the series (2.8)-(2.12) that there exists a relation as

$$\Psi_n(x,A)+a(x\sqrt{2A})\Psi_{n-1}(x,A)+[b\,I+c(x\sqrt{2A})^2]\Psi_{n-2}(x,A)+d(x\sqrt{2A})\Psi_{n-3}(x,A)=\theta \qquad (2.13)$$

in which the constants a, b, c, d are determined by the identity in k:

$$\alpha+k+a(n-2k)(\alpha+k)-b\,k(\beta+k)+c(n-2k)(n-2k-1)(\alpha+k)-d\,k(n-2k)(\beta+k)=0. \qquad (2.14)$$

The identity (2.14) yields

$$a=-\frac{2\beta+2n-1}{n(2\beta+n)}\;;\; b=\frac{2(2\alpha+n)}{n(2\beta+n)}\;;\; c=\frac{1}{n(2\beta+n)}\;;\; d=-\frac{2}{n(2\beta+n)}. \qquad (2.15)$$

Hence the $\Psi_n(x,A)$ satisfy

$$\begin{aligned} &n(2\beta+n)\Psi_n(x,A)-(2\beta+2n-1)(x\sqrt{2A})\Psi_{n-1}(x,A) \\ &+[2(2\alpha+n)I+(x\sqrt{2A})^2]\Psi_{n-2}(x,A)-2(x\sqrt{2A})\Psi_{n-3}(x,A)=\theta. \end{aligned} \qquad (2.16)$$

Since, by (2.7), $\Psi_n(x,A)=\frac{{}_HH_n(x,A)}{n!}$, we find that the polynomials ${}_HH_n(x,A)$ satisfy the pure recurrence relation

$$\begin{aligned} &(2\beta+n)\;{}_HH_n(x,A)-(2\beta+2n-1)(x\sqrt{2A})\;{}_HH_{n-1}(x,A) \\ &+(n-1)[2(2\alpha+n)I+(x\sqrt{2A})^2]\;{}_HH_{n-2}(x,A)-2(n-1)(n-2)(x\sqrt{2A})\;{}_HH_{n-3}(x,A)=\theta. \end{aligned} \qquad (2.17)$$

When $\alpha=\beta$, the ${}_HH_n(x,A)$ degenerates into the Hermite matrix polynomials $H_n(x,A)$, for which we already know the pure recurrence relation

$$H_n(x,A)-(x\sqrt{2A})H_{n-1}(x,A)+2(n-1)H_{n-2}(x,A)=\theta. \qquad (2.18)$$

The relation (2.17) may be put in the form

$$\begin{aligned} &(2\beta+n)[\;{}_HH_n(x,A)-(x\sqrt{2A})\;{}_HH_{n-1}(x,A)+2(n-1)\;{}_HH_{n-2}(x,A)] \\ &-(n-1)(x\sqrt{2A})[\;{}_HH_{n-1}(x,A)-(x\sqrt{2A})\;{}_HH_{n-2}(x,A)+2(n-2)\;{}_HH_{n-3}(x,A)] \qquad (2.19) \\ &+4(\alpha-\beta)(n-1)\;{}_HH_{n-2}(x,A)=\theta. \end{aligned}$$

It is now evident that if $\alpha=\beta$, (2.19) is an iteration of (2.18).

Theorem 2.1. *For a positive stable A in $\mathbb{C}^{N\times N}$, we get*

$$\begin{aligned} &(2\beta+n)\;{}_HH_n(x,A)=(2\beta+2n-1)x(\sqrt{2A})^2\;{}_HH_{n-1}(x,A) \\ &-(n-1)[2(2\beta+n)+(x\sqrt{2A})^2]\;{}_HH_{n-2}(x,A)-2(n-1)(n-2)x\sqrt{2A}\;{}_HH_{n-3}(x,A),\; n\geq 3. \end{aligned}$$

Now, we obtain some recurrence relations for generalized HHMP in this theorem.

Theorem 2.2. The generalized *HHMP satisfy the following relation*

$$\frac{d^r}{dx^r}\ {}_H H_n(x,A)=\frac{(\sqrt{2A})^r n!}{(n-r)!}\ {}_H H_{n-r}(x,A),\quad r=0,1\ldots,n. \tag{2.20}$$

Proof. Differentiating (2.2) with respect to x and using (2.2) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}\ {}_H H_n(x,A)\frac{t^n}{n!} &= t\sqrt{2A}\ {}_1F_1(\beta-\alpha;1+\beta;t^2)\exp(xt\sqrt{2A}-t^2 I)\\ &=\sqrt{2A}\sum_{n=0}^{\infty}\frac{1}{n!}\ {}_H H_n(x,A)t^{n+1}.\end{aligned} \tag{2.21}$$

Therefore, from identifying coefficients in t^n, we have

$$\frac{d}{dx}\ {}_H H_n(x,A)=n\sqrt{2A}\ {}_H H_{n-1}(x,A),\quad n\geq 1. \tag{2.22}$$

For $0\leq r\leq n$ make iteration (2.22) implies (2.20) for the proof.

In the next result, the generalized HHMP seem as finite series solution of third order matrix differential equation.

Theorem 2.3. *For a positive stable A in* $\mathbb{C}^{N\times N}$, the generalized *HHMP are solutions of the matrix differential equation of third order*

$$\begin{aligned}&\frac{d^3}{dx^3}\ {}_H H_n(x,A)+\left(\frac{(2\beta+n)}{x}I+\frac{x}{2}(\sqrt{2A})^2\right)\frac{d^2}{dx^2}\ {}_H H_n(x,A)\\ &-\left(\beta+n-\frac{1}{2}\right)(\sqrt{2A})^3\frac{d}{dx}\ {}_H H_n(x,A)+\frac{n(2\beta+n)}{2x}(\sqrt{2A})^2\ {}_H H_n(x,A)=\theta.\end{aligned} \tag{2.23}$$

Proof. Put $r=1,2,3$ in (2.20), we give

$$\begin{aligned}\frac{d^3}{dx^3}\ {}_H H_n(x,A)&=\frac{(\sqrt{2A})^3 n!}{(n-3)!}\ {}_H H_{n-3}(x,A),\quad 0\leq r\leq n,\\ {}_H H_{n-3}(x,A)&=\frac{(\sqrt{2A})^{-3}}{n(n-1)(n-2)}\frac{d^3}{dx^3}\ {}_H H_n(x,A),\\ {}_H H_{n-2}(x,A)&=\frac{(\sqrt{2A})^{-2}}{n(n-1)}\frac{d^2}{dx^2}\ {}_H H_n(x,A),\\ {}_H H_{n-1}(x,A)&=\frac{(\sqrt{2A})^{-1}}{n}\frac{d}{dx}\ {}_H H_n(x,A).\end{aligned} \tag{2.24}$$

Substituting from (2.24) into (2.17) yields

$$(2\beta+n)\ {}_H H_n(x,A)=(2\beta+2n-1)x(\sqrt{2A})^2\frac{(\sqrt{2A})^{-1}}{n}\frac{d}{dx}\ {}_H H_n(x,A)$$
$$-(n-1)[2(2\beta+n)I+(x\sqrt{2A})^2]\frac{(\sqrt{2A})^{-2}}{n(n-1)}\frac{d^2}{dx^2}\ {}_H H_n(x,A) \tag{2.25}$$
$$-2(n-1)(n-2)x\sqrt{2A}\frac{(\sqrt{2A})^{-3}}{n(n-1)(n-2)}\frac{d^3}{dx^3}\ {}_H H_n(x,A).$$

From (2.25), we obtain (2.23).

Now, we recall some significant properties of the generalized HHMP such as the addition and multiplication theorem.

Theorem 2.4. The generalized *HHMP satisfy the addition formula as follows: for $y \in \mathbb{R}$,*

$$ {}_H H_n(x+y,A) \quad = n!\sum_{k=0}^{n}\frac{(y\sqrt{2A})^{n-k}\ {}_H H_k(x,A)}{k!(n-k)!}. \tag{2.26}$$

Proof. By using the generating function (2.2) and (1.2), we have

$$\sum_{n=0}^{\infty} {}_H H_n(x+y)\frac{t^n}{n!} \quad = {}_1F_1(\beta-\alpha;1+\beta;t^2)\exp((x+y)t\sqrt{2A}-t^2I)$$
$$= {}_1F_1(\beta-\alpha;1+\beta;t^2)\exp(xt\sqrt{2A}-t^2I)e^{yt\sqrt{2A}}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_H H_k(x,A)\frac{t^k}{k!}\frac{(yt\sqrt{2A})^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{{}_H H_k(x,A)t^k}{k!}\frac{(yt\sqrt{2A})^{n-k}}{(n-k)!}.$$

Comparing the coefficients of t^n, (2.26) is derived.

Theorem 2.5. The generalized *HHMP satisfy the multiplication formula as: for $\mu \in \mathbb{R}$,*

$$ {}_H H_n(\mu x,A) \quad = n!\sum_{k=0}^{n}\frac{1}{k!(n-k)!}(x(\sqrt{2A})^2)^k(\mu-1)^k\ {}_H H_{n-k}(x,A). \tag{2.27}$$

Proof. Using (1.2) and (2.2), we have

$$\sum_{n=0}^{\infty} {}_H H_n(\mu x)\frac{t^n}{n!} \quad = {}_1F_1(\beta-\alpha;1+\beta;t^2)\exp(\mu xt\sqrt{2A}-t^2I)$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!n!}(x\sqrt{2A})^k(\mu-1)^k\ {}_H H_n(x,A)t^{n+k}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}(x\sqrt{2A})^k(\mu-1)^k\ {}_H H_{n-k}(x,A)t^n.$$

Therefore, we have the desired result.

In the next section, we have shown that the new integral representations are a fairly useful tool to obtain new families of matrix polynomials.

3. Generalized Legendre and Chebyshev matrix polynomials

We generate Legendre-Legendre and Chebyshev-Chebyshev matrix polynomials by using the properties in previous section.

Now, a version of Legendre-Legendre matrix polynomials will be given via the generalized HHMP. The Legendre matrix polynomials in [20] are defined by

$$P_n(x,A) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (x\sqrt{2A})^{n-2k}}{k!(n-2k)!} = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)!(x\sqrt{2A})^{n-2k}}{2^{2n-2k} k!(n-2k)!(n-k)!}. \tag{3.1}$$

By using (2.6), it follows that

$$\frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \; {}_H H_n(xt,A)dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (\alpha+1)_k (xt\sqrt{2A})^{n-2k}}{k!(n-2k)!(\beta+1)_k} dt$$

$$= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (\alpha+1)_k}{k!(n-2k)!(\beta+1)_k} (x\sqrt{2A})^{n-2k} \int_0^\infty e^{-t^2} t^{2n-2k} dt.$$

The series and the integral can be transputed since the summation in right is finite. Using Gamma function, one can see

$$\int_0^\infty e^{-t^2} t^{2n-2k} dt = \frac{1}{2}\Gamma\left(n-k+\frac{1}{2}\right) = \frac{1}{2}\frac{\sqrt{\pi}\,(2n-2k)!}{2^{2n-2k}(n-k)!}. \tag{3.2}$$

Then we get

$$\frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \; {}_H H_n(xt,A)dt = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (\alpha+1)_k (2n-2k)!(x\sqrt{2A})^{n-2k}}{2^{2n-2k} k!(n-2k)!(\beta+1)_k (n-k)!}.$$

Therefore, the Legendre-Legendre matrix polynomials are given as

$${}_P P_n(x,A) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (\alpha+1)_k (2n-2k)!(x\sqrt{2A})^{n-2k}}{2^{2n-2k} k!(n-2k)!(\beta+1)_k (n-k)!} \tag{3.3}$$

or

$${}_P P_n(x,A) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \; {}_H H_n(xt,A)dt \tag{3.4}$$

via the generalized HHMP.

Thus, the following theorem can be written.

Theorem 3.1. *The integral expressions (3.3) and (3.4) hold true for a positive stable A in* $\mathbb{C}^{N\times N}$.

In a similar manner, we can also define the generalized Legendre-Legendre-type matrix polynomials by using their integral representation.

For positive stable matrices A and B in $\mathbb{C}^{N\times N}$ and $AB = BA$, now we give generalized Legendre-Legendre-type matrix polynomials defined by the following relation:

$$ {}_P P_n(x,A,B) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k B^{k-n-\frac{1}{2}} (\alpha+1)_k (2n-2k)! (x\sqrt{2A})^{n-2k}}{2^{2n-2k} k!(n-2k)!(\beta+1)_k (n-k)!} \tag{3.5} $$

or

$$ {}_P P_n(x,A,B) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-Bt^2} t^n \; {}_H H_n(xt,A)dt. \tag{3.6} $$

Now, a version of the Chebyshev -Chebyshev matrix polynomials of the second kind will be given via the generalized HHMP. By (2.6), it follows that

$$ \frac{1}{n!}\int_0^\infty e^{-t} t^{\frac{n}{2}} \; {}_H H_n(x\sqrt{t},A)dt = \int_0^\infty \exp(-t) t^{\frac{n}{2}} \left(\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (\alpha+1)_k}{k!(n-2k)!(\beta+1)_k} (x\sqrt{t}\sqrt{2A})^{n-2k} \right) dt $$

$$ = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (\alpha+1)_k}{k!(n-2k)!(\beta+1)_k} (x\sqrt{2A})^{n-2k} \int_0^\infty \exp(-t) t^{n-k} dt. $$

By using Gamma function, we can write

$$ \frac{1}{n!}\int_0^\infty e^{-t} t^{\frac{n}{2}} \; {}_H H_n(x\sqrt{t},A)dt = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (n-k)! (\alpha+1)_k}{k!(n-2k)!(\beta+1)_k} (x\sqrt{2A})^{n-2k}. $$

Therefore, the Chebyshev-Chebyshev matrix polynomials of the second kind are given as

$$ {}_U U_n(x,A) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (n-k)! (\alpha+1)_k}{k!(n-2k)!(\beta+1)_k} (x\sqrt{2A})^{n-2k} \tag{3.7} $$

or

$$ {}_U U_n(x,A) = \frac{1}{n!}\int_0^\infty e^{-t} t^{\frac{n}{2}} \; {}_H H_n(x\sqrt{t},A)dt \tag{3.8} $$

via the integral representation for Chebyshev-Chebyshev matrix polynomials of the second kind. Here the Chebyshev matrix polynomials of the second kind in [1] are given as

$$U_n(x,A)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(n-k)!}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}.$$

Now, we give the Chebyshev-Chebyshev matrix polynomials of the first kind as

$$_{T}T_n(x,A)=n(\sqrt{2A})^{-1}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(n-k-1)!(\alpha+1)_k}{k!(n-2k)!(\beta+1)_k}(x\sqrt{2A})^{n-2k} \quad (3.9)$$

or

$$_{T}T_n(x,A)=\frac{1}{(n-1)!}(\sqrt{2A})^{-1}\int_0^{\infty}e^{-t}t^{\frac{n}{2}-1}\ {}_{H}H_n(x\sqrt{t},A)dt; n\geq 1,\ {}_{T}T_0(x,A)=\theta \quad (3.10)$$

via the integral transform of the generalized HHMP. Here the Chebyshev matrix polynomials of the first kind in [6] are given as

$$T_n(x,A)=n(\sqrt{2A})^{-1}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(n-k-1)!}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}.$$

Thus, the following theorem can be written.

Theorem 3.2. *The integral expressions (3.7), (3.8), (3.9) and (3.10) hold true for a positive stable A in* $\mathbb{C}^{N\times N}$.

In a similar manner, we can now generalize the above Chebyshev-Chebyshev matrix polynomials.

For positive stable matrices A and B in $\mathbb{C}^{N\times N}$ and $AB=BA$, then we define two new Chebyshev-Chebyshev-type matrix polynomials:

$$_{U}U_n(x,A,B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k B^{k-n-1}(n-k)!(\alpha+1)_k}{k!(n-2k)!(\beta+1)_k}(x\sqrt{2A})^{n-2k} \quad (3.11)$$

or

$$_{U}U_n(x,A,B)=\frac{1}{n!}\int_0^{\infty}e^{-Bt}t^{\frac{n}{2}}\ {}_{H}H_n(x\sqrt{t},A)dt \quad (3.12)$$

and

$$_{T}T_n(x,A,B)=n(\sqrt{2A})^{-1}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k B^{k-n}(n-k-1)!(\alpha+1)_k}{k!(n-2k)!(\beta+1)_k}(x\sqrt{2A})^{n-2k} \quad (3.13)$$

or

$$_{T}T_n(x,A,B)=\frac{1}{(n-1)!}(\sqrt{2A})^{-1}\int_0^{\infty}e^{-Bt}t^{\frac{n}{2}-1}\ {}_{H}H_n(x\sqrt{t},A)dt;\quad n\geq 1. \quad (3.14)$$

Taking $\alpha=\beta$ in (3.3), (3.7) and (3.9), the generalized Legendre and Chebyshev matrix polynomials reduce to the special case of the Legendre and Chebyshev matrix polynomials (see [1, 6, 20]).

In the next section, we have shown that new integral representations are a fairly useful tool to obtain new families of matrix polynomials.

4. Connections between Hermite-Hermite, Chebyshev-Hermite and Legendre-Hermite matrix polynomials

In this part, we generate Chebyshev-Hermite and Legendre-Hermite matrix polynomials by using properties in previous section.

The generalized HHMP of two variables for a positive stable A in $\mathbb{C}^{N\times N}$ and $x, y \in \mathbb{R}$ are defined as follows:

$$ {}_H H_n(x,y,A) = n!\sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k}{k!(n-2k)!(1+\beta)_k} y^k H_{n-2k}(x,A). \tag{4.1}$$

Now, a version of Legendre-Hermite matrix polynomials will be defined via the generalized HHMP of two variables. By using (4.1) and (2.1), it follows that

$$\begin{aligned}
\frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \, {}_H H_n\left(xt, \frac{1}{t^2}, A\right) dt &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k t^{-2k} H_{n-2k}(xt,A)}{k!(n-2k)!(1+\beta)_k} dt \\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k}{k!(n-2k)!(1+\beta)_k} \int_0^\infty e^{-t^2} t^{n-2k} H_{n-2k}(xt,A) dt \\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k}{k!(n-2k)!(1+\beta)_k} \left((n-2k)! \sum_{s=0}^{[\frac{1}{2}(n-2k)]} \frac{(-1)^s (x\sqrt{2A})^{n-2k-2s}}{s!(n-2k-2s)!} \int_0^\infty e^{-t^2} t^{2n-4k-2s} dt \right).
\end{aligned}$$

The series and the integral can be transputed since the summation in right is finite. Using Gamma function, we see

$$\int_0^\infty e^{-t^2} t^{2n-4k-2s} dt = \frac{1}{2}\Gamma\left(n-2k-s+\frac{1}{2}\right) = \frac{1}{2} \frac{\sqrt{\pi}\,(2n-4k-2s)!}{2^{2n-4k-2s}(n-2k-s)!}. \tag{4.2}$$

Then we have

$$\begin{aligned}
&\frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \, {}_H H_n\left(xt, \frac{1}{t^2}, A\right) dt \\
&= \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k}{k!(1+\beta)_k} \left(\sum_{s=0}^{[\frac{1}{2}(n-2k)]} \frac{(-1)^s (x\sqrt{2A})^{n-2k-2s}}{s!(n-2k-2s)!} \frac{(2n-4k-2s)!}{2^{2n-4k-2s}(n-2k-s)!} \right) \\
&= \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\beta-\alpha)_k P_{n-2k}(x,A)}{k!\,(1+\beta)_k} = {}_P H_n(x,A).
\end{aligned}$$

Thus, the Legendre-Hermite matrix polynomials are defined by

$$ {}_{P}H_n(x,A)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k P_{n-2k}(x,A)}{k!(1+\beta)_k} \tag{4.3} $$

or

$$ {}_{P}H_n(x,A) \quad = \quad \frac{2}{n!\sqrt{\pi}}\int_0^{\infty}e^{-t^2}t^n \ {}_{H}H_n\left(xt,\frac{1}{t^2},A\right)dt \tag{4.4} $$

via the generalized HHMP of two variables.

Thus, the following theorem can be written.

Theorem 4.1. *The integral expressions (4.3) and (4.4) hold true for a positive stable A in $\mathbb{C}^{N\times N}$.*

Now, we can also give generalized Legendre-Hermite-type matrix polynomials by using their integral representation.

Let A, B, AB^{-1} be positive stable in $\mathbb{C}^{N\times N}$, and $AB=BA$. Then we give generalized Legendre-Hermite-type matrix polynomials:

$$ {}_{P}H_n(x,A,B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{B^{k-\frac{n}{2}-\frac{1}{2}}(\beta-\alpha)_k P_{n-2k}(x,AB^{-1})}{k!(1+\beta)_k} \tag{4.5} $$

or

$$ {}_{P}H_n(x,A,B) \quad = \quad \frac{2}{n!\sqrt{\pi}}\int_0^{\infty}e^{-Bt^2}t^n \ {}_{H}H_n\left(xt,\frac{1}{t^2},A\right)dt. \tag{4.6} $$

Now, we define a version of Chebyshev-Hermite matrix polynomials via explict formula for the generalized HHMP of two variables. By (4.1) and (2.1), it follows that

$$ \frac{1}{n!}\int_0^{\infty}e^{-t}t^{\frac{n}{2}} \ {}_{H}H_n\left(x\sqrt{t},\frac{1}{t},A\right)dt=\int_0^{\infty}\exp(-t)t^{\frac{n}{2}}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k}{k!(n-2k)!(1+\beta)_k}t^{-k}\ H_{n-2k}(x\sqrt{t},A)dt $$

$$ =\sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k}{k!(1+\beta)_k}\left(\sum_{s=0}^{[\frac{1}{2}(n-2k)]}\frac{(-1)^s(x\sqrt{2A})^{n-2k-2s}}{s!(n-2k-2s)!}\int_0^{\infty}e^{-t}t^{n-2k-s}dt\right). $$

Using Gamma function, we can write

$$ \frac{1}{n!}\int_0^{\infty} \quad e^{-t}t^{\frac{n}{2}} \ {}_{H}H_n\left(x\sqrt{t},\frac{1}{t},A\right)dt=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k}{k!(1+\beta)_k}\left(\sum_{s=0}^{[\frac{1}{2}(n-2k)]}\frac{(-1)^s(n-2k-s)!(x\sqrt{2A})^{n-2k-2s}}{s!(n-2k-2s)!}\right) $$

$$ = \qquad \sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k}{k!(1+\beta)_k}U_{n-2k}(x,A). $$

Thus, the Chebyshev-Hermite matrix polynomials of the second kind is given as

$$ {}_U H_n(x,A)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k U_{n-2k}(x,A)}{k!(1+\beta)_k} \tag{4.7}$$

or

$$ {}_U H_n(x,A)=\frac{1}{n!}\int_0^\infty e^{-t}t^{\frac{n}{2}}\ {}_H H_n\left(x\sqrt{t},\frac{1}{t},A\right)dt. \tag{4.8}$$

In a similar manner, we have a definition of the Chebyshev-Hermite matrix polynomials of the first kind:

$$ {}_T H_n(x,A)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(\beta-\alpha)_k T_{n-2k}(x,A)}{k!(1+\beta)_k} \tag{4.9}$$

or

$$ {}_T H_n(x,A)=\frac{1}{(n-1)!}(\sqrt{2A})^{-1}\int_0^\infty e^{-t}t^{\frac{n}{2}-1}\ {}_H H_n\left(x\sqrt{t},\frac{1}{t},A\right)dt,\quad n\ge 1,\ {}_T H_0(x,A)=\theta. \tag{4.10}$$

Thus, the following theorem can be written.

Theorem 4.2. *The integral expressions (4.7), (4.8), (4.9) and (4.10) hold true for a positive stable A in* $\mathbb{C}^{N\times N}$.

In a similar manner, we can now generalize the above Chebyshev-Hermite matrix polynomials.

If A, B, AB^{-1} are positive stable in $\mathbb{C}^{N\times N}$ and $AB=BA$, we have two new Chebyshev-Hermite-type matrix polynomials:

$$ {}_U H_n(x,A,B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{B^{k-\frac{n}{2}-1}(\beta-\alpha)_k U_{n-2k}(x,AB^{-1})}{k!(1+\beta)_k} \tag{4.11}$$

or

$$ {}_U H_n(x,A,B)=\frac{1}{n!}\int_0^\infty e^{-Bt}t^{\frac{n}{2}}\ {}_H H_n\left(x\sqrt{t},\frac{1}{t},A\right)dt \tag{4.12}$$

and

$$ {}_T H_n(x,A,B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{B^{k-\frac{n}{2}}(\beta-\alpha)_k T_{n-2k}(x,AB^{-1})}{k!(1+\beta)_k} \tag{4.13}$$

or

$$ {}_T H_n(x,A,B)=\frac{1}{(n-1)!}(\sqrt{2A})^{-1}\int_0^\infty e^{-Bt}t^{\frac{n}{2}-1}\ {}_H H_n\left(x\sqrt{t},\frac{1}{t},A\right)dt\ ;\ n\ge 1. \tag{4.14}$$

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