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COMMON SOLUTIONS OF A SYSTEM OF VARIATIONAL INEQUALITY PROBLEMS

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In this paper, we introduce and study a new iterative scheme for a family of unrelated variational inequalities. The scheme is based on viscosity method. We obtain strong convergence of the proposed algorithm to common solutions to an infinite countable family of variational inequalities in a real Hilbert spaces. The results obtained in this paper extend and improve some recent known results.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be a nonlinear operator. It is well known that the *Variational Inequality Problem* (VIP) is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \qquad \forall v \in C.$$

The theory of variational inequalities has played an important role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, partial differential equation, operations research and engineering sciences. During the last decades this problem has been studied by many authors, (see [1-15]).

Recently, Censor, Gibali and Reich [16, 17] (see also [18]) introduced the *Common Solutions to Variational Inequality Problem (CSVIP)* which consists of finding common solutions to unrelated variational inequality. The general form of the *CSVIP* is the following:

Let H be a Hilbert space. Let there be given, for each $i = 1, 2, ..., N$, an operator $h_i : H \longrightarrow H$ and a nonempty, closed and convex subset $C_i \subset H$, with $\bigcap_{i=1}^N C_i \neq \emptyset$. The CSVIP *(for single-valued operators) is to find a point $z \in \bigcap_{i=1}^N C_i$ such that, for each i=1,2,...,N,*

$$\langle h_i z, x - z\rangle \geq 0, \qquad \forall x \in C_i, \quad 1 \leq i \leq N. \tag{1.1}$$

For $1 \leq i \leq N$, we denote by $SOL(C_i, h_i)$ the solution set of (1). We note that in *CSVIP*, if we choose all $h_i = 0$, then the problem reduces to that of finding a point $z \in \bigcap_{i=1}^N C_i$ in the nonempty intersection of a finite family of closed and convex set, which

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is the well known *Convex Feasibility Problem (CFP)*. Note that the CFP has received a lot of attention due to its extensive applications in many applied disciplines as diverse as approximation theory, image recovery and signal processing, control theory, biomedical engineering, communications, and geophysics (see [19, 20] and the references therein).

Definition 1.1. Let $h : H \longrightarrow H$ be an operator and let $C \subset H$. The operator h is called

(i) Lipschitz continuous on $C \subset H$ with constant $L > 0$ if

$$\|h(x) - h(y)\| \leq L\|x - y\|, \qquad \forall x, y \in C.$$

(ii) nonexpansive on C if it is Lipschitz continuous with constant $L = 1$.

(iii) a $k-$contraction if it is Lipschitz continuous with constant $k < 1$.

(iv) inverse strongly monotone with constant $\beta > 0$, $(\beta - ism)$ if

$$\langle h(x) - h(y), x - y\rangle \geq \beta\|h(x) - h(y)\|^2, \quad \forall x, y \in C.$$

Recently, Censor, Gibali and Reich [17], proved the following weak convergence theorem for solving the finite-set $CSVIP$ in a real Hilbert space.

Theorem 1.2. *Let H be a Hilbert space. For each $1 \leq i \leq N$, let an operator $h_i : H \longrightarrow H$ and a nonempty, closed and convex subset $C_i \subset H$ be given. Assume that $\bigcap_{i=1}^N C_i \neq \emptyset$, and $\Psi = \bigcap_{i=1}^N SOL(C_i, h_i) \neq \emptyset$ and that for each $1 \leq i \leq N$, h_i is α_i- ism. Set $\alpha := min_i\{\alpha_i\}$ and take $\lambda \in (0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \sum_{i=1}^N w_{n,i}(P_{C_i}(I - \lambda h_i)x_n), \qquad n \geq 0,$$

where $\sum_{i=1}^N w_{n,i} = 1$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in \Psi$, where $z = \lim_{n\longrightarrow\infty} P_\Psi x_n$.

In an infinite dimensional Hilbert space, this algorithm does not in general, have strong convergence. This bring us a natural question how to modify this algorithm so that strongly convergent sequence is guaranteed. In this paper, using the viscosity approximation method, we introduce a new iterative process for an infinite family-set $CSVIP$. Moreover, we establish strong convergence of the proposed algorithm to finding a common solution of an infinite countable family of variational inequalities in a Hilbert space.

2. Preliminaries

A bounded linear operator A on H is called strongly positive if there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \qquad (x \in H).$$

For a nonexpansive mapping T from a nonempty subset C of H into itself a typical problem is to minimize the quadratic function

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{2.1}$$

over the set of all fixed points $Fix(T)$ of T (see [21, 22]).

Lemma 2.1. *([23]) Let H be a Hilbert space and $\{x_n\}$ be a bounded sequence in H. Then for any given $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integer i, j with $i < j$,*

$$\|\sum_{n=1}^{\infty} \lambda_n x_n\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2.$$

Lemma 2.2. *([21]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \eta_n) a_n + \eta_n \delta_n, \qquad n \geq 0,$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$,
(ii) $\limsup_{n \longrightarrow \infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\eta_n \delta_n| < \infty$.

Then $\lim_{n \longrightarrow \infty} a_n = 0$.

Lemma 2.3. *([24]) Let $\{u_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which satisfies $u_{n_i} < u_{n_i+1}$ for all $i \geq 0$. Also consider the sequence of integers $\{\tau(n)\}$ defined by*

$$\tau(n) = \max\{k \leq n : u_k < u_{k+1}\}.$$

Then $\tau(n)$ is a nondecreasing sequence verifying $\lim_{n \longrightarrow \infty} \tau(n) \longrightarrow \infty$ and for all sufficiently large number n, it holds that $u_{\tau(n)} \leq u_{\tau(n)+1}$ and we have $u_n \leq u_{\tau(n)+1}$.

Definition 2.4. Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \qquad \forall y \in C.$$

P_C is called the metric projection of H onto C.

Lemma 2.5. *([18]) Let $C \subset H$ be a nonempty, closed and convex subset and let $h : H \longrightarrow H$ be an α-ism operator on H. Then for each $\lambda \in (0, 2\alpha)$,*

(i) *$P_C(I - \lambda h)$ is nonexpansive;*
(ii)

$$x^\star \in SOL(C, h) \iff x^\star \in Fix(P_C(I - \lambda h)).$$

Since the fixed point set of nonexpansive operators is closed and convex, the projection onto the solution set $SOL(C, h)$ is well defined whenever $SOL(C, h) \neq \emptyset$.

3. Main Result

In this section, we introduce a general algorithm for infinite family-set $CSVIP$. Then, we establish the strong convergence of the proposed algorithm to finding a common solution of an infinite countable family of variational inequalities in a Hilbert space.

Theorem 3.1. *Let H be a Hilbert space. Let C_i be a family of nonempty, closed and convex subset of H and $h_i : H \longrightarrow H$ be a family of operators. Assume that $\Psi = \bigcap_{i=1}^{\infty} SOL(C_i, h_i) \neq \emptyset$ and that for each $i \in \mathbb{N}$, h_i is θ_i- ism. Suppose that $\theta = inf_i\, \theta_i > 0$*

and take $\lambda \in (0, 2\theta)$. Assume that g be a k- contraction of H into itself and A be a self-adjoint strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^{\infty} \beta_{n,i} P_{C_i}(I - \lambda h_i)x_n, & n \geq 0, \\ x_{n+1} = \eta_n \gamma g(x_n) + (I - \eta_n A)y_n, & \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\alpha_n + \sum_{i=1}^{\infty} \beta_{n,i} = 1$ and $\{\alpha_n\}, \{\beta_{n,i}\}$ and $\{\eta_n\}$ satisfy the following conditions:

(i) $\{\eta_n\} \subset (0, 1)$, $\lim_{n \longrightarrow \infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0, 1)$ *and* $\liminf_{n \longrightarrow \infty} \alpha_n \beta_{n,i} > 0$, *for all* $i \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Psi$, which solves the variational inequality ;

$$\langle (A - \gamma g)z, x - z \rangle \geq 0, \qquad \forall x \in \Psi. \tag{3.2}$$

Proof. First, we show that there exists a unique $z \in \Psi$ such that $z = P_\Psi(I - A + \gamma g)z$. Indeed, since $\Psi = \bigcap_{i=1}^{\infty} SOL(C_i, h_i)$ is closed and convex, (see Lemma 2.5), we have that the projection P_Ψ is well defined. Now, let $Q = P_\Psi$, we show that $Q(I - A + \gamma g)$ is a contraction of H into itself. In fact,

$$\begin{aligned} \|Q(I - A + \gamma g)(x) - Q(I - A + \gamma g)(y)\| &\leq \|(I - A + \gamma g)(x) - (I - A + \gamma g)(y)\| \\ &\leq \|(I - A)x - (I - A)y\| + \gamma\|gx - gy\| \\ &\leq (1 - \overline{\gamma})\|x - y\| + \gamma k\|x - y\| \\ &\leq (1 - (\overline{\gamma} - \gamma k))\|x - y\|. \end{aligned}$$

Thus there exists a unique element $z \in \Psi$ such that $z = P_\Psi(I - A + \gamma g)z$. Since $\lim_{n \longrightarrow \infty} \eta_n = 0$, we may assume that $\eta_n \in (0, \|A\|^{-1})$ for all $n \geq 0$. Also we have $\|I - \eta_n A\| \leq 1 - \eta_n \overline{\gamma}$ (see [22] for details). Next, we show that $\{x_n\}$ is bounded. Since $z \in \Psi$ we have $P_{C_i}(I - \lambda h_i)z = z$. By Lemma 2.5, the operators $P_{C_i}(I - \lambda h_i)$ are nonexpansive and hence we have that

$$\begin{aligned} \|y_n - z\| &\leq \|\alpha_n x_n + \textstyle\sum_{i=1}^{\infty} \beta_{n,i} P_{C_i}(I - \lambda h_i)x_n - z\| \\ &\leq \alpha_n\|x_n - z\| + \textstyle\sum_{i=1}^{\infty} \beta_{n,i}\|P_{C_i}(I - \lambda h_i)x_n - P_{C_i}(I - \lambda h_i)z\| \\ &\leq \alpha_n\|x_n - z\| + \textstyle\sum_{i=1}^{\infty} \beta_{n,i}\|x_n - z\| \\ &\leq \|x_n - z\|. \end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned} \|x_{n+1} - z\| &= \|\eta_n(\gamma g(x_n) - Az) + ((I - \eta_n A)(y_n - z)\| \\ &\leq \eta_n\|\gamma g(x_n) - Az\| + \|I - \eta_n A\|\|y_n - z\| \\ &\leq \eta_n \gamma\|g(x_n) - g(z)\| + \eta_n\|\gamma g(z) - Az\| + (1 - \eta_n\overline{\gamma})\|x_n - z\| \\ &\leq \eta_n \gamma k\|x_n - z\| + \eta_n\|\gamma g(z) - Az\| + (1 - \eta_n\overline{\gamma})\|x_n - z\| \\ &\leq (1 - \eta_n(\overline{\gamma} - \gamma k))\|x_n - z\| + \eta_n\|\gamma g(z) - Az\| \\ &= (1 - \eta_n(\overline{\gamma} - \gamma k))\|x_n - z\| + \eta_n(\overline{\gamma} - \gamma k)\tfrac{1}{\overline{\gamma} - \gamma k}\|\gamma g(z) - Az\| \\ &\leq max\{\|x_n - z\|, \tfrac{1}{\overline{\gamma} - \gamma k}\|\gamma g(z) - Az\|\}. \end{aligned}$$

It follows by induction that

$$\|x_n - z\| \leq max\{\|x_0 - z\|, \frac{1}{\overline{\gamma} - \gamma k}\|\gamma g(z) - Az\|\}, \qquad \forall n \geq 0.$$

This shows that $\{x_n\}$ is bounded and so is $\{g(x_n)\}$. Next, we show that for each $i \in \mathbb{N}$,

$$\lim_{n \longrightarrow \infty} \|x_n - P_{C_i}(I - \lambda h_i)x_n\| = 0.$$

By using Lemma 2.1, for every $i \in \mathbb{N}$ we have that

$$\begin{aligned} &\|y_n - z\|^2 \leq \|\alpha_n x_n + \textstyle\sum_{i=1}^{\infty} \beta_{n,i} P_{C_i}(I - \lambda h_i)x_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + \textstyle\sum_{i=1}^{\infty} \beta_{n,i} \|P_{C_i}(I - \theta_i h_i)x_n - z\|^2 - \alpha_n \beta_{n,i} \|P_{C_i}(I - \lambda h_i)x_n - x_n\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + \textstyle\sum_{i=1}^{\infty} \beta_{n,i} \|x_n - z\|^2 - \alpha_n \beta_{n,i} \|P_{C_i}(I - \lambda h_i)x_n - x_n\|^2 \\ &\leq \|x_n - z\|^2 - \alpha_n \beta_{n,i} \|P_{C_i}(I - \lambda h_i)x_n - x_n\|^2. \end{aligned} \tag{3.4}$$

Hence we have from (3.4) that

$$\begin{aligned} &\|x_{n+1} - z\|^2 = \|\eta_n(\gamma g(x_n) - Az) + (I - \eta_n A)(y_n - z)\|^2 \\ &\leq \eta_n^2 \|\gamma g(x_n) - Az\|^2 + (1 - \eta_n \overline{\gamma})^2 \|y_n - z\|^2 + 2\eta_n(1 - \eta_n \overline{\gamma})\|\gamma g(x_n) - Az\|\|y_n - z\| \\ &leq \eta_n^2 \|\gamma g(x_n) - Az\|^2 + (1 - \eta_n \overline{\gamma})^2 \|x_n - z\|^2 + 2\eta_n(1 - \eta_n \overline{\gamma})\|\gamma g(x_n) - Az\|\|x_n - z\| \\ &\quad -(1 - \eta_n \overline{\gamma})^2 \alpha_n \beta_{n,i} \|P_{C_i}(I - \lambda h_i)x_n - x_n\|^2. \end{aligned} \tag{3.5}$$

By (3.5), we have that

$$\begin{aligned} &(1 - \eta_n \overline{\gamma})^2 \alpha_n \beta_{n,i} \|P_{C_i}(I - \lambda h_i)x_n - x_n\|^2 \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\eta_n(1 - \eta_n \overline{\gamma})\|\gamma g(x_n) - Az\|\|x_n - z\| + \eta_n^2 \|\gamma g(x_n) - Az\|^2. \end{aligned} \tag{3.6}$$

Case A: Put $\Gamma_n = \|x_n - z\|$ for all $n \in \mathbb{N}$. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \in \mathbb{N}$. In this case $\lim_{n \longrightarrow \infty} \Gamma_n$ exists. Since $\lim_{n \longrightarrow \infty} \eta_n = 0$, and $\{g(x_n)\}$ and $\{x_n\}$ are bounded, from (3.6) we have

$$\lim_{n \longrightarrow \infty} (1 - \eta_n \overline{\gamma})^2 \alpha_n \beta_{n,i} \|P_{C_i}(I - \lambda h_i)x_n - x_n\|^2 = 0,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|P_{C_i}(I - \lambda h_i)x_n - x_n\| = 0.$$

Next we show that $\limsup_{n \longrightarrow \infty} \langle (A - \gamma g)z, z - x_n \rangle \leq 0$. We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i \longrightarrow \infty} (\langle A - \gamma g)z, z - x_{n_i} \rangle = \limsup_{n \longrightarrow \infty} (\langle A - \gamma g)z, z - x_n \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to x^*. Without loss of generality, we may assume that x_{n_i} converges weakly to x^*. Since for each $i \in \mathbb{N}$, the operators $P_{C_i}(I - \lambda h_i)$ are nonexpansive and $\lim_{n \longrightarrow \infty} \|P_{C_i}(I - \lambda h_i)x_n - x_n\| = 0$, the demiclosed principle implies that $x^* \in \Psi$. Hence, from $z = P_\Psi(I - A + \gamma g)z$ and $x^* \in \Psi$, it follows that

$$\limsup_{n \longrightarrow \infty} \langle (A - \gamma g)z, z - x_n \rangle = \lim_{i \longrightarrow \infty} (\langle A - \gamma g)z, z - x_{n_i} \rangle = (\langle A - \gamma g)z, z - x^* \rangle \leq 0.$$

On the other hand, since

$$x_{n+1} - z = \eta_n(\gamma g(x_n) - Az) + (I - \eta_n A)(y_n - z).$$

It is known that in a Hilbert space H,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \qquad \forall x, y \in H.$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \le \|(I-\eta_n A)(y_n - z)\|^2 + 2\eta_n\langle \gamma g(x_n) - Az, x_{n+1}-z\rangle \\
&\le (1-\eta_n\overline{\gamma})^2\|x_n - z\|^2 + 2\eta_n\gamma\langle g(x_n) - g(z), x_{n+1}-z\rangle + 2\eta_n\langle \gamma g(z) - Az, x_{n+1}-z\rangle \\
&\le (1-\eta_n\overline{\gamma})^2\|x_n - z\|^2 + 2\eta_n k\gamma\|x_n - z\|\|x_{n+1}-z\| + 2\eta_n\langle \gamma g(z) - Az, x_{n+1}-z\rangle \\
&\le (1-\eta_n\overline{\gamma})^2\|x_n - z\|^2 + \eta_n k\gamma(\|x_n - z\|^2 + \|x_{n+1}-z\|^2) + 2\eta_n\langle \gamma g z - Az, x_{n+1}-z\rangle \\
&\le ((1-\eta_n\overline{\gamma})^2 + \eta_n k\gamma)\|x_n - z\|^2 + \eta_n\gamma k\|x_{n+1}-z\|^2 + 2\eta_n\langle \gamma g(z) - Az, x_{n+1}-z\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \tfrac{1-2\eta_n\overline{\gamma}+(\eta_n\overline{\gamma})^2+\eta_n\gamma k}{1-\eta_n\gamma k}\|x_n - z\|^2 + \tfrac{2\eta_n}{1-\eta_n\gamma k}\langle \gamma g z - Az, x_{n+1}-z\rangle \\
&= (1 - \tfrac{2(\overline{\gamma}-\gamma g)\eta_n}{1-\eta_n\gamma g})\|x_n - z\|^2 + \tfrac{(\eta_n\overline{\gamma})^2}{1-\eta_n\gamma g}\|x_n - z\|^2 + \tfrac{2\eta_n}{1-\eta_n\gamma k}\langle \gamma g z - Az, x_{n+1}-z\rangle \\
&\le (1 - \tfrac{2(\overline{\gamma}-\gamma k)\eta_n}{1-\eta_n\gamma k})\|x_n - z\|^2 + \tfrac{2(\overline{\gamma}-\gamma k)\eta_n}{1-\eta_n\gamma k}(\tfrac{(\eta_n\overline{\gamma}^2)M}{2(\overline{\gamma}-\gamma k)} + \tfrac{1}{\overline{\gamma}-\gamma k})\langle \gamma g z - Az, x_{n+1}-z\rangle) \\
&= (1-\gamma_n)\|x_n - z\|^2 + \gamma_n\delta_n,
\end{aligned}$$

where $M = sup\{\|x_n - z\|^2 : n \ge 0\}$, $\gamma_n = \frac{2(\overline{\gamma}-\gamma k)\eta_n}{1-\eta_n\gamma k}$ and

$$\delta_n = \frac{(\eta_n\overline{\gamma}^2)M}{2(\overline{\gamma}-\gamma k)} + \frac{1}{\overline{\gamma}-\gamma k}\langle \gamma g z - Az, x_{n+1}-z\rangle).$$

It is easy to see that $\gamma_n \longrightarrow 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\longrightarrow\infty}\delta_n \le 0$. Now, by Lemma 2.2, we conclude that the sequence $\{x_n\}$ converges strongly to z.

Case B: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$\tau(n) = max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.3 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. So we have from (3.6) that

$$\begin{aligned}
&(1-\eta_{\tau(n)}\overline{\gamma})^2\alpha_{\tau(n)}\beta_{\tau(n),i}\|P_{C_i}(I-\lambda h_i)x_{\tau(n)} - x_{\tau(n)}\|^2 \\
&\qquad \le 2\eta_{\tau(n)}(1-\eta_{\tau(n)}\overline{\gamma})\|\gamma g(x_{\tau(n)}) - Az\|\|x_{\tau(n)} - z\| \\
&\qquad + \eta_{\tau(n)}^2\|\gamma g(x_{\tau(n)}) - Az\|^2 + \|x_{\tau(n)} - z\|^2 - \|x_{\tau(n)+1} - z\|^2 \\
&\qquad \le 2\eta_{\tau(n)}(1-\eta_{\tau(n)}\overline{\gamma})\|\gamma g(x_{\tau(n)}) - Az\|\|x_{\tau(n)} - z\| + \eta_{\tau(n)}^2\|\gamma g(x_{\tau(n)}) - Az\|^2. \qquad (3.7)
\end{aligned}$$

By our assumption that $\lim_{n\longrightarrow\infty}\eta_n = 0$, we get that

$$\lim_{n\longrightarrow\infty}\|P_{C_i}(I-\lambda h_i)x_{\tau(n)} - x_{\tau(n)}\| = 0.$$

Following the same argument as the proof of *Case A* for $\{x_{\tau(n)}\}$ we have that

$$\|x_{\tau(n)+1} - z\|^2 \le (1-\eta_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \eta_{\tau(n)}\delta_{\tau(n)},$$

where $\eta_{\tau(n)} \longrightarrow 0, \sum_{n=1}^{\infty}\eta_{\tau(n)} = \infty$ and $\limsup_{n\longrightarrow\infty}\delta_{\tau(n)} \le 0$. Hence, by Lemma 2.2, we obtain $\lim_{n\longrightarrow\infty}\|x_{\tau(n)} - z\| = 0$ and $\lim_{n\longrightarrow\infty}\|x_{\tau(n)+1} - z\| = 0$. Thus by Lemma 2.3 we have

$$0 \le \|x_n - z\| \le \max\{\|x_{\tau(n)} - z\|, \|x_n - z\|\} \le \|x_{\tau(n)+1} - z\|.$$

Therefore $\{x_n\}$ converges strongly to $z = P_\Psi(I - A + \gamma g)z$. $\square$

Putting $A = I$ and $\gamma = 1$ in Theorem 3.1, for a finite-set *CSVIP* we obtain the following result.

Theorem 3.2. *Let H be a Hilbert space. Let C_i be a family of nonempty, closed and convex subset of H and $h_i : H \longrightarrow H$ be a family of operators. Assume that $\Psi = \bigcap_{i=1}^{N} SOL(C_i, h_i) \neq \emptyset$ and that for each $i = 1, 2, ..., N$, h_i is θ_i- ism. Set $\theta = min_i \theta_i$ and take $\lambda \in (0, 2\theta)$. Assume that g be a k- contraction of H into itself. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and*

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^{N} \beta_{n,i} P_{C_i}(I - \lambda h_i)x_n, & n \geq 0, \\ x_{n+1} = \eta_n g(x_n) + (1 - \eta_n)y_n, & \forall n \geq 0, \end{cases} \tag{3.8}$$

where $\alpha_n + \sum_{i=1}^{\infty} \beta_{n,i} = 1$ and $\{\alpha_n\}, \{\beta_{n,i}\}$ and $\{\eta_n\}$ satisfy the following conditions:

(i) $\{\eta_n\} \subset (0, 1)$, $\lim_{n\longrightarrow\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0, 1)$ *and* $\liminf_{n\longrightarrow\infty} \alpha_n \beta_{n,i} > 0$, *for all* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Psi$, which solves the variational inequality ;

$$\langle z - gz, x - z\rangle \geq 0, \qquad \forall x \in \Psi. \tag{3.9}$$

Now we consider a new iterative scheme. By using similar argument as in the proof of Theorem 3.1 we can prove the following theorem.

Theorem 3.3. *Let H be a Hilbert space. Let C_i be a family of nonempty, closed and convex subset of H and $h_i : H \longrightarrow H$ be a family of operators. Assume that $\Psi = \bigcap_{i=1}^{\infty} SOL(C_i, h_i) \neq \emptyset$ and that for each $i \in \mathbb{N}$, h_i is θ_i- ism. Suppose g be a k-contraction of H into itself. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and*

$$x_{n+1} = \alpha_n x_n + \beta_n g(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \theta_i h_i)x_n, \qquad n \geq 0,$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If $\lim_{n\longrightarrow\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, and $\liminf_n \alpha_n \gamma_{n,i} > 0$. Then the sequence $\{x_n\}$ converges strongly to $z \in \Psi$, where $z = P_\Psi g(z)$.

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