NON-UNIFORM HAAR WAVELETS METHOD FOR SOLVING LINEAR STOCHASTIC ITO - VOLTERRA INTEGRAL EQUATIONS

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In this paper, a non-uniform Haar wavelets method is developed to numerically solve stochastic Volterra integral equations. By using collocation points, the non-uniform Haar wavelet coefficients are obtained. Moreover, numerical examples are given to show the accuracy and efficiency of the proposed method.

Keywords: Non-uniform Haar wavelets; Stochastic Volterra integral equations; Collocation points; Brownian motion process

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1. Introduction

Many problems in finance, mechanics, biology, medical, social sciences and other disciplines can be modeled by stochastic integral equations (SIEs). Given the wide range of applications of SIEs, solving these type of equations is a great importance. Clearly, obtaining the analytic solution of SIEs is often either complicated or impossible. Therefore, the development of numerical methods for solving these types of equations is inevitable. Hence, many authors have proposed several numerical approaches for solving these equations. In [11], authors applied triangular functions for solving SIEs. Asgari et al. suggested stochastic operational matrix based on Bernstein polynomials for obtaining numerical solution of nonlinear SIEs [12]. Cheraghi et al. [2], used new basis functions for solving linear stochastic Volterra integral equations. Authors in [1] used stochastic operational matrix based on Haar wavelets for obtaining numerical solution of nonlinear SIEs. To see another methods for solving SIEs, one can refer to [3, 5, 6, 8, 10, 13, 14, 15, 16].

The orthogonal basis function such as uniform Haar functions and non-uniform Haar functions are used to estimate the solution of SIEs, that by using these orthogonal functions the SIEs reduced to a linear or nonlinear system of algebraic equations which can be solved by using known methods. Some of advantage of applying wavelet functions is their efficiency and simple applicability. The conventional form of the uniform Haar wavelet approach is applicable for the range of the argument $x \in [0, 1]$, besides it is assumed that this interval is distributed into subintervals of equal length. If we want to raise the exactness of the results, we must increase the number of the grid points. In the course of the solution, we have to invert some matrices, but by increasing the number of calculation points these matrices become nearly singular and therefore the inverse matrices cannot be evaluated with

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necessary accuracy. One possibility to find a way out of these difficulties is to make use of the non-uniform Haar method for which the length of the subintervals is unequal. This idea was proposed in [4].
In this paper, non-uniform Haar functions will be used to solve the following linear stochastic Volterra integral equation

$$Y(x) = Y_0(x) + \lambda_1 \int_0^x f_1(s,x)Y(s)ds + \lambda_2 \int_0^x f_2(s,x)Y(s)dB(s), \quad (\mathbf{1}.1)$$

where $Y_0(x)$, $f_1(s,x)$ and $f_2(s,x)$ for $s,x \in [0,1)$ are given stochastic processes defined on the same probability space (Ω, F, P), λ_1, λ_2 are known parameters, and $Y(x)$ is the unknown. Additionally, $B(x)$ and $\int_0^x f_2(s,x)Y(s)dB(s)$ are a Brownian motion process and the Ito integral, respectively. We first describe non-uniform Haar wavelets and their properties. Then by using non-uniform Haar wavelet, we offer a numerical method for approximate solution of SIEs. Error analysis and convergence of the proposed method are also investigated. Illustrative examples are included to demonstrate the applicability and validity of the technique.
This article is organized as follows: In Section 2 basic properties of the non-uniform Haar wavelets are described. In Section 3 function approximation is described. In Section 4 a new computational method is proposed for solving stochastic Volterra integral equation (1). Error analysis in Section 5 was given. Also, numerical examples are presented in Section 6. Finally in Section 7 the conclusion is given.

2. Non-uniform Haar wavelets

The following analysis is based on the paper [4]. Non-uniform Haar wavelets are characterized by two numbers: the dilation parameter $j = 0,1,...,J$ (J is maximal level of resolution) and the translation parameter $k = 0,1,...,n-1$, where the integer $n = 2^j$. The number of wavelet is identified as $i = n + k + 1$. Also the maximal value is $i = 2N$, where $N = 2^J$. We divide the interval $[0,1]$ into $2N$ subinterval of unequal lengths with the division points $0 = \tilde{x}(0) < \tilde{x}(1) < ... < \tilde{x}(2N) = 1$. Consider the following non-uniform Haar wavelet family $\{H_i\}_{i\in\mathbb{N}}$

$$H_i(x) = \begin{cases} 1, & \vartheta_1(i) \le x \le \vartheta_2(i), \\ -p_i, & \vartheta_2(i) \le x \le \vartheta_3(i), \\ 0, & elsewhere, \end{cases} \quad (\mathbf{2}.1)$$

where

$$\vartheta_1(i) = \tilde{x}(2k\xi),\ \vartheta_2(i) = \tilde{x}((2k+1)\xi),$$
$$\vartheta_3(i) = \tilde{x}((2k+2)\xi),\ \xi = N/n.$$

With the requirement

$$\int_0^1 H_i(x)dx = 0,$$

we have

$$p_i = \frac{\vartheta_2(i) - \vartheta_1(i)}{\vartheta_3(i) - \vartheta_2(i)}. \quad (\mathbf{2}.2)$$

Clearly, these equations hold when $i > 2$. For the case $i = 1$ and $i = 2$, we have $\vartheta_1(1) = 0$, $\vartheta_2(1) = \vartheta_3(1) = 1$, $\vartheta_1(2) = 0$, $\vartheta_2(2) = \frac{\tilde{x}(2N)}{2}$, $\vartheta_3(2) = 1$. Then, for $i = 1$ the corresponding scaling function in interval $[0,1]$ is :

$$H_1(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & elsewhere. \end{cases}$$

The Haar wavelets are piecewise orthogonal

$$\int_0^1 H_r(x)H_s(x)dx = \begin{cases} 2^{-j}, & r = s, \\ 0, & r \neq s. \end{cases}$$

Similarly, the non-uniform Haar wavelets in $[0,1]$ are piecewise orthogonal

$$\int_0^1 H_i(x)H_j(x)dx = \begin{cases} \delta_i, & i = j, \\ 0, & i \neq j, \end{cases} \tag{2.3}$$

with

$$\delta_i = p_i(\vartheta_3(i) - \vartheta_1(i)). \tag{2.4}$$

3. Function approximation

Each square integrable function $Y(x)$ an $[0,1]$ can be expanded in terms of the non-uniform Haar wavelets as

$$Y(x) = q_1 H_1(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^{j-1}} q_{2^j+k+1} H_{2^j+k+1}(x),\ x \in [0,1], \tag{3.1}$$

with coefficients q_i given by

$$q_1 = \frac{1}{\delta_1}\int_0^1 Y(x)H_1(x)dx,\ q_i = \frac{1}{\delta_i}\int_0^1 Y(x)H_i(x)dx,$$

where $i = 2^j + k + 1$, $j \geq 0$ and $0 \leq k < 2^j$, such that the square error Υ as

$$\Upsilon = \int_0^1 \left(Y(x) - \sum_{i=1}^{n} q_i H_i(x) \right)^2 dx,\ j \in \mathbb{N} \cup \{0\},\ n = 2^j,$$

is minimized. By using Eq. (2), the non-uniform Haar coefficients q_i can be rewritten as

$$q_i = \frac{1}{\delta_i} \left(\int_{\vartheta_1(i)}^{\vartheta_2(i)} Y(x)dx - p_i \int_{\vartheta_2(i)}^{\vartheta_3(i)} Y(x)dx \right),$$

where $j, k = 0, 1, 2, ...,$ and $0 \leq k < 2^j$. Usually the series expansion of Eq. (6) contains infinite terms. If $Y(x)$ is piecewise constant or can be approximated as piecewise constant on each subinterval, then $Y(x)$ will be terminated at n finite terms. This means

$$\begin{aligned} Y(x) &\cong q_1 H_1(x) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} q_{2^j+k+1} H_{2^j+k+1}(x) \\ &= q^T H(x),\ x \in [0,1], \end{aligned}$$

where

$$q^T = [q_1, q_2, q_3, ..., q_{2N}],$$
$$H(x) = [H_1(x), H_2(x), H_3(x), ..., H_{2N}(x)]^T.$$

A two-variable function $f(s,x) \in L^2[0,1) \times L^2[0,1)$ can be approximated with respect to the non-uniform Haar wavelets as

$$f(s,x) \approx H^T(s) F H(x),$$

where $H(x)$ is the non-uniform Haar wavelets vector and $F = f_{i,j}$ is the $n \times n$ non-uniform Haar wavelet coefficients matrix with (i,j)-th element given by

$$f_{i,j} = \frac{1}{{\delta_i}^2} \int_0^1 \int_0^1 f(s,x) H_i(x) H_j(s) dxds,\ i,j = 1, 2, ..., 2N.$$

4. Stochastic non-uniform Haar wavelets

We choose the division points on $[0, L]$ as follows. Let us denote the length of the u-th subinterval by $\Delta x_u = x_u - x_{u-1}, u = 1, 2, ..., 2N$. We assume that $\Delta x_{u+1} = r\Delta x_u$, where $r > 1$ is constant. If we sum the lengths of these subintervals, we have

$$\Delta x_1(1 + r + r^2 + ... + r^{2N-1}) = L,$$

or

$$\Delta x_1 = L\frac{r-1}{r^{2N}-1}.$$

Since

$$\tilde{x}(u) = \Delta x_1 \left(1 + r + ... + r^{u-1}\right) = \Delta x_1 \frac{r^u - 1}{r-1},\ u = 1, 2, ..., 2N, \quad (4.1)$$

we have obtained the grid points as

$$\tilde{x}(u) = L\frac{r^u - 1}{r^{2N}-1},\ u = 1, 2, ..., 2N. \quad (4.2)$$

Let us consider a 8th-order system. By using the grid points defined in Eq. (8), if $r = 2$ the first eight bases non-uniform Haar functions are given by

$$H_1(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & elsewhere, \end{cases} \qquad H_2(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & elsewhere, \end{cases}$$

$$H_3(x) = \begin{cases} 1, & 0 \le x \le \frac{3}{255}, \\ -\frac{1}{4}, & \frac{3}{255} \le x \le \frac{15}{255}, \\ 0, & elsewhere, \end{cases} \qquad H_4(x) = \begin{cases} 1, & \frac{15}{255} \le x \le \frac{63}{255}, \\ -\frac{1}{4}, & \frac{63}{255} \le x \le 1, \\ 0, & elsewhere, \end{cases}$$

$$H_5(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{255}, \\ -\frac{1}{2}, & \frac{1}{255} \le x \le \frac{3}{255}, \\ 0, & elsewhere, \end{cases} \qquad H_6(x) = \begin{cases} 1, & \frac{3}{255} \le x \le \frac{7}{255}, \\ -\frac{1}{2}, & \frac{7}{255} \le x \le \frac{15}{255}, \\ 0, & elsewhere, \end{cases}$$

$$H_7(x) = \begin{cases} 1, & \frac{15}{255} \le x \le \frac{31}{255}, \\ -\frac{1}{2}, & \frac{31}{255} \le x \le \frac{63}{255}, \\ 0, & elsewhere, \end{cases} \qquad H_8(x) = \begin{cases} 1, & \frac{63}{255} \le x \le \frac{127}{255}, \\ -\frac{1}{2}, & \frac{127}{255} \le x \le 1, \\ 0, & elsewhere. \end{cases}$$

For instance, the non-uniform Haar coefficients are given in the 8×8 square matrix

$$H(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

On the other hand, we know that each function $Y(x)$, square integrable on $[0, 1]$, can be approximated by

$$Y(x) \simeq \sum_{i=1}^{2N} q_i H_i(x). \quad (4.3)$$

Next we refer to Eq. (1) and consider the collocation points

$$x_l = \frac{x(l-1)+x(l)}{2},\ l = 1,2,3,...,2N.$$

We obtain

$$Y(x_l) = Y_0(x_l) + \lambda_1 \int_0^{x_l} f_1(s,x_l)Y(s)ds + \lambda_2 \int_0^{x_l} f_2(s,x_l)Y(s)dB(s).$$

By substituting approximation (9) of $Y(x)$ in the above equation, we deduce

$$\begin{array}{l}\sum\limits_{i=1}^{2N} q_i H_i(x_l) = Y_0(x_l) + \sum\limits_{i=1}^{2N} q_i\lambda_1 \int_0^{x_l} f_1(s,x_l)H_i(s)ds \\ \quad + \sum\limits_{i=1}^{2N} q_i\lambda_2 \int_0^{x_l} f_2(s,x_l)H_i(s)dB(s),\end{array}$$

or

$$\sum_{i=0}^{2N} q_i[H_i(x_l) - \lambda_1 I_{il} - \lambda_2 {I^S}_{il}] = Y_0(x_l),\ l = 0,1,2,...,2N, \qquad (\mathbf{4}.4)$$

where

$$I_{il} = \int_0^{x_l} f_1(s,x_l)H_i(s)ds,\ {I^S}_{il} = \int_0^{x_l} f_2(s,x_l)H_i(s)dB(s).$$

We express the integrals I_{il} and ${I^S}_{il}$ as:

$$I_{il} = \begin{cases} \int_{\vartheta_1(i)}^{x_l} f_1(s,x_l)ds, & \vartheta_1(i) \le x_l < \vartheta_2(i), \\ \int_{\vartheta_1(i)}^{\vartheta_2(i)} f_1(s,x_l)ds - p_i \int_{\vartheta_2(i)}^{x_l} f_1(s,x_l)ds, & \vartheta_2(i) \le x_l < \vartheta_3(i), \\ \int_{\vartheta_1(i)}^{\vartheta_2(i)} f_1(s,x_l)ds - p_i \int_{\vartheta_2(i)}^{\vartheta_3(i)} f_1(s,x_l)ds, & \vartheta_3(i) \le x_l < 1, \\ 0, & x_l < \vartheta_1(i), \end{cases}$$

and

$${I^S}_{il} = \begin{cases} \int_{\vartheta_1(i)}^{x_l} f_2(s,x_l)dB(s), & \vartheta_1(i) \le x_l < \vartheta_2(i), \\ \int_{\vartheta_1(i)}^{\vartheta_2(i)} f_2(s,x_l)dB(s) - p_i \int_{\vartheta_2(i)}^{x_l} f_2(s,x_l)dB(s), & \vartheta_2(i) \le x_l < \vartheta_3(i), \\ \int_{\vartheta_1(i)}^{\vartheta_2(i)} f_2(s,x_l)dB(s) - p_i \int_{\vartheta_2(i)}^{\vartheta_3(i)} f_2(s,x_l)dB(s), & \vartheta_3(i) \le x_l < 1, \\ 0, & x_l < \vartheta_1(i), \end{cases}$$

By solving Eq. (10), the coefficients q_i are calculated. By inserting them in Eq. (9) the numerical solution $Y(x)$ is obtained.

5. Error analysis

In this section, we investigate the convergence and perform the error analysis of the proposed method in the previous section for solving stochastic Volterra integral equations. To this purpose we need the following theorems.

Theorem 5.1. *Suppose that $g(x) \in L^2[0,1)$ is an arbitrary function with bounded first derivative, $|g'(x)| \le D$, and consider the error function*

$$e_m(x) = g(x) - \sum_{i=0}^{m-1} g_i H_i(x),$$

where $i = 2^j + k + 1$, $m = 2^{J+1}$, $J > 0$, and

$$g_i = \frac{1}{\delta_i}\int_0^1 H_i(x)g(x) = \frac{1}{\delta_i}\left(\int_{\vartheta_1(i)}^{\vartheta_2(i)} g(x)dx - p_i \int_{\vartheta_2(i)}^{\vartheta_3(i)} g(x)dx\right).$$

Then, we have

$$\|e_m\|_2 = O(\frac{1}{m}).$$

Proof. By the definition of $e_m(x)$, we can write

$$\|e_m\|_2^2 = \int_0^1 \left(g(x) - \sum_{i=0}^{m-1} g_i H_i(x) \right)^2 dx$$
$$= \int_0^1 \left(\sum_{i=m}^{\infty} g_i H_i(x) \right)^2 dx = \sum_{i=m}^{\infty} g_i^2 \int_0^1 H_i^2(x) dx.$$

By the mean value theorem for integrals, there are $\alpha_1 \in (\vartheta_1(i), \vartheta_2(i))$, $\alpha_2 \in (\vartheta_2(i), \vartheta_3(i))$, such that

$$g_i = \frac{1}{\delta_i} \left(g(\alpha_1)(\vartheta_2(i) - \vartheta_1(i)) - p_i g(\alpha_2)(\vartheta_3(i) - \vartheta_2(i)) \right)$$
$$= \frac{1}{\delta_i} \left(g(\alpha_1)(\vartheta_2(i) - \vartheta_1(i)) - \frac{\vartheta_2(i) - \vartheta_1(i)}{\vartheta_3(i) - \vartheta_2(i)} g(\alpha_2)(\vartheta_3(i) - \vartheta_2(i)) \right)$$
$$= \frac{1}{\delta_i} \left((\vartheta_2(i) - \vartheta_1(i))(g(\alpha_1) - g(\alpha_2)) \right) = \frac{1}{\delta_i} \left((\vartheta_2(i) - \vartheta_1(i))(\alpha_1 - \alpha_2) g'(\alpha) \right), \ \alpha \in (\alpha_1, \alpha_2).$$

From Eq. (4) and definitions of α_1, α_2, it follows that

$$\|e_m\|_2^2 = \sum_{i=m}^{\infty} \frac{1}{{\delta_i}^2} \Big((\vartheta_2(i) - \vartheta_1(i))^2 (\alpha_1 - \alpha_2)^2 (g'(\alpha))^2 \Big) \delta_i$$
$$\leq \sum_{i=m}^{\infty} \frac{1}{\delta_i} \Big((\vartheta_2(i) - \vartheta_1(i))^2 (\vartheta_3(i) - \vartheta_1(i))^2 D^2 \Big). \qquad (\mathbf{5}.1)$$

Now, by definitions of $\vartheta_1(i)$, $\vartheta_2(i)$, $\vartheta_3(i)$, and Eq. (7) we have

$$\vartheta_1(i) = \frac{2kN}{n} \Delta x_1,$$
$$\vartheta_2(i) = \frac{(2k+1)N}{n} \Delta x_1,$$

and

$$\vartheta_3(i) = \frac{(2k+2)N}{n} \Delta x_1,$$

therefore, we get

$$\vartheta_2(i) - \vartheta_1(i) = \frac{N \Delta x_1}{n}, \qquad (\mathbf{5}.2)$$
$$\vartheta_3(i) - \vartheta_1(i) = \frac{2N \Delta x_1}{n}, \qquad (\mathbf{5}.3)$$

and $\vartheta_3(i) - \vartheta_2(i) = \frac{N \Delta x_1}{n}$. Since $\vartheta_2(i) - \vartheta_1(i) \leq \vartheta_3(i) - \vartheta_1(i)$, we have

$$\|e_m\|_2^2 \leq D^2 \sum_{i=m}^{\infty} \frac{1}{\delta_i} \Big((\vartheta_3(i) - \vartheta_1(i))^4 \Big). \qquad (\mathbf{5}.4)$$

Now, by using Eqs. (3) and (5), we get

$$\frac{1}{\delta_i} = \frac{\vartheta_3(i) - \vartheta_2(i)}{(\vartheta_2(i) - \vartheta_1(i))(\vartheta_3(i) - \vartheta_1(i))}. \qquad (\mathbf{5}.5)$$

With $\vartheta_3(i) - \vartheta_2(i) \leq \vartheta_3(i) - \vartheta_1(i)$ and Eq. (12) in Eq. (15), we can write

$$\frac{1}{\delta_i} \leq \frac{1}{\vartheta_2(i) - \vartheta_1(i)} = \frac{1}{\frac{N(\Delta x_1)}{n}} = \frac{1}{\frac{N(\Delta x_1)}{2^j}}. \qquad (\mathbf{5}.6)$$

Using Eqs. (13) and (14), Eq. (16) implies

$$\|e_m\|_2^2 \le D^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{2N^3(\Delta x_1)^3}{(2^j)^3}$$
$$= 2D^2N^3(\Delta x_1)^3 \sum_{j=J+1}^{\infty} \frac{1}{(2^j)^3} \times 2^j = 2D^2N^3(\Delta x_1)^3 \sum_{j=J+1}^{\infty} \frac{1}{(2^j)^2},$$

that is,

$$\|e_m\|_2^2 \le \frac{8D^2N^3(\Delta x_1)^3}{3}\left(\frac{1}{2^{J+1}}\right)^2 = A\left(\frac{1}{2^{J+1}}\right)^2, \qquad (\mathbf{5}.7)$$

where $A = \frac{8D^2N^3(\Delta x_1)^3}{3}$. Since $m = 2^{J+1}$, we have $\|e_m\|_2 = O(\frac{1}{m})$. □

Theorem 5.2. *Suppose that $g(s,x) \in L^2[0,1)^2$ is a function with bounded partial derivative, $\left|\frac{\partial^2 g}{\partial s \partial x}\right| < V$, and let e_m be defined by $e_m(s,x) = g(s,x) - \sum_{i=0}^{m-1}\sum_{l=0}^{m-1} g_{i,l}H_i(s)H_l(x)$, where $i = 2^{j_1}+k+1$, $l = 2^{j_2}+k+1$, $m = 2^{J+1}$, $J > 0$, and $g_{i,l} = \frac{1}{\delta_i{}^2}\int_0^1\int_0^1 H_i(s)H_l(x)g(s,x)dsdx$. Then, we have $\|e_m\|_2 = O(\frac{1}{m^2})$.*

Proof. We can write

$$\|e_m\|_2^2 = \int_0^1\int_0^1 \left(g(s,x) - \sum_{i=0}^{m-1}\sum_{l=0}^{m-1} g_{i,l}H_i(s)H_l(x)\right)^2 dsdx$$
$$= \int_0^1\int_0^1 \left(\sum_{i=m}^{\infty}\sum_{l=m}^{\infty} g_{i,l}H_i(s)H_l(x)\right)^2 dsdx = \sum_{i=m}^{\infty}\sum_{l=m}^{\infty}\int_0^1\int_0^1 \left(g_{i,l}^2 H_i^2(s)H_l^2(x)\right)dsdx.$$

From the non-uniform wavelet definition, the mean value theorem and Theorem 5.1, there exist $\alpha, \alpha_1, \alpha_2$, and β, β_1, β_2 such that

$$g_{i,l} = \frac{1}{\delta_i{}^2}\int_0^1 H_i(s)\left(\int_0^1 H_l(x)g(s,x)dx\right)ds$$
$$= \frac{1}{\delta_i{}^2}\int_0^1 H_i(s)\left(\left(\vartheta_2(l) - \vartheta_1(l)\right)\left(\beta_1 - \beta_2\right)\frac{\partial g(s,\beta)}{\partial x}\right)ds$$
$$= \frac{1}{\delta_i{}^2}\left(\vartheta_2(l) - \vartheta_1(l)\right)\left(\beta_1 - \beta_2\right)\int_0^1 \left(H_i(s)\frac{\partial g(s,\beta)}{\partial x}\right)ds$$
$$= \frac{1}{\delta_i{}^2}\left(\vartheta_2(l) - \vartheta_1(l)\right)\left(\beta_1 - \beta_2\right)\left(\vartheta_2(i) - \vartheta_1(i)\right)\left(\alpha_1 - \alpha_2\right)\frac{\partial^2 g(\alpha,\beta)}{\partial s \partial x}.$$

So, we obtain

$$\|e_m\|_2^2 = \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} \frac{1}{\delta_i{}^4}(\vartheta_2(l) - \vartheta_1(l))^2(\beta_1 - \beta_2)^2(\vartheta_2(i) - \vartheta_1(i))^2(\alpha_1 - \alpha_2)^2\left|\frac{\partial^2 g(\alpha,\beta)}{\partial s \partial x}\right|^2 \delta_i{}^2$$
$$\le V^2 \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} \frac{1}{\delta_i{}^2}(\vartheta_2(i) - \vartheta_1(i))^2(\alpha_1 - \alpha_2)^2(\vartheta_2(l) - \vartheta_1(l))^2(\beta_1 - \beta_2)^2,$$

and with Eq. (17), we get $\|e_m\|_2^2 \le V^2A^2 \times \frac{1}{m^4} = \frac{C}{m^4}$, $C = V^2A^2$. In the other words, we have $\|e_m\|_2 = O(\frac{1}{m^2})$. □

Lemma 5.1. *For any $x > 0$, we have $P\left(M(t) \geq x\right) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-\frac{z^2}{2}} dz = 2\left(1 - \varphi(\frac{x}{\sqrt{t}})\right)$, where $M(t) = \sup\limits_{0 \leq s \leq t} B(s)$, and φ is the cumulative standard normal distribution function. So, for $x \geq 4$ we have $\sup\limits_{0 \leq t \leq 1} B(t) < \infty$, with probability one.*

Proof. See [7] □

Theorem 5.3. *Suppose that $Y(x)$ is the exact solution of Eq. (1), and $\hat{Y}(x)$ is the non-uniform Haar wavelet series approximate solution (9) of Eq. (1). Moreover, assume that*
1) $\|Y\|_2 \leq \Phi$,
2) $\|f_i\|_2 \leq \theta_i, \; i = 1, 2,$
3) $\left(\lambda_1 \left(\theta_1 + \frac{\sqrt{C_1}}{m^2}\right) + \lambda_2 \sup\limits_{x \in [0,1)} |B(x)| \left(\theta_2 + \frac{\sqrt{C_2}}{m^2}\right)\right) < 1,$

where the functions f_1, f_2 and the parameters λ_1, λ_2 are considered in Eq. (1), and C_1, C_2 are based on definition of C at Theorem 5.2. Then,

$$\left\|Y - \hat{Y}\right\|_2 \leq \frac{\frac{\sqrt{A}}{m} + \frac{\lambda_1 \sqrt{C_1}}{m^2}\Phi + \lambda_2 \sup\limits_{x \in [0,1)} |B(x)| \frac{\sqrt{C_2}}{m^2}\Phi}{1 - \left(\lambda_1 \left(\theta_1 + \frac{\sqrt{C_1}}{m^2}\right) + \lambda_2 \sup\limits_{x \in [0,1)} |B(x)| \left(\theta_2 + \frac{\sqrt{C_2}}{m^2}\right)\right)}.$$

Proof. From Eq. (1) we have

$$Y(x) - \hat{Y}(x) = Y_0(x) - \hat{Y}_0(x) + \lambda_1 \int_0^x \left(f_1(s,x)Y(s) - \hat{f}_1(s,x)\hat{Y}(s)\right) ds$$
$$+ \lambda_2 \int_0^x \left(f_2(s,x)Y(s) - \hat{f}_2(s,x)\hat{Y}(s)\right) dB(s),$$

so, by the mean value theorem we obtain

$$\left\|Y - \hat{Y}\right\|_2 \leq \left\|Y_0 - \hat{Y}_0\right\|_2 + \lambda_1 x \left\|f_1.Y - \hat{f}_1.\hat{Y}\right\|_2 + \lambda_2 B(x) \left\|f_2.Y - \hat{f}_2.\hat{Y}\right\|_2.$$

Using the assumptions 1) and 2) and Theorem 5.2 we get

$$\left\|f_1.Y - \hat{f}_1.\hat{Y}\right\|_2 \leq \|f_1\|_2 \left\|Y - \hat{Y}\right\|_2 + \left\|f_1 - \hat{f}_1\right\|_2 \left(\left\|Y - \hat{Y}\right\|_2 + \|Y\|_2\right) \tag{5.8}$$
$$\leq \theta_1 \left\|Y - \hat{Y}\right\|_2 + \frac{\sqrt{C_1}}{m^2}\left(\left\|Y - \hat{Y}\right\|_2 + \Phi\right).$$

Also we have

$$\left\|f_2.Y - \hat{f}_2\hat{Y}\right\|_2 \leq \theta_2 \left\|Y - \hat{Y}\right\|_2 + \frac{\sqrt{C_2}}{m^2}\left(\left\|Y - \hat{Y}\right\|_2 + \Phi\right). \tag{5.9}$$

By using Eqs. (19) and (20) in Eq. (18) and Theorem 5.1 we obtain

$$\left\|Y - \hat{Y}\right\|_2 \leq \frac{\sqrt{A}}{m} + \lambda_1 x \left(\theta_1 \left\|Y - \hat{Y}\right\|_2 + \frac{\sqrt{C_1}}{m^2}\left(\left\|Y - \hat{Y}\right\|_2 + \Phi\right)\right)$$
$$+ \lambda_2 B(x) \left(\theta_2 \left\|Y - \hat{Y}\right\|_2 + \frac{\sqrt{C_2}}{m^2}\left(\left\|Y - \hat{Y}\right\|_3 + \Phi\right)\right).$$

By taking sup we can write

$$\left\|Y-\hat{Y}\right\|_2 \leq \frac{\sqrt{A}}{m} + \lambda_1 \sup_{x\in[0,1)} x\left(\left(\theta_1 + \frac{\sqrt{C_1}}{m^2}\right)\sup_{x\in[0,1)}\left\|Y-\hat{Y}\right\|_2 + \frac{\sqrt{C_1}}{m^2}\Phi\right)$$
$$+\lambda_2 \sup_{x\in[0,1)} |B(x)|\left(\left(\theta_2 + \frac{\sqrt{C_2}}{m^2}\right)\sup_{x\in[0,1)}\left\|Y-\hat{Y}\right\|_2 + \frac{\sqrt{C_2}}{m^2}\Phi\right),$$

so

$$\left\|Y-\hat{Y}\right\|_2 \leq \frac{\frac{\sqrt{A}}{m} + \frac{\lambda_1\sqrt{C_1}}{m^2}\Phi + \lambda_2 \sup_{x\in[0,1)} |B(x)|\frac{\sqrt{C_2}}{m^2}\Phi}{1-\left(\lambda_1\left(\theta_1+\frac{\sqrt{C_1}}{m^2}\right)+\lambda_2\sup_{x\in[0,1)}|B(x)|\left(\theta_2+\frac{\sqrt{C_2}}{m^2}\right)\right)},$$

and Lemma 5.1 proves the desired result. □

6. Numerical examples

In this section, we consider numerical examples to illustrate the efficiency and reliability of the non-uniform Haar wavelets in solving stochastic Volterra integral equations. Here we have: $\bar{X}_E$ is the Error's mean, $\bar{X}_A$ is the mean of analytical solution, $\bar{X}_N$ is the mean of the numerical solution and 95% confidence interval for Error's mean is calculated. Also L and U are the confidence interval lower and upper bounds, respectively.

Example 6.1. We consider the following stochastic Volterra integral equation $Y(x) = -\frac{1}{8} - \int_0^x \frac{s}{4}Y(s)ds - \int_0^x \frac{1}{40}Y(s)dB(s)$. Its analytical solution is $Y(x) = -\frac{1}{8}\exp(-\frac{1}{40}B(x) - \frac{x^2}{8} - \frac{x}{3200})$. For r = 2, the results are shown in Table 1. A comparison between the numerical solutions given by the non-uniform Haar wavelet method (NHWM) and the uniform Haar wavelet method (UHWM) are shown in Table 2.

Table 1: The result of Example 1.

x	$\bar{X}_E$	$\bar{X}_A$	$\bar{X}_N$	95% confidence interval for $\bar{X}_E$ L	U
0	0	-0.1250	-0.1250	0	0
0.1	0.0018	-0.1249	-0.1267	0.0017	0.0019
0.2	0.0025	-0.1241	-0.1216	0.0003	0.0046
0.3	0.0021	-0.1243	-0.1243	0.0020	0.0021
0.4	0.0021	-0.1273	-0.1252	-0.0004	0.0047
0.5	0.0058	-0.1245	-0.1187	0.0050	0.0065
0.6	0.0090	-0.1244	-0.1154	0.0087	0.0093
0.7	0.0083	-0.1262	-0.1179	0.0079	0.0087
0.8	0.0082	-0.1204	-0.1122	0.0077	0.0087
0.9	0.0086	-0.1232	-0.1147	0.0078	0.0094

Table 2: Comparison of Error's mean and length of the confidence interval (LCI) for NHWM with UHWM.

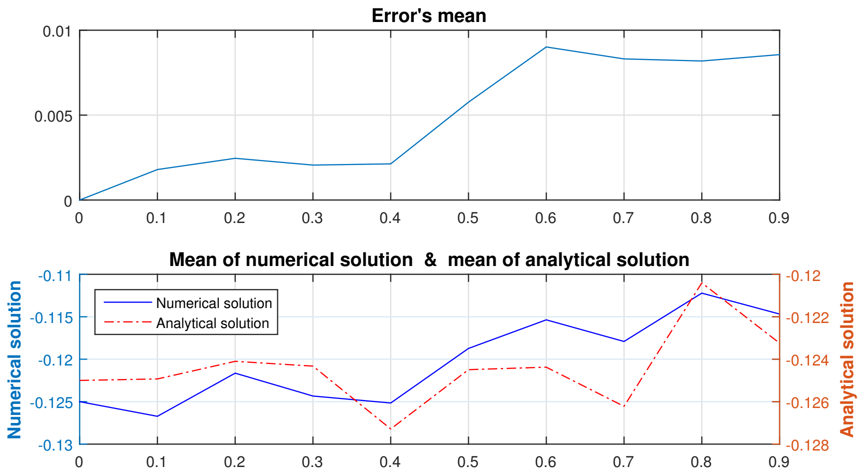

Fig. 1. Error's mean, analytical and approximate solutions for $J = 2$.

x	$Method$	$\bar{X}_E$	LCI
0.1	NHWM	0.0018	0.0002
	UHWM	0.0104	0.0028
0.3	NHWM	0.0021	0.0004
	UHWM	0.0103	0.0063
0.5	NHWM	0.0058	0.0015
	UHWM	0.0098	0.0030
0.7	NHWM	0.0083	0.0007
	UHWM	0.0091	0.0011

Example 6.2. [16]We consider the following stochastic Volterra integral equation $Y(x) = 1 + \int_0^x s^2 Y(s)ds + \int_0^x sY(s)dB(s)$. Its solution is $Y(x) = \exp\left(\frac{x^3}{6} + \int_0^x sdB(s)\right)$. For r = 2, the results are shown in Table 3. A comparison between the numerical solutions given by the non-uniform Haar wavelet method (NHWM) and methot of used [16] Haar Block pulse operational matrix (HBOM) are shown in Table 4.

Table 3: The result of Example 2.

x	$\bar{X}_E$	$\bar{X}_A$	$\bar{X}_N$	95% confidence interval for $\bar{X}_E$ L	U
0.1	0.0051	0.9112	0.0962	0	0.0102
0.2	0.0073	0.1061	0.1125	0.0014	0.0132
0.3	0.0153	0.0851	0.0912	0.0065	0.0242
0.4	0.0146	0.0800	0.0873	0.0035	0.0258
0.5	0.0125	0.0726	0.0767	0.0046	0.0204
0.6	0.0158	0.0857	0.0933	0.0054	0.0261
0.7	0.0173	0.0869	0.1009	-0.0014	0.0362
0.8	0.0213	0.0918	0.1130	-0.0044	0.0471

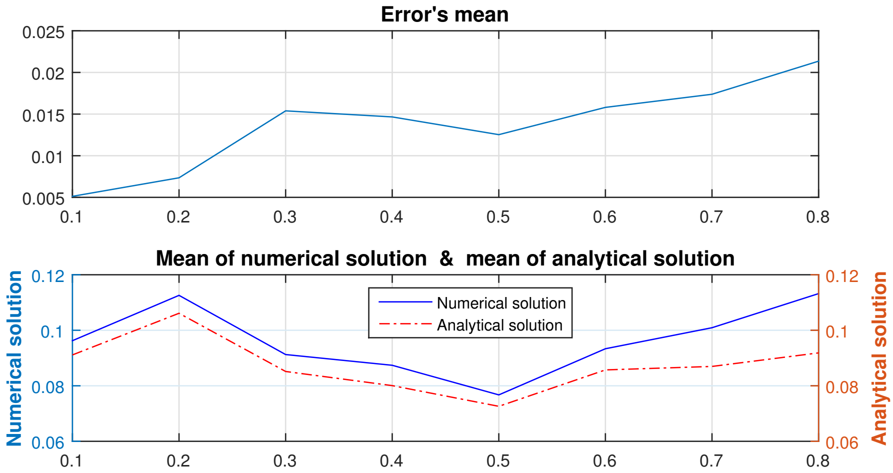

Fig. 2. Error's mean, analytical and approximate solutions for $J = 2$.

Table 4: Comparison of Error's mean for NHWM with HBOM.

x	$Method$	$\bar{X}_E$	n
0.1	NHWM	0.00512551	2^3
	HBOM	0.00584517	2^4
0.3	NHWM	0.01538704	2^3
	HBOM	0.03623059	2^4
0.5	NHWM	0.01253040	2^3
	HBOM	0.02501065	2^4
0.7	NHWM	0.01737967	2^3
	HBOM	0.13708159	2^4

A review of Table 4. shows that the numerical results of the NHWM are better than the numerical results of the HBOM. Obviously, with respect to n, the computational complexity of our method is less than the computational complexity of the HBOM which demonstrate the validity of this method.

7. Conclusion

In this paper, we have successfully used the non-uniform Haar wavelet orthogonal basis functions to approximate the solution of the stochastic Volterra integral equation. The error analysis and the numerical examples confirm that the method is highly accurate. The typical convergence rate of the method is $O(\frac{1}{m^2})$. We can also use this method for solving stochastic Fredholm or Volterra-Fredholm integral equations.

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