ON SOME RATIONAL ZETA SERIES INVOLVING $\zeta(2n)$ AND BINOMIAL COEFFICIENTS

Cezar LUPU[1], Vlad MATEI[2]

In this note, we give an exact formula for a general family of rational zeta series involving the coefficient $\zeta(2n)$ in terms of Hurwitz zeta values. This formula generalizes two previous formulas from a paper in [5]*. Our method will involve derivatives polynomials for the cotangent function.*

Keywords: Riemann zeta function, Hurwitz zeta values, rational zeta series
MSC2020: Primary 11M06, 11M35, 40A05.

1. **Introduction**

The Riemann zeta function is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \operatorname{Re} s > 1.$$

In 1882, Hurwitz defined the following "shifted" zeta function,

$$\zeta(s;a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \operatorname{Re} s > 1, 0 < a \leq 1.$$

Both of them have similar properties in many aspects. For example, both of them are analytic and they have analytic continuation to the whole complex plane except for the pole $s = 1$. Some particular values include $\zeta(-n;a) = -\dfrac{B_{n+1}(a)}{n+1}$, where $B_k(a)$ is the Bernoulli polynomial which is defined by the power series

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Also, as a special case, we have $\zeta(0;a) = \frac{1}{2} - a$. Other obvious values include $\zeta\left(s; \frac{1}{2}\right) = (2^s - 1)\zeta(s)$ and $\zeta(s; a+1) = \zeta(s,a) - a^s$. For more details, one can consult [1, 3, 7].

A classical problem which goes back to Goldbach and Bernoulli asserts that

$$\sum_{\omega \in S} (\omega - 1)^{-1} = 1,$$

where $S = \{n^k : n, k \in \mathbb{Z}_{\geq 0} - \{1\}\}$. In terms of the Riemann zeta function $\zeta(s)$, the above problem reads as,

Assistant Professor, Beijing Institute of Mathematical Sciences and Applications (BIMSA), Yau Mathematical Sciences Center (YMSC), Tsinghua University, Beijing, People's Republic of China, e-mail: `lupucezar@gmail.com, lupucezar@bimsa.cn` (corresponding author)

Scientific Researcher III, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania, e-mail: `vmatei@imar.ro`

$$\sum_{n=2}^{\infty}(\zeta(n)-1)=1.$$

Also, there are other representations for $\log 2$ and γ (Euler-Mascheroni constant) such as,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-1}{n}=\log 2,$$

and

$$\sum_{n=2}^{\infty}\frac{\zeta(n)-1}{n}=1-\gamma.$$

For instance, one way to generate rational zeta series involving $\zeta(2n)$ is by looking at the cotangent power series formula in the form:

$$\sum_{n=1}^{\infty}\zeta(2n)x^{2n}=\frac{1}{2}(1-\pi x\cot(\pi x)), |x|<1.$$

Dividing by x and integrating once, we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n}x^{2n}=\log\left(\frac{\pi x}{\sin(\pi x)}\right), |x|<1.$$

For $x=\frac{1}{2}$ and $x=\frac{1}{4}$ in the above formulas, we obtain the following representations:

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}}=\frac{1}{2} \tag{1}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{4n}}=\frac{4-\pi}{8} \tag{2}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n2^{2n}}=\log\pi-\log 2 \tag{3}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n2^{4n}}=\log\pi-\frac{3}{2}\log 2 \tag{4}$$

Moreover, integrating from 0 to $\frac{1}{2}$ the last power series equality, we derive

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)2^{2n}}=\log\pi-1 \tag{5}$$

which can be found in [6].

This type of rational zeta series and many others are treated in [2]. In [5] there are given exact formulas for the following rational zeta series

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^{n}}\binom{2n}{m} \tag{6}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n}\binom{2n}{m} \tag{7}$$

in terms of zeta values. In this note, we give an exact formula for a more general rational zeta series which encompasses the two series above.

The main result of this note is the following

Theorem 1.1. *Let $\zeta(s)$ and $\zeta(s;r)$ be the Riemann and Hurwitz zeta functions. For $a \in (0,1)$ we have*

$$\sum_{n=1}^{\infty} \frac{a^{2n}\zeta(2n)}{n}\binom{2n}{m} = \frac{a^m}{m}\Bigg((-1)^m\zeta(m;a) + \zeta(m;1-a)\Bigg) + \frac{(-1)^{m-1}}{m}. \tag{8}$$

The main idea of the proof is a combination of expressing the rational zeta series from the left-hand side as the nth derivative of the cotangent function (a polynomial $P_n(\cot \pi x)$) and a surprising result of Hoffman [4] which relates this polynomial $P_n(\cot x)$ in terms of Hurwitz zeta values.

Acknowledgement. The second author was supported by the project “Group schemes, root systems, and related representations” founded by the European Union - NextGenerationEU through Romania’s National Recovery and Resilience Plan (PNRR) call no. PNRR-III-C9-2023- I8, Project CF159/31.07.2023, and coordinated by the Ministry of Research, Innovation and Digitalization (MCID) of Romania.

2. **The proof of Theorem 1.1**

First proof. Before we dive into the proof of the main result, let us recall a result of Hoffman [4] which will be an essential ingredient for our purpose.

Lemma 2.1 (M.E. Hoffman, 1995)**.** *For real $0 < a < 1$ and integer $n \geq 0$,*

(a)

$$\sum_{k=0}^{\infty}\left[\frac{1}{(k+a)^{n+1}} + \frac{(-1)^{n+1}}{(k+1-a)^{n+1}}\right] = \frac{\pi^{n+1}}{n!}P_n(\cot a\pi).$$

and

(b)

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+a)^{n+1}} + (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1-a)^{n+1}} = \frac{\pi^{n+1}}{n!}\csc a\pi P_n(\cot a\pi)$$

For a function α we will denote by $\alpha^{(k)}$ its kth derivative. Our purpose is to explore sums of the type

$$g(a) = \sum_{n=1}^{\infty}\frac{a^{2n}\zeta(2n)}{n}\binom{2n}{m}$$

where $a \in (0,1)$ is a general real parameter.

We start with the following well known cotangent power series expansion written in the form

$$\pi\cot(\pi x) = \frac{1}{x} - 2\sum_{k=1}^{\infty}\zeta(2k)x^{2k-1}.$$

To connect the above with this expansion, note that

$$\frac{1}{n}\binom{2n}{m} = \frac{2\cdot(2n-1)\dots(2n-m+1)}{m!}.$$

Thus $g(a)$ can be computed as

$$\frac{a^m}{m!}h^{(m-1)}(a)$$

where $h(x) = -\pi\cot(\pi x) + \frac{1}{x}$.

In [4], Hoffman computes the nth derivative of $-\cot x$ in terms of a polynomial P_n. More precisely

$$\frac{d^n}{dx^n}\cot x = (-1)^n P_n(\cot x).$$

For our problem we have $\frac{d^n}{dx^n}\cot(\pi x) = (-1)^n\pi^n P_n(\cot(\pi x))$. Moreover for $0 < a < 1$ by Lemma 2.1 (part (a)), we have

$$\sum_{k=0}^{\infty}\left[\frac{1}{(k+a)^{n+1}} + \frac{(-1)^{n+1}}{(k+1-a)^{n+1}}\right] = \frac{\pi^{n+1}}{(n+1)!}P_n(\cot(\pi a)).$$

Putting everything together we get the following expression for $g(a)$, which is valid for $0 < a < 1$

$$g(a) = \frac{a^m}{m}\sum_{k=0}^{\infty}\left[\frac{(-1)^m}{(k+a)^m} + \frac{1}{(k+1-a)^m}\right] + \frac{(-1)^{m-1}}{m}.$$

Note that we can compactly write the above in terms of Hurwitz zeta function,

$$g(a) = \frac{a^m}{m}\Bigg((-1)^m\zeta(m,a) + \zeta(m,1-a)\Bigg) + \frac{(-1)^{m-1}}{m}.$$

Second proof. This is more direct proof without using Lemma 2.1. In fact, we prove our equality in the following form,

$$\sum_{n=1}^{\infty}\frac{a^{2n}\zeta(2n)}{n}\binom{2n}{m} - \frac{(-1)^{m-1}}{m} = \frac{a^m}{m}\Bigg((-1)^m\zeta(m;a) + \zeta(m;1-a)\Bigg). \tag{9}$$

The strategy is the following. We start with the power series formula of cotangent in the form

$$2\sum_{n=1}^{\infty}\zeta(2n)x^{2n-1} - \frac{1}{x} = -\pi\cot(\pi x). \tag{10}$$

Afterwards, we take on both sides of (10) the $(m-1)$-st derivative, evaluating at $x = a$, and multiplying both sides of the resulting equality by $\frac{a^m}{m!}$ immediately leads to the left-hand side of (9). On the other hand, we are left to show that the $(m-1)$-st derivative of $-\pi\cot(\pi x)$ (after evaluation at $x = a$ and multiplication by $\frac{a^m}{m!}$) equals

$$\frac{a^m}{m}\Bigg((-1)^m\zeta(m,a) + \zeta(m,1-a)\Bigg) + \frac{(-1)^{m-1}}{m}.$$

Now, denote the functions $\psi(x) = 2\sum_{n=1}^{\infty}\zeta(2n)x^{2n-1} - \frac{1}{x}$ and $\theta(x) = -\pi\cot(\pi x)$, respectively. As it was outlined in the strategy above, by taking the $(m-1)$-st derivative of $\psi(x)$ and evaluating at $x = a$ yelds

$$\begin{aligned}
\frac{d^{m-1}}{dx^{m-1}}\psi(x)|_{x=a} &= 2\sum_{n=1}^{\infty}\zeta(2n)a^{2n-m}(2n-1)\cdot\ldots\cdot(2n-m+1) - \frac{(-1)^{m-1}(m-1)!}{a^m} \\
&= \sum_{n=1}^{\infty}\frac{a^{2n-m}\zeta(2n)}{n}\cdot\frac{2n(2n-1)\cdot\ldots\cdot(2n-m+1)}{m!}\cdot m! - \frac{(-1)^{m-1}(m-1)!}{a^m} \\
&= \sum_{n=1}^{\infty}\frac{a^{2n-m}\zeta(2n)}{n}\binom{2n}{m}m! - \frac{(-1)^{m-1}(m-1)!}{a^m}
\end{aligned}$$

which gives us the claimed left-hand side after multiplication by $\frac{a^m}{m!}$. On the other hand, we compute the $(m-1)$-st derivative of $\theta(x)$. For this purpose, we employ the following partial fraction expansion of the cotangent function,

$$-\pi\cot(\pi x) = -\frac{1}{x} - \sum_{n=1}^{\infty}\left(\frac{1}{x+n} + \frac{1}{x-n}\right) \tag{11}$$

Again, taking the $(m-1)$-st derivative of $\theta(x)$ at evaluating at $x = a$ we have

$$\frac{d^{m-1}}{dx^{m-1}}\theta(x)|_{x=a} = \frac{(-1)^{m-1}(m-1)!}{a^m} - \sum_{n=1}^{\infty}\left(\frac{(-1)^{m-1}(m-1)!}{(a+n)^m} + \frac{(-1)^{m-1}(m-1)!}{(a-n)^m}\right).$$

Assuming from now on that $m \geq 2$, the latter formula can be rewritten as

$$\begin{aligned}
\frac{d^{m-1}}{dx^{m-1}}\theta(x)|_{x=a} &= (-1)^m(m-1)!\sum_{n=0}^{\infty}\frac{1}{(n+a)^m} + (-1)^m(m-1)!\sum_{n=1}^{\infty}\frac{(-1)^m}{((n-1)+(1-a))^m} \\
&= (-1)^m(m-1)!\zeta(m;a) + (m-1)!\zeta(m;1-a).
\end{aligned}$$

After multiplying the latter quantity by $\frac{a^m}{m!}$, we arrive at the claimed right-hand side of (9). □

Remark 1. Let us start by pointing out that in the case $m = 1$ the individual terms on the right-hand side of formula are not well-defined, since the Hurwitz zeta functions $\zeta(s;a)$ and $\zeta(s;1-a)$ each have a pole of order one with residue one at $s = 1$. However, from this we conclude that the difference $-\zeta(1;a) + \zeta(1;1-a)$ is indeed a well-defined quantity.

Remark 2. Using Wikipedia for the Hurwitz zeta functions [7] one can express for $0 < p < q$ and $\gcd(p,q) = 1$

$$\zeta\left(s, \frac{p}{q}\right) = \frac{q^s}{\varphi(q)}\sum_{\chi}\overline{\chi}(p)L(s,\chi)$$

where the sum runs over all Dirichlet characters mod q.

Thus one can express concretely $g(\frac{p}{q})$ in terms of Dirichlet's L-functions. Dirichlet L-functions [1, 3] are defined as follows. First, consider χ to be a homomorphism from the units of $\mathbb{Z}/k\mathbb{Z}$ to $\mathbb{C}^*$. Now, we can extend χ to a function on $\mathbb{Z}$ called Dirichlet character modulo q as follows

$$\chi(n) = \begin{cases} \chi(q\mathbb{Z}+n) & \gcd(n,k)=1, \\ 0 & \text{otherwise}. \end{cases}$$

Then for $s \in \mathbb{C}$, the Dirichlet L-series corresponding to the character χ is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

As corollaries of the main theorem (Theorem 1.1), we have the following representations from [5].

Corollary 2.1.

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\binom{2n}{m} = \begin{cases} \dfrac{1}{m} & m \text{ odd}, \\ \dfrac{1}{m}\left(2\zeta(m)\left(1-\dfrac{1}{2^m}\right)-1\right) & m \text{ even}. \end{cases} \tag{12}$$

and

Corollary 2.2. *We have the following series representation*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n}\binom{2n}{m} = \begin{cases} \dfrac{1}{m}\left(1-\beta(m)\right) & m \text{ odd}, \\ \dfrac{1}{m}\left(\zeta(m)\left(1-\dfrac{1}{2^m}\right)-1\right) & m \text{ even}, \end{cases} \tag{13}$$

where $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$ *is the Dirichlet's beta function.*

As it has been showed in [5], many well-known rational zeta series can be obtained from the last two corollaries which are similar with (1), (2), (3), (4), (5). We refer the readers to [5] for more details.

REFERENCES

[1] T. Apostol, *Introduction to Analytic Number Theory*, 3rd ed., Springer-Verlag, New York, 1986.

[2] Jonathan M. Borwein, David M. Bradley, Richard E. Crandall, Computational Strategies for the Riemann Zeta Function, *J. Comput. Appl. Math.* **121** (2000), 247—296.

[3] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, 4th edn. Oxford University Press, London (1960).

[4] M. E. Hoffman, Derivative polynomials for tangent and secant, *Amer. Math. Monthly* **102** (1995), 23–30.

[5] C. Lupu, D. Orr, Series representations for the Apery constant $\zeta(3)$ involving the values $\zeta(2n)$, *Ramanujan J.* **45** (2019), 477–494.

[6] D. Tyler, P. R. Chernoff, An old sum reappears-Elementary problem 3103, *Amer. Math. Monthly* **92** (1985), 507.

[7] Wikipedia-Hurwitz zeta function, `https : //en.wikipedia.org/wiki/Hurwitz_zeta_function`