FUZZY SOFT HYPERIDEALS IN HYPERLATTICES

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Firstly, soft hyperideals of hyperlattices are introduced and several interesting examples of them are given. Secondly, fuzzy soft hyperideals are proposed, which are generalizations of hyperideals and soft hyperideals in hyperlattices. After that, equivalent conditions of fuzzy soft hyperideals are given by using level soft sets. Finally, under the fuzzy soft homomorphism of hyperlattices, the image and pre-image of fuzzy soft hyperideals are studied.

Keywords: Soft hyperideal, Fuzzy soft hyperideal, Level soft set, Fuzzy soft homomorphism

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1. Introduction

The concept of hyperstructure was introduced in 1934 by a French mathematician, Marty [1]. Algebraic hyperstructures are suitable generalizations of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. There appeared many components of hyperalgebras such as hypergroups in [2], hyperrings in [3] etc. Konstantinidou and Mittas have introduced the concept of hyperlattices in [4] and superlattices in [5], also see [6, 7, 8]. In particular, Rasouli and Davvaz further studied the theory of hyperlattices and obtained some interesting results [9, 10], which enrich hyperlattice theory.

Recently, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Also, a recent book [11] contains a wealth of applications on geometry, binary relations, lattices, fuzzy sets and rough sets, automata, combinatorics, codes, artificial intelligence and probabilistic. Another book [12] is devoted especially to the study of hyperring theory, written by Davvaz and Leoreanu-Fotea. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

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Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these kinds of difficulties, Molodtsov[13] proposed a completely new approach, which was called soft set theory, for modeling uncertainty. At present, works on the soft set theory are making progress rapidly. Maji et al.[14] described the application of soft set theory to a decision making problem and studied several operations on the theory of soft sets. The algebraic structure of soft sets has been studied by some authors. They also discussed the notion of soft groups [15] and drove their basic properties using Molodtsov's definition of the soft sets. Furthermore, Jun et al.[16, 17] discussed the applications of soft sets in theory of BCK/BCI-algebras. Yamak et al.[18] studied soft hyperstructure.

Maji et al.[19] extended the study of soft sets to fuzzy soft sets. As soft set theory, the theory of fuzzy soft sets turned out to have applications. Roy and Maji[20] presented some applications of this notion to decision-making problems, while Cagman et al.[21] mentioned applications of fuzzy parametrized fuzzy soft set theory in the so-called fpfs decision-making. Subsequently, various fuzzy soft structures have been considered. The notion of a fuzzy soft group is introduced in [22], which is a parametrized family of fuzzy subgroups. Jun et al.[23, 24] discussed the applications of fuzzy soft sets in theory of BCK/BCI-algebras. Fuzzy soft hypergroups, fuzzy soft Γ-hyperrings and fuzzy soft polygroups were defined and analysed by Leoreanu-Fotea and Zhan et al.[25, 26, 27, 29, 30]. In this paper, applying the notions of soft sets and fuzzy soft sets to the theory of hyperlattices, we introduce soft hyperideals and fuzzy soft hyperideals in hyperlattices, and study some properties of them.

2. **Preliminaries**

In this section, we recall some notions and definitions (see [13, 14]) that will be used in the sequel.

Let X be a universe set and E be a set of parameters. Let $P(X)$ be the power set of X and $A \subseteq E$. The notion of soft sets is defined as follows.

Definition 2.1. [13] *A pair (F, A) is called a soft set over X, where A is a subset of the set of parameters E and $F : A \to P(X)$ is a set-valued mapping.*

In other words, a soft set over X is a parameterized family of subsets of X. For all $a \in A$, the subset $F(a)$ can be considered as the set of a-approximate elements of (F, A).

For a soft set (F, A), the set $Supp(F, A) = \{x \in A | F(x) \neq \phi\}$ is called the support of the soft set (F, A). If $Supp(F, A) \neq \phi$, then a soft set (F, A) is called non-null.

Fuzzy soft sets extend the notion of the soft set. Let $I = [0, 1]$ and E be all convenient parameter sets for the universe X. Let I^X denote the set of all fuzzy sets on X and $A \subseteq E$.

Definition 2.2. [19] *A pair (f, A) is called a fuzzy soft set over X, where A is a subset of the set of parameters E and $f : A \to I^X$ is a mapping.*

That is, for all $a \in A$, $f(a) = f_a : X \to I$ is a fuzzy set on X.

Obviously, a soft set (F, A) over X can be seen as a fuzzy soft set (f, A) according to this manner, for $a \in A$, the image of $a \in A$ under f is defined as the characteristic function of the set $F(a)$, that is, $f_a(b) = \chi_{F(a)}(b) = \begin{cases} 1, & if\ b \in F(a) \\ 0, & otherwise. \end{cases}$

Definition 2.3. [19] *Let (f, A) and (g, B) be two fuzzy soft sets over X. Then*
(1) (f, A) is said to be a fuzzy soft subset of (g, B), denoted $(f, A) \subseteq (g, B)$, if $A \subseteq B$ and $f_a \subseteq g_a$ for all $a \in A$, that is, f_a is a fuzzy subset of g_a.
(2) (f, A) and (g, B) are said to be equal, denoted $(f, A) = (g, B)$, if $(f, A) \subseteq (g, B)$ and $(g, B) \subseteq (f, A)$.

Definition 2.4. [19] *Let (f, A) and (g, B) be two fuzzy soft sets over X such that $A \cap B \neq \phi$. The restricted intersection of (f, A) and (g, B) is the fuzzy soft set (h, C), where $C = A \cap B$ and $h_c = f_c \cap g_c$, for all $c \in C$. This is denoted by $(h, C) = (f, A) \sqcap (g, B)$.*

Definition 2.5. [30] *The extended intersection of two fuzzy soft sets (f, A) and (g, B) over X is the fuzzy soft set (h, C), where $C = A \cup B$, $h_c = \begin{cases} f_c, & if\ c \in A - B \\ g_c, & if\ c \in B - A \\ f_c \cap g_c, & if\ c \in A \cap B \end{cases}$ for all $c \in C$. This is denoted by $(f, A)\tilde{\cap}(g, B) = (h, C)$.*

Definition 2.6. [19] *If (f, A) and (g, B) are two fuzzy soft sets. Then $(f, A)\tilde{\wedge}(g, B)$ is defined as $(h, A \times B)$, where $h(a, b) = f_a \cap g_b$, for all $(a, b) \in A \times B$.*

Definition 2.7. [30] *If (f, A) and (g, B) are two fuzzy soft sets. Then $(f, A)\tilde{\vee}(g, B)$ is defined as $(\tilde{O}, A \times B)$, where $\tilde{O}(a, b) = f_a \cup g_b$, for all $(a, b) \in A \times B$.*

3. **Soft hyperideals in hyperlattices**

In this section, we will introduce soft hyperideals in hyperlattices and give several interesting examples of them.

Let L be a nonempty set and $P^*(L)$ be the set of all nonempty subsets of L. A hyperoperation on L is a map $\circ : L \times L \longrightarrow P^*(L)$, which associates a nonempty subset $a \circ b$ with any pair (a, b) of elements of $L \times L$. The couple $(L, \circ)$ is called a hypergroupoid.

If A and B are nonempty subsets of L, for $a, b, x \in L$, then we denote:
(1) $x \circ A = \{x\} \circ A = \bigcup_{a \in A} x \circ a$, $A \circ x = A \circ \{x\} = \bigcup_{a \in A} a \circ x$;
(2) $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$.

There are several kinds of hyperlattices that can be defined on a nonempty set, you can see [4, 5, 6, 7, 8]. Throughout the paper, we shall consider one of the most general types of hyperlattices, as follows.

Definition 3.1. [8] *Let L be a nonempty set endowed with two hyperoperations "$\otimes$" and "$\oplus$". The triple $(L, \otimes, \oplus)$ is called a hyperlattice if the following relations hold: for all $a, b, c \in L$,*
(1) $a \in a \otimes a$, $a \in a \oplus a$;
(2) $a \otimes b = b \otimes a$, $a \oplus b = b \oplus a$;

(3) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(4) $a \in a \otimes (a \oplus b)$, $a \in a \oplus (a \otimes b)$.

Let $(L, \wedge, \vee)$ be a lattice. Define the hyperoperations "$\otimes$" and "$\oplus$" on L as follows: for all $a, b \in L$, $a \otimes b = \{a \wedge b\}$, $a \oplus b = \{a \vee b\}$, then $(L, \otimes, \oplus)$ is a hyperlattice. From this fact, we can see that hyperlattices are generalizations of lattices.

Now, we give some new examples of hyperlattices.

Example 3.1. *Let $(L, \leq)$ be a partially ordered set. Define the following hyperoperations on L: for all $a, b \in L$, $a \otimes b = \{x \in L | x \leq a, x \leq b\}$, $a \oplus b = \{x \in L | a \leq x, b \leq x\}$. Then $(L, \otimes, \oplus)$ is a hyperlattice.*

Example 3.2. *Let $(L, \wedge, \vee)$ be a lattice. Define the following hyperoperations on L: for all $a, b \in L$, $a \otimes b = \{x \in L | a \vee x = b \vee x = a \vee b\}$, $a \oplus b = \{x \in L | a \wedge x = b \wedge x = a \wedge b\}$. Then $(L, \otimes, \oplus)$ is a hyperlattice.*

In what follows, let us recall the notion of hyperideals in hyperlattices.

Definition 3.2. [31] *Let $(L, \otimes, \oplus)$ be a hyperlattice and A be a non-empty subset of L.*
(1) *A is called a $\oplus$-hyperideal of L if for all $a, b \in A$ and $x \in L$,*
(i) $a \otimes b \subseteq A$, *(ii)* $a \oplus x \subseteq A$;
(2) *A is called a $\otimes$-hyperideal of L if for all $a, b \in A$ and $x \in L$,*
(i) $a \oplus b \subseteq A$, *(ii)* $a \otimes x \subseteq A$.

Now, we introduce the notion of soft hyperideals.

Definition 3.3. *Let $(L, \otimes, \oplus)$ be a hyperlattice and (F, A) be a soft set over L.*
(1) *(F, A) is called a soft $\oplus$-hyperideal over L if $F(x)$ is a $\oplus$-hyperideal of L for all $x \in Supp(F, A)$.*
(2) *(F, A) is called a soft $\otimes$-hyperideal over L if $F(x)$ is a $\otimes$-hyperideal of L for all $x \in Supp(F, A)$.*

Let us illustrate this definition by the following examples.

Example 3.3. (1) *Let μ be a fuzzy $\oplus$-hyperideal of the hyperlattice $(L, \otimes, \oplus)$. That is, a fuzzy set μ satisfies the following conditions: for all $x, y \in L$,*
(i) $\bigwedge_{z \in x \otimes y} \mu(z) \geq \mu(x) \wedge \mu(y)$, *(ii)* $\bigwedge_{z \in x \oplus y} \mu(z) \geq \mu(x) \vee \mu(y)$.
It is easy to verify that μ is a fuzzy $\oplus$-hyperideal of L if and only if for all $t \in [0, 1]$ with $\mu_t \neq \phi$, the t-level set $\mu_t = \{x \in L | \mu(x) \geq t\}$ is a $\oplus$-hyperideal of L. Let us consider the family of t-level sets for μ, given by $F(t) = \{x \in L | \mu(x) \geq t\}$, where $t \in [0, 1]$. Then for all $t \in [0, 1]$, $F(t)$ is a $\oplus$-hyperideal of L. Therefore, $(F, [0, 1])$ is a soft $\oplus$-hyperideal of L.

(2) *Let μ be a fuzzy $\otimes$-hyperideal of the hyperlattice $(L, \otimes, \oplus)$. That is, a fuzzy set μ satisfies the following conditions: for all $x, y \in L$,*
(i) $\bigwedge_{z \in x \oplus y} \mu(z) \geq \mu(x) \wedge \mu(y)$, *(ii)* $\bigwedge_{z \in x \otimes y} \mu(z) \geq \mu(x) \vee \mu(y)$.
One can verify that μ is a fuzzy $\otimes$-hyperideal of L if and only if for all $t \in [0, 1]$ with

$\mu_t \neq \phi$, the t-level set $\mu_t = \{x \in L|\mu(x) \geq t\}$ is a $\otimes$-hyperideal of L. Let us consider the family of t-level sets for μ, given by $F(t) = \{x \in L|\mu(x) \geq t\}$, where $t \in [0,1]$. Then for all $t \in [0,1]$, $F(t)$ is a $\otimes$-hyperideal of L. Therefore, $(F,[0,1])$ is a soft $\otimes$-hyperideal of L.

From above examples, we can see that every fuzzy $\oplus$-hyperideal ($\otimes$-hyperideal) can be interpreted as a soft $\oplus$-hyperideal ($\otimes$-hyperideal).

The next example shows that soft hyperideals can be connected to other algebraic structures, for example, lattice theory.

Example 3.4. (1) *Let $(L,\wedge,\vee)$ be a lattice. Define the following hyperoperations on L: for all $a,b \in L$, $a \otimes b = \{x \in L|a \vee x = b \vee x = a \vee b\}$, $a \oplus b = \{x \in L|x \leq a \wedge b\}$. Then $(L,\otimes,\oplus)$ is a hyperlattice. For all a of the lattice L, $I(a)$ denotes the principal ideal generated by a, which means that $I(a) = \{x \in L|x \leq a\} =\downarrow a$, then it is easy to check that $I(a)$ is a $\oplus$-hyperideal of the hyperlattice $(L,\otimes,\oplus)$. We define a map $F: L \to P(L)$ as follows: $F(a) = I(a) =\downarrow a$, for all $a \in L$. Then (F,L) is a soft $\oplus$-hyperideal over L.*

(2) *Let $(L,\wedge,\vee)$ be a lattice. We define two hyperoperations on L: for all $a,b \in L$, $a \otimes b = \{x \in L|a \vee b \leq x\}$, $a \oplus b = \{x \in L|a \wedge x = b \wedge x = a \wedge b\}$. Then $(L,\otimes,\oplus)$ is a hyperlattice. For all a of the lattice L, $F(a)$ denotes the principal filter generated by a, which means that $F(a) = \{x \in L|x \geq a\} =\uparrow a$, then it is easy to check that $F(a)$ is a $\otimes$-hyperideal of the hyperlattice $(L,\otimes,\oplus)$. We define a map $F': L \to P(L)$ as follows: $F'(a) = F(a) =\uparrow a$, for all $a \in L$. Then (F',L) is a soft $\otimes$-hyperideal over L.*

Example 3.5. *Let $L = \{a,b,c,d\}$ and the hyperoperations "$\otimes$" and "$\oplus$" on L be defined as follows:*

$\otimes$	a	b	c	d
a	{a}	{a}	{a}	{a}
b	{a}	{a,b}	{a}	{a,b}
c	{a}	{a}	{c}	{c}
d	{a}	{a,b}	{c}	{c,d}

$\oplus$	a	b	c	d
a	{a,b}	{b}	{c,d}	{d}
b	{b}	{b}	{d}	{d}
c	{c,d}	{d}	{c,d}	{d}
d	{d}	{d}	{d}	{d}

Then $(L,\otimes,\oplus)$ is a hyperlattice. One can check that $\{a,b\}$ and $\{c,d\}$ are a $\otimes$-hyperideal and a $\oplus$-hyperideal, respectively.

(1) *Consider the set $A = \{x,y\}$ and the map $F: A \to P(L)$ defined by $F(x) = \{a,b\}$, $F(y) = L$, then (F,A) is a soft $\otimes$-hyperideal of L.*

(2) *Consider the set $A = \{x,y\}$ and the map $F: A \to P(L)$ defined by $F(x) = \{c,d\}$, $F(y) = L$, then (F,A) is a soft $\oplus$-hyperideal of L.*

4. **Fuzzy soft hyperideals in hyperlattices**

In this section, we introduce fuzzy soft hyperideals, and study some properties of them.

Definition 4.1. *Let $(L, \otimes, \oplus)$ be a hyperlattice and (f, A) be a fuzzy soft set over L.*
(1) (f, A) is called a fuzzy soft $\oplus$-hyperideal over L if for all $a \in A$ and $x, y \in L$,
(i) $\bigwedge_{z \in x \otimes y} f_a(z) \geq f_a(x) \wedge f_a(y)$, (ii) $\bigwedge_{z \in x \oplus y} f_a(z) \geq f_a(x) \vee f_a(y)$.
That is, for each $a \in A$, f_a is a fuzzy $\oplus$-hyperideal of L.
(2) (f, A) is called a fuzzy soft $\otimes$-hyperideal over L if for all $a \in A$ and $x, y \in L$,
(i) $\bigwedge_{z \in x \oplus y} f_a(z) \geq f_a(x) \wedge f_a(y)$, (ii) $\bigwedge_{z \in x \otimes y} f_a(z) \geq f_a(x) \vee f_a(y)$.
That is, for each $a \in A$, f_a is a fuzzy $\otimes$-hyperideal of L.

Next, let us illustrate this definition by the following examples.

Example 4.1. *Since each soft set can be considered as a fuzzy soft set and each characteristic function of a $\oplus$-hyperideal in Definition 3.4 is a fuzzy $\oplus$-hyperideal of the hyperlattice, each soft $\oplus$-hyperideal is a fuzzy soft $\oplus$-hyperideal. Similarly, each soft $\otimes$-hyperideal is a fuzzy soft $\otimes$-hyperideal.*

Example 4.2. *A fuzzy soft $\oplus$-hyperideal (f, A), for which A is a singleton, is a fuzzy $\oplus$-hyperideal in Example 3.6, hence a fuzzy $\oplus$-hyperideal is a particular type of fuzzy soft $\oplus$-hyperideals. In a similar way, a fuzzy $\otimes$-hyperideal is a particular type of fuzzy soft $\otimes$-hyperideals.*

From above two examples, we can see that fuzzy soft $\oplus$-hyperideals ($\otimes$-hyperideals) generalize soft $\oplus$-hyperideals and fuzzy $\oplus$-hyperideals ($\otimes$-hyperideals).

In the following example, we give a fuzzy soft hyperideal, which is neither a fuzzy hyperideal nor a soft hyperideal.

Example 4.3. *Let $(L, \otimes, \oplus)$ be the hyperlattice in Example 3.8. Set $A = \{\alpha, \beta\}$.*
(1) Let (f, A) be a fuzzy soft set on L, where fuzzy sets f_α and f_β are as follows:

$$f_\alpha(x) = \begin{cases} 0.8, x \in \{a, b\} \\ 0.4, x \in \{c, d\} \end{cases}, \ f_\beta(x) = \begin{cases} 0.6, x \in \{a, b\} \\ 0.3, x \in \{c, d\} \end{cases}.$$

Then (f, A) is a fuzzy soft $\otimes$-hyperideal over L.
(2) Let (f, A) be a fuzzy soft set on L, where fuzzy sets f_α and f_β are as follows:

$$f_\alpha(x) = \begin{cases} 0.5, x \in \{a, b\} \\ 0.7, x \in \{c, d\} \end{cases}, \ f_\beta(x) = \begin{cases} 0.2, x \in \{a, b\} \\ 0.4, x \in \{c, d\} \end{cases}.$$

Then (f, A) is a fuzzy soft $\oplus$-hyperideal over L.

In what follows, we shall investigate some properties of fuzzy soft hyperideals.

Proposition 4.1. *Let (f, A) and (g, B) be two fuzzy soft $\oplus$-hyperideals ($\otimes$-hyperideals) over the hyperlattice $(L, \otimes, \oplus)$. Then $(f, A) \sqcap (g, B)$ is a fuzzy soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L.*

Proof. By Definition 2.4, we can write $(f, A) \sqcap (g, B) = (h, C)$, where $C = A \cap B$ and $h_c = f_c \cap g_c$, that is, $h_c(x) = f_c(x) \wedge g_c(x)$ for all $c \in C$ and $x \in L$.

Now, suppose that (f, A) and (g, B) are two fuzzy soft $\oplus$-hyperideals over the hyperlattice $(L, \otimes, \oplus)$. If any $x, y \in L$ and $z \in x \otimes y$, for all $c \in C$, we have $h_c(z) = f_c(z) \wedge g_c(z) \geq (f_c(x) \wedge f_c(y)) \wedge (g_c(x) \wedge g_c(y)) \geq (f_c(x) \wedge g_c(x)) \wedge (f_c(y) \wedge g_c(y)) = h_c(x) \wedge h_c(y)$. Then we obtain $\bigwedge_{z \in x \otimes y} h_c(z) \geq h_c(x) \wedge h_c(y)$ for all $c \in C$.

On the other hand, for all $z \in x \oplus y$ and $c \in C$, we have $h_c(z) = f_c(z) \wedge g_c(z) \geq (f_c(x) \vee f_c(y)) \wedge (g_c(x) \vee g_c(y)) = (f_c(x) \wedge g_c(x)) \vee (f_c(y) \wedge g_c(y)) = h_c(x) \vee h_c(y)$, which implies $\bigwedge\limits_{z \in x \oplus y} h_c(z) \geq h_c(x) \vee h_c(y)$ for all $c \in C$. Therefore, $(f, A) \sqcap (g, B)$ is a fuzzy soft $\oplus$-hyperideal over L.

The case for $\otimes$-hyperideals can be similarly proved. $\square$

Proposition 4.2. *Let (f, A) and (g, B) be two fuzzy soft $\oplus$-hyperideals ($\otimes$-hyperideals) over the hyperlattice $(L, \otimes, \oplus)$. Then $(f, A)\tilde{\cap}(g, B)$ is a fuzzy soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L.*

Proof. Suppose that (f, A) and (g, B) are two fuzzy soft $\oplus$-hyperideals over the hyperlattice $(L, \otimes, \oplus)$.

By Definition 2.5, we can write $(f, A)\tilde{\cap}(g, B) = (h, C)$, where $C = A \cup B$ and

$$h_c = \begin{cases} f_c, \ if \ c \in A - B \\ g_c, \ if \ c \in B - A \\ f_c \cap g_c, \ if \ c \in A \cap B \end{cases}, \text{ for all } c \in C.$$

Now, for all $c \in C$ and $x, y \in L$, we consider the following cases.

Case 1: $c \in A - B$, then $h_c = f_c$. Since (f, A) is a fuzzy soft $\oplus$-hyperideal over the hyperlattice $(L, \otimes, \oplus)$, h_c is a fuzzy soft $\oplus$-hyperideal over $(L, \otimes, \oplus)$.

Case 2: $c \in B - A$, then $h_c = g_c$. Analogous to the proof of Case 1, we have h_c is a fuzzy soft $\oplus$-hyperideal over $(L, \otimes, \oplus)$.

Case 3: $c \in A \cap B$, then $h_c = f_c \cap g_c$. Similar to the proof of Proposition 4.5, it follows that h_c is a fuzzy soft $\oplus$-hyperideal over $(L, \otimes, \oplus)$.

Combining the above arguments, we have that $(f, A)\tilde{\cap}(g, B)$ is a fuzzy soft $\oplus$-hyperideal over L.

The case for $\otimes$-hyperideals can be similarly proved. $\square$

Proposition 4.3. *Let (f, A) and (g, B) be two fuzzy soft $\oplus$-hyperideals ($\otimes$-hyperideals) over the hyperlattice $(L, \otimes, \oplus)$. Then $(f, A)\tilde{\wedge}(g, B)$ is a fuzzy soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L.*

Proof. By Definition 2.6, we denote $(f, A)\tilde{\wedge}(g, B) = (h, A \times B)$. We know that for all $a \in A, b \in B$, f_a and g_b are fuzzy $\oplus$-hyperideals of L and so is $h(a, b) = f_a \cap g_b$, for $(a, b) \in A \times B$, because intersection of two fuzzy $\oplus$-hyperideals is also a fuzzy $\oplus$-hyperideal. Therefore, $(h, A \times B) = (f, A)\tilde{\wedge}(g, B)$ is a fuzzy soft $\oplus$-hyperideal over L. Similarly, we can prove that $(h, A \times B) = (f, A)\tilde{\wedge}(g, B)$ is a fuzzy soft $\otimes$-hyperideal over L. $\square$

Proposition 4.4. *Let (f, A) and (g, B) be two fuzzy soft $\oplus$-hyperideals ($\otimes$-hyperideals) over the hyperlattice $(L, \otimes, \oplus)$. If for all $a \in A$ and $b \in B$, $f_a \subseteq g_b$ or $g_b \subseteq f_a$, then $(f, A)\tilde{\vee}(g, B)$ is a fuzzy soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L.*

Proof. By Definition 2.7, we can write $(\tilde{O}, C) = (f, A)\tilde{\vee}(g, B)$, where $C = A \times B$. For all $(a, b) \in C$, we have $\tilde{O}(a, b) = f_a \cup g_b$. By hypothesis, for all $(a, b) \in C$, $f_a \subseteq g_b$ or $g_b \subseteq f_a$. Now, we assume that $f_a \subseteq g_b$.

For any $x, y \in L$ and $z \in x \otimes y$, we have $\tilde{O}(a, b)(z) = f_a(z) \vee g_b(z) = g_b(z) \geq g_b(x) \wedge g_b(y) = (f_a(x) \vee g_b(x)) \wedge (f_a(y) \vee g_b(y)) = \tilde{O}(a, b)(x) \wedge \tilde{O}(a, b)(y)$. Then we obtain $\bigwedge\limits_{z \in x \otimes y} \tilde{O}(a, b)(z) \geq \tilde{O}(a, b)(x) \wedge \tilde{O}(a, b)(y)$.

On the other hand, for all $z \in x \oplus y$, we have $\tilde{O}(a,b)(z) = f_a(z) \vee g_b(z) = g_b(z) \geq g_b(x) \vee g_b(y) = (f_a(x) \vee g_b(x)) \vee (f_a(y) \vee g_b(y)) = \tilde{O}(a,b)(x) \vee \tilde{O}(a,b)(y)$. Hence, $\bigwedge_{z \in x \oplus y} \tilde{O}(a,b)(z) \geq \tilde{O}(a,b)(x) \vee \tilde{O}(a,b)(y)$. Therefore, $(f,A)\tilde{\vee}(g,B)$ is a fuzzy soft $\oplus$-hyperideal over L.

The case for $\otimes$-hyperideals can be similarly proved. □

Definition 4.2. *Let (f,A) be a fuzzy soft set over L. The soft sets $(f,A)_t = \{(f_a)_t : a \in A\}$ for all $t \in [0,1]$ and $(f,A)_{(t)} = \{(f_a)_{(t)} : a \in A\}$ for all $t \in [0,1)$, are called the t-level soft set and strong t-level soft set of the fuzzy soft set (f,A), respectively, where $(f_a)_t$ and $(f_a)_{(t)}$ are the t-level set and strong t-level set of the fuzzy set f_a, respectively.*

Next, we can characterize fuzzy soft hyperideals by level soft hyperideals.

Theorem 4.1. *Let (f,A) be a fuzzy soft set over the hyperlattice $(L,\otimes,\oplus)$. Then (f,A) is a fuzzy soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L if and only if for all $a \in A$ and $t \in [0,1]$ with $(f_a)_t \neq \phi$, the t-level soft set $(f,A)_t$ is a soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L.*

Proof. $\Longrightarrow$ Let (f,A) be a fuzzy soft $\oplus$-hyperideal over the hyperlattice $(L,\otimes,\oplus)$. Then for all $a \in A$, f_a is a fuzzy $\oplus$-hyperideal of L. For $t \in [0,1]$ with $(f_a)_t \neq \phi$, let $x, y \in (f_a)_t$, then $f_a(x) \geq t$ and $f_a(y) \geq t$. Hence, for all $z \in x \otimes y$, we have $f_a(z) \geq f_a(x) \wedge f_a(y) \geq t \wedge t = t$, that is, $z \in (f_a)_t$, which implies $x \otimes y \subseteq (f_a)_t$. On the other hand, let $b \in L$, for all $z \in b \oplus x$, we have $f_a(z) \geq f_a(b) \vee f_a(x) \geq t$, that is, $z \in (f_a)_t$, which implies $b \oplus x \subseteq (f_a)_t$. Then we obtain that $(f_a)_t$ is a $\oplus$-hyperideal of L, for all $a \in A$. Therefore, $(f,A)_t$ is a soft $\oplus$-hyperideal over L.

$\Longleftarrow$ For all $x, y \in L, a \in A$, let $\alpha = f_a(x) \wedge f_a(y)$, then we have $f_a(x) \geq \alpha, f_a(y) \geq \alpha$, which implies $x, y \in (f_a)_\alpha$. Since $(f_a)_\alpha$ is a $\oplus$-hyperideal of L, then $x \otimes y \subseteq (f_a)_\alpha$. Hence, for all $z \in x \otimes y$, we have $z \in (f_a)_\alpha$. Thus we can obtain $f_a(z) \geq \alpha = f_a(x) \wedge f_a(y)$, which implies $\bigwedge_{z \in x \otimes y} f_a(z) \geq f_a(x) \wedge f_a(y)$. On the other hand, let $\beta = f_a(x)$, then we have $f_a(x) \geq \beta$, that is, $x \in (f_a)_\beta$ and $\beta \in [0,1]$. Then for all $b \in L$, $b \oplus x \subseteq (f_a)_\beta$. Hence, for all $z \in b \oplus x$, we have $z \in (f_a)_\beta$. Thus, we have $f_a(z) \geq \beta = f_a(x)$. Similarly, $f_a(z) \geq f_a(b)$, which implies $\bigwedge_{z \in b \oplus x} f_a(z) \geq f_a(b) \vee f_a(x)$. Therefore, (f,A) is a fuzzy soft $\oplus$-hyperideal over L.

The case for $\otimes$-hyperideals can be similarly proved. □

Theorem 4.2. *Let (f,A) be a fuzzy soft set over the hyperlattice $(L,\otimes,\oplus)$. Then (f,A) is a fuzzy soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L if and only if for all $a \in A$ and $t \in [0,1)$ with $(f_a)_{(t)} \neq \phi$, the strong t-level soft set $(f,A)_{(t)}$ is a soft $\oplus$-hyperideal ($\otimes$-hyperideal) over L.*

Proof. $\Longrightarrow$ The proof is similar to that of the necessity in Theorem 4.10.

$\Longleftarrow$ Now, assume that (f,A) is not a fuzzy soft $\oplus$-hyperideal over L. Then there exists $a \in A$ such that f_a is not a fuzzy $\oplus$-hyperideal of L. That is, there exists $x_0, y_0 \in L$, such that $\bigwedge_{z \in x_0 \otimes y_0} f_a(z) < f_a(x_0) \wedge f_a(y_0)$ or $\bigwedge_{z \in x_0 \oplus y_0} f_a(z) < f_a(x_0) \vee f_a(y_0)$.

Now, we consider the following cases.

(i) If $\bigwedge_{z\in x_0\otimes y_0} f_a(z) < f_a(x_0)\wedge f_a(y_0)$, let $t = \bigwedge_{z\in x_0\otimes y_0} f_a(z)$. Then there exists $z_0 \in x_0\otimes y_0$ such that $f_a(z_0) = t$. Hence $t = f_a(z_0) < f_a(x_0)\wedge f_a(y_0)$. Then we get $x_0, y_0 \in (f_a)_{(t)}$, but $z_0 \notin (f_a)_{(t)}$. Thus, we obtain $x_0\otimes y_0 \nsubseteq (f_a)_{(t)}$.

(ii) If $\bigwedge_{z\in x_0\oplus y_0} f_a(z) < f_a(x_0)\vee f_a(y_0)$, let $t = \bigwedge_{z\in x_0\oplus y_0} f_a(z)$. Then there exists $z_0 \in x_0\oplus y_0$ such that $f_a(z_0) = t$. Hence $t = f_a(z_0) < f_a(x_0)\vee f_a(y_0)$. Then we have $t\in[0,1)$ and $f_a(x_0) > t$ or $f_a(y_0) > t$, that is, $x_0 \in (f_a)_{(t)}$ or $y_0 \in (f_a)_{(t)}$, but $z_0 \notin (f_a)_{(t)}$. Thus, we also obtain $x_0\oplus y_0 \nsubseteq (f_a)_{(t)}$.

Thus, results in case (i) and (ii) contradict the fact that $(f, A)_{(t)}$ is a soft $\oplus$-hyperideal over L. Therefore, (f, A) is a fuzzy soft $\oplus$-hyperideal over L.

The case for $\otimes$-hyperideals can be similarly proved. □

Let $(L_1, \otimes_1, \oplus_1)$ and $(L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A map $f : L_1 \longrightarrow L_2$ is said to be a homomorphism if $f(a\otimes_1 b) = f(a)\otimes_2 f(b)$ and $f(a\oplus_1 b) = f(a)\oplus_2 f(b)$ for all $a, b \in L_1$.

If such a homomorphism f is surjective, injective or bijective, then f is called an epimorphism, a monomorphism or an isomorphism from $(L_1, \otimes_1, \oplus_1)$ to $(L_2, \otimes_2, \oplus_2)$, respectively.

Next, we shall study the image and pre-image of fuzzy soft hyperideals under the fuzzy soft homomorphism of hyperlattices. We start by recalling the following concept, refer to [22].

Let (f, A) and (g, B) be two fuzzy soft sets over L_1 and L_2, respectively, $\varphi : L_1 \to L_2$, $\psi : A \to B$ be two functions. Then the pair (φ, ψ) is said to be a fuzzy soft function from L_1 to L_2.

Definition 4.3. *Let (f, A) and (g, B) be two fuzzy soft sets over L_1 and L_2, respectively and let (φ, ψ) be a fuzzy soft function from L_1 to L_2.*

(1) The image of (f, A) under (φ, ψ), denoted by $(\varphi, \psi)(f, A)$, is a fuzzy soft set over L_2 defined by $(\varphi, \psi)(f, A) = (\varphi(f), \psi(A))$, where

$$\varphi(f)_k(y) = \begin{cases} \bigvee_{\varphi(x)=y}\ \bigvee_{\psi(a)=k} f_a(x), & if x \in \varphi^{-1}(y) \\ 0, & otherwise \end{cases} \quad \text{for all } k \in \psi(A),\ y \in L_2.$$

(2) The pre-image of (g, B) under (φ, ψ), denoted by $(\varphi, \psi)^{-1}(g, B)$, is a fuzzy soft set over L_1, defined by $(\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))$, where $\varphi^{-1}(g)_a(x) = g_{\psi(a)}(\varphi(x))$, for all $a \in \psi^{-1}(B)$, $x \in L_1$.

In what follows, we give the notion of fuzzy soft homomorphisms of hyperlattices.

Definition 4.4. *Let (f, A) and (g, B) be two fuzzy soft sets over the hyperlattice L_1 and the hyperlattice L_2, respectively. Let (φ, ψ) be a fuzzy soft function from L_1 to L_2. If φ is a homomorphism from L_1 to L_2, then (φ, ψ) is said to be a fuzzy soft homomorphism from L_1 to L_2.*

Theorem 4.3. *Let (φ, ψ) be a fuzzy soft homomorphism from the hyperlattice $(L_1, \otimes_1, \oplus_1)$ to the hyperlattice $(L_2, \otimes_2, \oplus_2)$. If (f, A) is a fuzzy soft $\oplus_1$-hyperideal ($\otimes_1$-hyperideal) over L_1, then $(\varphi, \psi)(f, A)$ is a fuzzy soft $\oplus_2$-hyperideal ($\otimes_2$-hyperideal) over L_2.*

Proof. Let $k \in \psi(A)$ and $y_1, y_2 \in L_2$. If $\varphi^{-1}(y_1) = \phi$ or $\varphi^{-1}(y_2) = \phi$, the proof is straightforward. We only verify the case when $\varphi^{-1}(y_1) \neq \phi$ and $\varphi^{-1}(y_2) \neq \phi$. Suppose that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Since (f, A) is a fuzzy soft $\oplus_1$-hyperideal over L_1, it follows that $f_a(x_1) \wedge f_a(x_2) \leq f_a(t)$, for all $t \in x_1 \otimes_1 x_2$. Let $z \in y_1 \otimes_2 y_2 = \varphi(x_1 \otimes_1 x_2)$, we obtain $z = \varphi(t)$, for some $t \in x_1 \otimes_1 x_2$. Then we have $(\bigvee_{\varphi(x_1)=y_1} f_a(x_1)) \wedge (\bigvee_{\varphi(x_2)=y_2} f_a(x_2)) \leq \bigvee_{\varphi(x_1)=y_1} \bigvee_{\varphi(x_2)=y_2} f_a(t)$, hence $(\varphi(f)_k(y_1)) \wedge (\varphi(f)_k(y_2)) \leq \bigvee_{\psi(a)=k} \bigvee_{\varphi(x_1)=y_1} \bigvee_{\varphi(x_2)=y_2} f_a(t) = \bigvee_{\psi(a)=k} \bigvee_{\varphi(t)=z} f_a(t) = \varphi(f)_k(z)$. Then we have $\bigwedge_{z \in y_1 \otimes_2 y_2} \varphi(f)_k(z) \geq (\varphi(f)_k(y_1)) \wedge (\varphi(f)_k(y_2))$, for all $k \in \psi(A)$. Similarly, we can obtain that $\bigwedge_{z \in y_1 \oplus_2 y_2} \varphi(f)_k(z) \geq (\varphi(f)_k(y_1)) \vee (\varphi(f)_k(y_2))$, for all $k \in \psi(A)$. Therefore, $(\varphi, \psi)(f, A)$ is a fuzzy soft $\oplus_2$-hyperideal over L_2.

In a similar way, we have that $(\varphi, \psi)(f, A)$ is a fuzzy soft $\otimes_2$-hyperideal over L_2. □

Theorem 4.4. *Let (φ, ψ) be a fuzzy soft homomorphism from the hyperlattice $(L_1, \otimes_1, \oplus_1)$ to the hyperlattice $(L_2, \otimes_2, \oplus_2)$. If (g, B) is a fuzzy soft $\oplus_2$-hyperideal ($\otimes_2$-hyperideal) over L_2, then $(\varphi, \psi)^{-1}(g, B)$ is a fuzzy soft $\oplus_1$-hyperideal ($\otimes_1$-hyperideal) over L_1.*

Proof. Let $a \in \psi^{-1}(B)$ and $x_1, x_2 \in L_1$, $z \in x_1 \otimes_1 x_2$. Suppose that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Since (g, B) is a fuzzy soft $\oplus_2$-hyperideal over L_2, we have $\varphi^{-1}(g)_a(x_1) \wedge \varphi^{-1}(g)_a(x_2) = g_{\psi(a)}(\varphi(x_1)) \wedge g_{\psi(a)}(\varphi(x_2)) = g_{\psi(a)}(y_1) \wedge g_{\psi(a)}(y_2) \leq g_{\psi(a)}(t)$, for all $t \in y_1 \otimes_2 y_2 = \varphi(x_1 \otimes_1 x_2)$. Hence, for all $z \in x_1 \otimes_1 x_2$, $\varphi^{-1}(g)_a(x_1) \wedge \varphi^{-1}(g)_a(x_2) \leq g_{\psi(a)}(\varphi(z)) = \varphi^{-1}(g)_a(z)$, that is, $\bigwedge_{z \in x_1 \otimes_1 x_2} \varphi^{-1}(g)_a(z) \geq \varphi^{-1}(g)_a(x_1) \wedge \varphi^{-1}(g)_a(x_2)$. Similarly, we obtain $\bigwedge_{z \in x_1 \oplus_1 x_2} \varphi^{-1}(g)_a(z) \geq (\varphi^{-1}(g)_a(x_1)) \vee (\varphi^{-1}(g)_a(x_2))$. Therefore, $(\varphi, \psi)^{-1}(g, B)$ is a fuzzy soft $\oplus_1$-hyperideal over L_1.

Similarly, we can prove that $(\varphi, \psi)^{-1}(g, B)$ is a fuzzy soft $\otimes_1$-hyperideal over L_1. □

5. Conclusion

In this paper, we apply the notion of soft sets and fuzzy soft sets to the theory of hyperlattices. We introduce soft hyperideals and fuzzy soft hyperideals, and study some properties of them. This study is just at the begining and it can be continuated in many directions:

(1) to do some further work on the properties of fuzzy soft hyperideals, which may be useful to characterize the structure of hyperlattices;

(2) to study the construction the quotient hyperlattices in the mean of fuzzy soft structures and fuzzy soft isomorphism theorems of hyperlattices;

(3) to apply the fuzzy soft set theory of hyperlattices to some applied fields, such as decision making, data analysis and forecasting and so on.

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