g-ORTHONORMAL BASES, g-RIESZ BASES AND g-DUAL OF g-FRAMES

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The duals of frames, duals of g-frames, and orthonormal bases are widely used in mathematics, physics, signal processing, and many other areas, in which elements need to be represented in terms of the frame, g-frame, and basis elements. In this paper, we give equivalent conditions for a g-orthonormal basis and characterize all g-Riesz bases for a separable Hilbert space $\mathcal{H}$, starting with a given g-orthonormal basis. Also, we introduce the notion of generalized dual of a g-frame in a separable Hilbert space $\mathcal{H}$ and provide some characterizations of the generalized dual of g-frames. Our results generalize and improve the results obtained by Dehgan and Hasankhani Fard in [1].

Keywords: g-frame; g-orthonormal basis; g-Riesz basis; g-dual g-frame.
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1. Introduction

Frames for Hilbert space were formally defined by Duffin and Schaeffer [5] in 1952 in order to study some problems in non-harmonic Fourier series. These frames were improved in 1986 by Daubechies, Grossman and Meyer [3]. Wenchang Sun [7] developed a generalization of frame (g-frame) including more other concepts and proved that many basic properties can be derived from this more general context and he also presented a generalization of orthonormal bases or simply g-orthonormal bases. In this paper, we define the notion of g-orthonormal system and give some equivalent conditions for a g-orthonormal system $\{\xi_j\}_{j\in J}$ to be a g-orthonormal basis. M. A. Dehghan and M. A. Hasankhani Fard [1] determined and characterized g-duals of a frame in a separable Hilbert space $\mathcal{H}$ and obtained more reconstruction formulas of vectors (or signals) in terms of the frame elements by using the g-dual frames. In this work, we introduce the generalized dual of generalized frames or simply g-dual of g-frames and extend the reconstruction formulas of vectors in terms of the adjoint of g-frame elements using the g-dual of g-frames. Then, some properties are presented.

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2. **Preliminary Notes**

Below, we will briefly recall some definitions and basic properties of g-frames in Hilbert spaces. We first give some notations which are needed later. Throughout this paper, J is a finite or a countable index set. $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces and $\{\mathcal{K}_j\}_{j\in J}$ is a sequence of closed Hilbert subspaces of $\mathcal{K}$. For each $j \in J$, $\mathcal{B}(\mathcal{H}, \mathcal{K}_j)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}_j$. We also denote:

$$\bigoplus_{j\in J} \mathcal{K}_j = \{g = \{g_j\} : g_j \in \mathcal{K}_j \quad \text{and} \quad \sum_{j\in J} |\langle g_j, g_j\rangle| < \infty\}.$$

Definition 2.1. *We call a sequence $\{\Lambda_j \in \mathcal{B}(\mathcal{H}, \mathcal{K}_j) : j \in J\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ if there exist two positive constants C and D such that:*

$$C\|f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le D\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1}$$

We call C and D the lower and upper g-frame bounds, respectively. If only the right-hand inequality of (1) is satisfied, we call $\{\Lambda_j\}_{j\in J}$ the g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ with g-Bessel bound D. If $C = D = \lambda$, we call $\{\Lambda_j\}_{j\in J}$ the λ-tight g-frame. Moreover, if $\lambda = 1$, we call $\{\Lambda_j\}_{j\in J}$ the Parseval g-frame.

The bounded linear operator T_Λ defined by:

$$T_\Lambda : \bigoplus_{j\in J} \mathcal{K}_j \longrightarrow \mathcal{H}, \qquad T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j,$$

is called the pre-frame operator of $\{\Lambda_j\}_{j\in J}$. Also, the bounded linear operator S_Λ defined by:

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S_\Lambda(f) = \sum_{j\in J} \Lambda_j^* \Lambda_j f,$$

is called the frame operator of $\{\Lambda_j\}_{j\in J}$.

Example 2.1. *If $\mathcal{K}_j$=$\mathbb{C}$ for any $j \in J$ and $\Lambda_j f = \langle f, f_j\rangle$ for any $f \in \mathcal{H}$, in this case, the g-frame is just a frame for $\mathcal{H}$.*

3. g-**orthonormal basis and** g-**Riesz basis**

In this section, we give equivalent conditions for a g-orthonormal basis and characterize all g-Riesz bases for a separable Hilbert space $\mathcal{H}$, starting with one g-orthonormal basis.

Definition 3.1. *A g-sequence $\{\xi_j \in \mathcal{B}(\mathcal{H}, \mathcal{K}_j) : j \in J\}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ is said to be a g-orthonormal basis if it satisfies the following cases:*

(i) $\langle \xi_j^* g_j, \xi_k^* f_k\rangle = \delta_{jk}\langle g_j, f_k\rangle$, $\qquad (j, k \in J), \quad (g_j \in \mathcal{K}_j, f_k \in \mathcal{K}_k)$.

(ii) $\|f\|^2 = \sum_{j\in J} \|\xi_j f\|^2$, $\qquad (f \in \mathcal{H})$.

If only the condition (i) is satisfied, we call $\{\xi_j\}_{j\in J}$ the g-orthonormal system for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. Note that a g-orthonormal system $\{\xi_j\}_{j\in J}$ is a g-Bessel sequence. In fact, if $\{g_j\}_{j\in J}\in\bigoplus_{j\in J}\mathcal{K}_j$ and $m,n\in J,\ \ n>m$, then:

$$\|\sum_{k=1}^{n}\xi_k^*g_k-\sum_{k=1}^{m}\xi_k^*g_k\|^2=\langle\sum_{k=m+1}^{n}\xi_k^*g_k,\sum_{k=m+1}^{n}\xi_k^*g_k\rangle$$
$$=\sum_{k=m+1}^{n}\sum_{j=m+1}^{n}\delta_{jk}\langle g_k,g_j\rangle=\sum_{k=m+1}^{n}\|g_k\|^2.$$

Since $\{g_j\}_{j\in J}\in\bigoplus_{j\in J}\mathcal{K}_j$, we know that $\{\sum_{k=1}^{n}\|g_k\|^2\}_{j\in J}$ is a Cauchy sequence in $\mathbb{C}$. The above calculation shows that $\{\sum_{k=1}^{n}\xi_k^*g_k\}_{j\in J}$ is a Cauchy sequence in $\mathcal{H}$ and therefore convergent. The next theorem gives equivalent conditions for a g-orthonormal system $\{\xi_j\}_{j\in J}$ to be a g-orthonormal basis.

Theorem 3.1. *For a g-orthonormal system $\{\xi_j\}_{j\in J}$, the following cases are equivalent:*

(i) $\{\xi_j\}_{j\in J}$ *is a g-orthonormal basis.*

(ii) $f=\sum_{j\in J}\xi_j^*\xi_jf$, *for all* $f\in\mathcal{H}$.

(iii) $\langle f,g\rangle=\sum_{j\in J}\langle\xi_jf,\xi_jg\rangle$, *for all* $f,g\in\mathcal{H}$.

(iv) $\overline{\mathrm{Im}T_\xi}=\mathcal{H}$.

(v) If $\xi_jf=0$ *for all* $j\in J$, *then* $f=0$.

Proof. For the proof of $(i)\Rightarrow(ii)$, let $f\in\mathcal{H}$ then we have:

$$\langle f,f\rangle=\|f\|^2=\sum_{j\in J}\|\xi_jf\|^2=\sum_{j\in J}\langle\xi_jf,\xi_jf\rangle=\langle\sum_{j\in J}\xi_j^*\xi_jf,f\rangle$$

therefore $f=\sum_{j\in J}\xi_j^*\xi_jf$. (iii) is an obvious consequence of (ii). For the proof of $(iii)\Rightarrow(iv)$, let $W=\overline{\mathrm{Im}T_\xi}$ and $W\neq\mathcal{H}$; there exist $f\in\mathcal{H}$ and $f\neq0$ such that $f\perp W$. So, for $T_\xi(\{\xi_jf\}_{j\in J})\in W$, we have:

$$0=\langle T_\xi(\{\xi_jf\}_{j\in J}),f\rangle=\sum_{j\in J}\langle\xi_jf,\xi_jf\rangle=\langle f,f\rangle.$$

This contradiction shows that $\overline{\mathrm{Im}T_\xi}=\mathcal{H}$. To prove $(iv)\Rightarrow(v)$, note that by assuming $T_\xi^*f=0$ and $\overline{\mathrm{Im}T_\xi}=\mathcal{H}$ we infer $\ker T_\xi^*=\overline{\mathrm{Im}T_\xi}^{\perp}=0$. Therefore, $f=0$. For the proof of $(v)\Rightarrow(i)$, let $f\in\mathcal{H}$. Since $\{\xi_j\}_{j\in J}$ is a g-Bessel sequence, we know that $g:=\sum_{j\in J}\xi_j^*\xi_jf$ is well defined; furthermore, for any $k\in J$ and $g_k\in\mathcal{K}_k$, we have:

$$\langle\xi_kg,g_k\rangle=\sum_{j\in J}\langle\xi_j^*\xi_jf,\xi_k^*g_k\rangle=\sum_{j\in J}\delta_{jk}\langle\xi_jf,g_k\rangle=\langle\xi_kf,g_k\rangle.$$

Therefore, $\xi_j(f-g)=0$ for all $j\in J$, so by (v), $f=g=\sum_{j\in J}\xi_j^*\xi_j f$. On the other hand:

$$\|f\|^2=\langle\sum_{j\in J}\xi_j^*\xi_j f,\sum_{k\in J}\xi_k^*\xi_k f\rangle=\sum_{j\in J}\sum_{k\in J}\delta_{jk}\langle\xi_j f,\xi_k f\rangle=\sum_{j\in J}\langle\xi_j f,\xi_j f\rangle,$$

and this completes the proof. □

Definition 3.2. *A g-frame $\{\Lambda_j\in\mathcal{B}(\mathcal{H},\mathcal{K}_j):j\in J\}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ is said to be a g-Riesz basis if it satisfies:*

(i) $\Lambda_j\neq 0$;

(ii) If a linear combination $\sum_{j\in K}\Lambda_j^* g_j$ is equal to zero, then every summand $\Lambda_j^* g_j$ equals zero, where $g_j\in\mathcal{K}_j$ and $K\subseteq J$.

Example 3.1. *As in Example 2.1, the induced functionals of any orthonormal basis (respectively Riesz basis) form a g-orthonormal basis (respectively g-Riesz basis).*

The following Theorem characterizes all g-Riesz bases for $\mathcal{H}$, starting with one g-orthonormal basis.

Theorem 3.2. *A g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ is a family with the form $\{\xi_j U^*\}_{j\in J}$, where $\{\xi_j\}_{j\in J}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ and U is a bounded bijective operator.*

Proof. Let $\{\Lambda_j\}_{j\in J}$ be a g-Riesz basis and $\{\xi_j\}_{j\in J}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. Define the operator:

$$U:\mathcal{H}\longrightarrow\mathcal{H},\quad U(\sum_{j\in J}\xi_j^* g_j)=\sum_{j\in J}\Lambda_j^* g_j,\quad \left(\{g_j\}_{j\in J}\in\bigoplus_{j\in J}\mathcal{K}_j\right).$$

Then, U boundedly and bijectively maps $\mathcal{H}$ onto $\mathcal{H}$. Furthermore, for any $\{g_j\}_{j\in J}\in\bigoplus_{j\in J}\mathcal{K}_j$ and $f\in\mathcal{H}$, we have:

$$\begin{aligned}\langle\{g_j\}_{j\in J},\{\Lambda_j f\}_{j\in J}\rangle&=\left\langle\sum_{j\in J}\Lambda_j^* g_j,f\right\rangle=\left\langle U(\sum_{j\in J}\xi_j^* g_j),f\right\rangle\\&=\langle\{g_j\}_{j\in J},\{\xi_j U^* f\}_{j\in J}\rangle.\end{aligned}$$

So $\{\Lambda_j f\}_{j\in J}=\{\xi_j U^* f\}_{j\in J}$. Conversely, assume that $\sum_{j\in J}U\xi_j^* g_j=0$ for some elements $\{g_j\}\in\bigoplus_{j\in J}\mathcal{K}_j$. Then $\sum_{j\in J}\xi_j^* g_j=0$and for $g_i\in\mathcal{K}_i$, we have:

$$\|g_i\|^2=\langle g_i,g_i\rangle=\sum_{j\in J}\delta_{ij}\langle g_j,g_i\rangle=\langle\sum_{j\in J}\xi_j^* g_j,\xi_i^* g_i\rangle=0.$$

So, every summand $\xi_j^* g_j$ equals zero. □

4. Generalized dual of a g-frame

Dehghan [1] introduced generalized dual frames; however, we will introduce generalized dual of generalized frames or simply g-dual of g-frames.

Definition 4.1. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. A g-frame $\{\Gamma_j\}_{j\in J}$ is called a generalized dual g-frame or g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that, for all $f \in \mathcal{H}$,*

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j A(f). \tag{2}$$

If $\mathcal{W}$ is a closed subspace of $\mathcal{H}$, a g-frame $\{\Gamma_j\}_{j\in J}$ is called a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{W}$ if (2) holds for all $f \in \mathcal{W}$ and for some invertible operator $A \in \mathcal{B}(\mathcal{W})$.

When $A = I$, then $\{\Gamma_j\}_{j\in J}$ is an ordinary dual frame of $\{\Lambda_j\}_{j\in J}$. If S_Λ is the g-frame operator of the g-frame $\{\Lambda_j\}_{j\in J}$, then for all $f \in \mathcal{H}$, we have:

$$f = \sum_{j\in J} \Lambda_j^* \Lambda_j S_\Lambda^{-1}(f).$$

Hence, each g-frame is a g-dual g-frame to itself. The operator A in (2) is unique, since for all $f \in \mathcal{H}$,

$$A^{-1} f = \sum_{j\in J} \Lambda_j^* \Gamma_j(f).$$

Hence, $A^{-1} = T_\Lambda U_\Gamma^*$, where T_Λ and U_Γ are the pre-frame operators of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$, respectively. Also, we say the g-frame $\{\Gamma_j\}_{j\in J}$ is a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ with the corresponding invertible operator A.

Lemma 4.1. *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be g-Bessel sequences in $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. Then the following cases are equivalent:*

(1) There exists an invertible operator $A\in\mathcal{B}(\mathcal{H})$ such that $f = \sum_{j\in J} \Gamma_j^* \Lambda_j A(f)$.

(2) There exists an invertible operator $A\in\mathcal{B}(\mathcal{H})$ such that $f = \sum_{j\in J} \Lambda_j^* \Gamma_j A^*(f)$.

(3) There exists an invertible operator $A\in\mathcal{B}(\mathcal{H})$ such that $\langle f, g\rangle = \sum_{j\in J} \langle \Lambda_j f, \Gamma_j g\rangle$.

In case that one of the equivalent conditions is satisfied, $\{\Gamma_j\}_{j\in J}$ is a g-dual of $\{\Lambda_j\}_{j\in J}$, and vice versa.

Proof. Let (1) be satisfied and let $f \in \mathcal{H}$. Then, there exists $g \in \mathcal{H}$ such that $f = Ag$ and $g = \sum_{j\in J} \Gamma_j^* \Lambda_j A(g)$. Therefore, $f = Ag = \sum_{j\in J} A\Gamma_j^* \Lambda_j(f)$. Since $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j A^*\}_{j\in J}$ are g-Bessel sequences, using Proposition 3.6 [6], we have:

$$f = \sum_{j\in J} A\Gamma_j^* \Lambda_j(f) = \sum_{j\in J} \Lambda_j^* \Gamma_j(A^* f).$$

Hence, (2) holds. A similar argument shows that (2) implies (1). The rest of the proof is obvious. If the conditions are satisfied for $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$, and C_Λ is the g-Bessel bound for $\{\Lambda_j\}_{j\in J}$ then:

$$\|f\|^4 = |\sum_{j\in J}\langle \Lambda_j Af, \Gamma_j f\rangle|^2 \leq C_\Lambda \|A\|^2 \|\|f\|^2 \sum_{j\in J} \|\Gamma_j f\|^2,$$

Therefore, $\{\Gamma_j\}_{j\in J}$ is a g-frame. Since (1) and (2) are equivalent so $\{\Lambda_j\}_{j\in J}$ is also a g-frame. □

Proposition 4.1. *Every two g-Riesz bases are g-dual g-frames.*

Proof. Let$\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two g-Riesz bases for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. There exist an orthonormal basis $\{\xi_j\}_{j\in J}$ and bounded invertible operators U and V on $\mathcal{H}$ such that $\Lambda_j = \xi_j U^*$ and $\Gamma_j = \xi_j V^*$. Since U and V are invertible, there exists a bounded invertible operator T on $\mathcal{H}$ such that $UV^*T = I$; hence, for all $f \in \mathcal{H}$, we have:

$$f = UV^*T(f) = \sum_{j\in J} U\xi_j^*\xi_j V^*(T(f)) = \sum_{j\in J} \Lambda_j^*\Gamma_j(T(f)). \qquad \square$$

Now, we are going to give a simple way for the construction of infinitely many g-dual g-frames of a given g-frame.

Proposition 4.2. *Assume that $\{\Gamma_j\}_{j\in J}$ is a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ with invertible operator $U \in \mathcal{B}(\mathcal{H})$ and let α be a complex number. Then, the sequence $\{\Delta_j\}_{j\in J}$ defined by:*

$$\Delta_j = \alpha\Gamma_j + (1-\alpha)\Lambda_j S_\Lambda^{-1} U^{-1},$$

is a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ with invertible operator U, where S_Λ is the g-frame operator of $\{\Lambda_j\}_{j\in J}$.

Proof. For all $f \in \mathcal{H}$, we have:

$$\begin{aligned}\sum_{j\in J}\Lambda_j^*\Delta_j(Uf) &= \sum_{j\in J}\Lambda_j^*\Big(\alpha\Gamma_j + (1-\alpha)\Lambda_j S_\Lambda^{-1}U^{-1}\Big)(Uf)\\ &= \alpha\sum_{j\in J}\Lambda_j^*\Gamma_j Uf + (1-\alpha)\sum_{j\in J}\Lambda_j^*\Lambda_j S_\Lambda^{-1} f = f. \qquad \square\end{aligned}$$

In the next Proposition, we obtain a g-dual g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ from a g-dual g-frame for a subspace of $\mathcal{H}$.

Proposition 4.3. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ with frame operator S and let $\{\Gamma_j\}_{j\in J}$ be a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{W} = \overline{\mathrm{Im}T_\Gamma}$ with invertible operator $B \in \mathcal{B}(\mathcal{W})$. Then, the sequence $\{\Delta_j\}_{j\in J}$ defined by $\Delta_j = \Gamma_j B_1 + \Lambda_j S_\Lambda^{-1}$ is a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$, where the operator $B_1 : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by $B_1 := BP + Q$ and P and Q are the orthogonal projection onto $\mathcal{W}$ and $\mathcal{W}^\perp$, respectively.*

Proof. Let $A := I - \frac{1}{2}P$, where I denotes the identity operator on $\mathcal{H}$. Since $\|I-A\| < 1$, the operator A is invertible. Let $f \in \mathcal{H}$; then there exist the unique vectors $g \in \mathcal{W}$

and $h \in \mathcal{W}^{\perp}$ such that $f = g + h$. Therefore,

$$\begin{aligned}\sum_{j\in J} \Lambda_j^* \Delta_j A f &= \sum_{j\in J} \Lambda_j^* \Delta_j h + \frac{1}{2} \sum_{j\in J} \Lambda_j^* \Delta_j g \\ &= \sum_{j\in J} \Lambda_j^* \Gamma_j B_1 h + \sum_{j\in J} \Lambda_j^* \Lambda_j S_\Lambda^{-1} h + \frac{1}{2} \sum_{j\in J} \Lambda_j^* \Gamma_j B g + \frac{1}{2} \sum_{j\in J} \Lambda_j^* \Lambda_j S_\Lambda^{-1} g \\ &= T_\Lambda T_\Gamma^* h + h + \frac{1}{2} g + \frac{1}{2} g = f.\end{aligned}$$

Note that $\ker T_\Gamma^* = \overline{\mathrm{Im} T_\Gamma}^{\perp} = \mathcal{W}^{\perp}$. □

Corollary 4.1. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ with g-frame operator S_Λ and let $\{\Gamma_j\}_{j\in J}$ be a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{W} = \overline{\mathrm{Im} T_\Gamma}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. Then, the g-sequence $\{\Delta_j\}_{j\in J}$ defined by:*

$$\Delta_j = \Gamma_j + \Lambda_j S^{-1}$$

is a g-dual g-frame of $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$.

In the following Proposition, a necessary condition for g-duality of the sum of two g-dual g-frames is given.

Proposition 4.4. *Let $\{\Gamma_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ be two g-dual g-frames of $\{\Lambda_j\}_{j\in J}$ with the corresponding invertible operators A and B, respectively. If $A^{-1} + B^{-1}$ is an invertible operator, then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j + \Theta_j\}_{j\in J}$ are g-dual g-frames.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be the inverse operator of $A^{-1} + B^{-1}$. For all $f \in \mathcal{H}$, we have:

$$\begin{aligned}\sum_{j\in J} \Lambda_j^* (\Gamma_j + \Theta_j)(Tf) &= \sum_{j\in J} \Lambda_j^* \Gamma_j A (A^{-1} T f) + \sum_{j\in J} \Lambda_j^* \Theta_j B (B^{-1} T f) \\ &= A^{-1} T f + B^{-1} T f = (A^{-1} + B^{-1}) T f = f.\end{aligned}$$ □

Invertible operators preserve the g-duality property of g-frames.

Proposition 4.5. *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two g-sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$ and let $U, V \in \mathcal{B}(\mathcal{H})$ be two invertible operators on $\mathcal{H}$. Then, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-dual g-frames for $\mathcal{H}$ if and only if $\{\Lambda_j U^*\}_{j\in J}$ and $\{\Gamma_j V^*\}_{j\in J}$ are g-dual g-frames for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$.*

Proof. Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be g-dual g-frames for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$. Then, there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that:

$$f = \sum_{j\in J} \Gamma_j^* \Lambda_j A f,$$

for all $f \in \mathcal{H}$. Hence:

$$f = V V^{-1} f = \sum_{j\in J} V \Gamma_j^* \Lambda_j U^* (U^*)^{-1} A V^{-1} f = \sum_{j\in J} V \Gamma_j^* \Lambda_j U^* B f,$$

where $B = (U^*)^{-1} A V^{-1}$ is an invertible operator on $\mathcal{H}$. The converse is obtained by applying the operators U^{*-1} and V^{*-1}. □

5. **Conclusions**

In this article g-orthonormal systems are independently defined, and we provide some conditions such that a g-orthonormal system will be a g-orthonormal basis. Also we characterize all g-Riesz bases for Hilbert space $\mathcal{H}$, starting with one g-orthonormal basis. In the second part we introduce the notion "the g-dual g-frame" and we obtain some results of g-dual g-frame. As a matter of fact our definition and results are an extension of definition and results obtained in [1, 2].

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