U.P.B. Sci. Bull., Series A, Vol. 75, Iss. 2, 2013 ISSN 1223-7027

LINEAR STABILITY ANALYSIS OF A PLANE-POISEUILLE HYDROMAGNETIC FLOW USING ADOMIAN DECOMPOSITION METHOD

Samuel O. ADESANYA[1]

In this paper, the small-disturbances stability of plane-Poiseuille flow of an electrically conducting fluid in the presence of a transverse magnetic field is studied. Using the mode approach, the fourth order differential equation for the flow is derived and solved analytically using Adomian decomposition method (ADM) and the result obtained showed a good agreement with previously obtained result by Multideck asymptotic techniques.

Keywords: ADM, Magneto-Hydrodynamics, Stability Analysis, Hartmann's Number

1. Introduction

Adomian decomposition method [1, 5, 10-16] was developed in the 80's. It possesses great potential in solving different kinds of differential equations. The major strength of the method is that it avoids linearization, transformation and discretization. The main objective of this paper is to apply Adomian decomposition method (ADM) to study the temporal stability analysis for hydromagnetic Plane – Poiseuille flow.

Now consider the standard operator

$$Lu + Ru + Nu = g\,, \tag{1.1}$$

Where u is the unknown function, L is the highest order derivative, which is assumed easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (1.1) and using the given conditions we obtain

$$u = h(y) - L^{-1}(Ru) - L^{-1}(Nu), \tag{1.2}$$

where $h(y)$ represents the terms arising from integrating the source term g and from the boundary conditions, The standard ADM defines the solution u by the series

$$u = \sum_{n=0}^{\infty} u_n\,, \tag{1.3}$$

moreover, the nonlinear term series

[1]PhD., Department of Mathematical Sciences, Redeemer's University, Redemption City, Nigeria, e-mail: adesanyaolumide@yahoo.com

$$Nu = \sum_{n=0}^{\infty} A_n \ , \tag{1.4}$$

where A_n are the Adomian polynomials determined formally from the relation

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0} . \tag{1.5}$$

If the nonlinear term is expressed as a nonlinear function *f(u),* the Adomian polynomials are arranged into the form

$$\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1 f^{(1)}(u_0) \\ A_2 &= u_2 f^{(1)}(u_0) + \frac{1}{2!}u_1^2 f^{(2)}(u_0) \\ A_3 &= u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{1}{3!}u_1^3 f^{(3)}(u_0) \\ &.... \end{aligned} \tag{1.6}$$

The components $u_0, u_1, u_2, ...$ are then determined recursively by using the relation

$$\begin{cases} u_0 = h(y) \\ u_{k+1} = -L^{-1}Ru_k - L^{-1}A_{k,} \quad k \geq 0 \end{cases}, \tag{1.7}$$

where u_0 is referred to as the zeroth component. The partial sum of the series is thus obtained as

$$u(y) = \sum_{n=0}^{k} u_n(y) \tag{1.8}$$

The convergence results for the Adomian decomposition method has been studied extensively in the works by Cherraultz et al [6, 7 and 8].

2. Mathematical formulation

Consider the flow of an electrically conducting viscous incompressible fluid through a two – dimensional channel under the influence of a transverse magnetic field. The governing equations are [2];

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\text{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - H^2 u \tag{2.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad (2.3)$$

together with the boundary conditions

$$u(\pm 1)=0, v(\pm 1)=0 \quad (2.4)$$

The flow quantities in equation (2.1)-(2.4) have been non-dimensionalized as follows:

$$u=\frac{\bar{u}}{U_0}, v=\frac{\bar{v}}{U_0}, x=\frac{\bar{x}}{a}, y=\frac{\bar{y}}{a}, p=\frac{\bar{p}}{\rho U_0^2}, t=\frac{U_0\bar{t}}{a}, \mathrm{Re}=\frac{U_0 a}{\nu}, H=\sqrt{\frac{\sigma_e B_0^2 a}{\rho U_0}} \quad (2.5)$$

Where *x* and *y* are the streamwise and normal coordinates respectively, *u* and *v* are the streamwise and normal velocity respectively, *t*-time, *p* – pressure, *Re* – Reynolds number and *H* is the Hartmann's number, U_0 is the characteristic velocity of the fluid, *a* is the characteristic half width of the channel, υ is the kinematic fluid viscosity and ρ is the fluid density, σ_e is the fluid electrical conductivity.

The equation and the boundary conditions for the basic flow are

$$\frac{d^2U}{dy^2}-H^2U=-A;\ U(\pm 1)=0 \quad (2.6)$$

By Adomian decomposition, equation (2.6) admits a series solution of the form

$$U(y)=\sum_{n=0}^{\infty}U_n(y) \quad (2.7)$$

using (2.7) in (2.6) we obtain the zeroth component as

$$U_0(y)=a_0-\int_0^y\int_0^y A\,dy\,dy \quad (2.8)$$

While other components can be easily obtained using the recursive relation

$$U_{n+1}(y)=\int_0^y\int_0^y H^2U_n\,dy\,dy \quad (2.9)$$

obtaining few terms of (2.9) we get

$$U_0(y) = a_0 - \frac{A}{2}y^2$$
$$U_1(y) = a_0 H^2 \frac{y^2}{2} - AH^2 \frac{y^4}{4!}$$
$$U_2(y) = a_0 H^4 \frac{y^4}{4!} - AH^4 \frac{y^6}{6!} \tag{2.10}$$
$$U_3(y) = a_0 H^6 \frac{y^6}{6!} - AH^6 \frac{y^8}{8!}.$$

Summing up (2.10) leads to the partial sum

$$\sum_{n=0}^{3} U_n(y) \tag{2.11}$$

as the approximate solution.

Using $U(1) = 0$, the unknown constant is determined to be

$$a_0 = \frac{A\left(\frac{1}{2} + \frac{H^2}{4!} + \frac{H^4}{4!} + \frac{H^6}{6!}\right)}{\left(1 + H^2 + \frac{H^4}{4!} + \frac{H^6}{6!}\right)} \tag{2.12}$$

Therefore, the series converges to the exact solution obtained in [2]

$$U(y) = \frac{A}{H^2}\left(1 - \frac{Cosh(Hy)}{Cosh(H)}\right) \tag{2.13}$$

If we assume that $0 < H << 1$ then (2.13) leads to

$$U(y, H << 1) = \frac{A}{2}\left(1 - y^2\right) + \frac{AH^2}{24}\left(-5 + 6y^2 - y^4\right) + \frac{AH^4}{720}\left(61 - 75y^2 + 15y^4 - y^6\right) + O\left(H^6\right) \tag{2.14}$$

3. Computational Approach

By Squire Theorem [3, 4, 9, 17], we impose a 2-Dimensional disturbance in the form

$$u(x,y,t) = U(y) + u'(x,y,t),$$
$$v(x,y,t) = 0 + v'(x,y,t), \tag{2.15}$$
$$p(x,y,t) = P(x) + p'(x,y,t),$$

Where $U(y)$ is the solution of the basic flow equation (2.13) and u', v', p' are the small disturbances, substituting (2.15) into (2.1)-(2.3) and neglecting all quadratic terms, we get

$$\frac{\partial u'}{\partial v}+\frac{\partial v'}{\partial y}=0$$
$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{\partial U}{\partial y}=-\frac{\partial p'}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}\right)-H^2u' \tag{2.16}$$
$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}=-\frac{\partial p'}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}\right)$$

We now seek a mode solution in the form

$$\Psi(x,y,t)=\phi(y)e^{i\alpha(x-ct)}=\phi(y)e^{i\alpha(x-c_Rt)}e^{\alpha c_I t} \tag{2.17}$$

Where $c=c_R+ic_I$ is complex valued function and α is real, it is clear from (2.17) that when $c_I>0$ the disturbance grows and the flow become unstable. For $c_I<0$ the disturbance decays and the flow become stable and neutrally stable when $c_I=0$. Additionally $c_R>0$ enhances the flow stability, So that the velocity components can be obtained as

$$u'(x,y,t)=\phi'(y)e^{i\alpha(x-ct)}$$
$$v'(x,y,t)=-i\alpha\phi(y)e^{i\alpha(x-ct)} \tag{2.18}$$
$$p'(x,y,t)=h(y)e^{i\alpha(x-ct)}.$$

Putting (2.18) in (2.16) and eliminating $p'(x,y,t)$, we obtain the fourth order ordinary differential equation

$$\phi^{iv}=\left(2\alpha^2-i\alpha\,\mathrm{Re}\,c\right)\phi''+\left(i\alpha U+H^2\right)\mathrm{Re}\,\phi'' + \left(i\alpha^3\,\mathrm{Re}\,c-\alpha^4\right)\phi-i\alpha\,\mathrm{Re}\left(\alpha^2U+U''\right)\phi \tag{2.19}$$

subject to the boundary conditions

$$\phi(-1)=\phi'(-1)=0 \tag{2.20}$$
$$\phi'(1)=\phi(1)=0 \tag{2.21}$$

In the limiting case as $H\to 0$, equation (2.19) reduces to the well-known Orr-Sommerfield equation.

By ADM the solution of (2.19) - (2.21) can be written as

$$\varphi_0(y)=\int_{-1}^{y}\int_{-1}^{y} b_0\,dydy+\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y} b_1\,dydydy$$

$$\varphi_{n+1}(y)=$$

$$\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\left(\left(2\alpha^2-i\alpha Rc\right)\frac{d^2\varphi_n}{dy^2}+\left(H^2+i\alpha U\right)R\frac{d^2\varphi_n}{dy^2}+\left(i\alpha^3Rc-\alpha^4\right)\varphi_n-i\alpha R\left(\alpha^2U+U''\right)\varphi_n\right)dydydydy$$

(2.22)

where the unknown constants are to be evaluated using the boundary condition (2.21).

To obtain the eigenvalues of the approximate solution, the partial sum $\phi(y)=\sum_{n=0}^{k}\phi_n(y)$ is solved using the boundary conditions (2.21). This returns two equations as functions of b_0 and b_1. Using Mathematica, the two constants are eliminated, and we obtain the following results for the wave speed (*c*) when $k=4$. The numerical results of (2.22) are shown as Tables 1- 3 for different parameter values.

4. Results and Discussion

Table 1 shows the effect of an increase in Hartmann's number on the flow stability. The result shows that the value of c_i reduces with an increase in Hartmann's number in a quadratic manner, this is true due to the retarding effect of Lorentz forces on the flow applied across the channel. Therefore, increasing magnetic field intensity enhances the flow stability. This behaviour validates the previously obtained result by [2].

Table 1

Computation showing variations in wave speed $\alpha=1,\ \mathrm{Re}=10^4$

H	c_r	c_i
1	1.88324	-1.00754i
5	18.743	-25.2347i
10	321.096	-100.938i
15	1647.78	-227.109i
20	5232.96	-403.749i
25	12804.4	-630.858i
30	26583.7	-908.434i
35	49285.8	-1236.48i
40	84119.7	-1614.99i
45	134788.0	-2043.98i
50	205486.0	-2523.43i

Table 2

Computation showing variations in wave speed $\alpha = 1,\ H = 1$

Re	c_r	c_i
10,000	0.188324	-1.00754i
20,000	0.188331	-1.00737i
30,000	0.188334	-1.00731i
40,000	0.188335	-1.00729i
50,000	0.188336	-1.00727i
60,000	0.188337	-1.00726i
70,000	0.188337	-1.00725i
80,000	0.188337	-1.00724i
90,000	0.188338	-1.00724i
1,000,000	0.188339	-1.0072i
1,000,000,000	0.188339	-1.0072i

Table 3

Computation showing variations in wave speed $H = 1,\ \mathrm{Re} = 10^4$

α	c_r	c_i
1	0.188324	-1.00754i
5	0.323685	-0.328911i
10	0.283335	-0.13684i
15	0.238024	0.0291119i
20	0.260771	0.0143511i
25	0.264298	0.00594748i
30	0.26487	0.00189868i
35	0.264779	-0.000385081i
40	0.26456	-0.00187533i
45	0.264341	-0.0029701i
50	0.264154	-0.00385024i

In Table 2, it is observed that as the Reynolds number increases there is increase in the c_r while there is decrease in the value of c_i this brings about instability. Finally, Table 3 shows that both c_r and c_i oscillates with increase in the wave number α .

5. Conclusion

In this paper, the ADM is used to study the temporal development of small disturbances in hydromagnetic fluid flow. The criteria for the onset of instability have been presented theoretically and confirmed analytically. It is observed that increase in Hartmann’s number stabilizes the flow while the Reynolds number has destabilizing effect on the flow.

REFERENCES

[1] *G. Adomian*, Solving Frontier problems in Physics, kluver publisher 1994.

[2] *O.D. Makinde*, Magneto-Hydrodynamic stability of plane-poiseuille flow using Multideck Asymptotic technique, Mathematical and Computer Modelling 37, 251-259, 2003

[3] *P. K Kundu and I. M. Cohen*. Fluid Mechanics, Acadamic press, page 479-480, 2002

[4] *P. Drazin*. Introduction to Hydrodynamics Stability, Cambridge University Press2002

[5] *H. Haddadpour*''An exact solution for variable coefficients fourth-order wave equation using the Adomian method'', Mathematical and Computer Modelling 44, 2006,1144–1152,

[6] *N. Himoun, K. Abbaoui and Y. Cherruault*, ''New results of convergence of Adomian's method Kybernetes, Vol. 28 No. 4, 1999, pp. 423-429, © MCB University Press, 0368-492X

[7] *N. Himoun, K. Abbaoui, Y. Cherruault,*''New results on Adomian method Kybernetes Vol. 32 No. 4, 2003,pp. 523-539 q MCB UP Limited 0368-492X DOI 10.1108/03684920310463911

[8] *R. Z Ouedraogo, Y. Cherruault K. Abbaoui*. ''Convergence of Adomian's method applied to algebraic equations''Kybernetes, Vol. 29 No. 9/10, 2000, 1298-1305. MCB University Press, 0368-492X

[9] *P. Sibanda, Makinde O.D.*''Incompressible flow theory''Zimbakwe University press 2000

[10] *A. M Wazwaz*,''Pade´ approximants and Adomian decomposition method for solving the Flierl–Petviashivili equation and its variants'' Applied Mathematics and Computation 182, 2006, 1812–1818

[11] *A. M Wazwaz, El-Sayed*,''A new modification of Adomian decomposition method for linear and non-Linear operators'' Applied Mathematics and Computation 122, 2001, 393–405

[12] *A. M Wazwaz*,''Necessary Conditions for the Appearance of Noise Terms in decomposition Solution Series''Applied mathematics and computation 81, 1997, 265-274

[13] *A. M Wazwaz*,''The modified decomposition method and Pade´ approximants for a boundary layer equation in unbounded domain'' Applied Mathematics and Computation 177, 2006, 737–744

[14] *A. M Wazwaz*,''Analytical solution for the time-dependent Emden–Fowler type of equations by Adomian decomposition method'' Applied Mathematics and Computation 166, 2005, 638–651

[15] *A. M Wazwaz*. ''A new algorithm for calculating Adomian polynomials for nonlinear operators'', Applied Mathematics and Computation 111.2000, 53-69

[16] *A. M Wazwaz,* Adomian decomposition method for a reliable treatment of the Emden–Fowler equation Applied Mathematics and Computation 161,2005, 543–560

[17] *E. Grenier*, Handbook Of Mathematical Fluid Dynamics, Volume III Elsevier B.V2004