FIXED POINT RESULTS IN EXTENDED RECTANGULAR b-METRIC SPACES WITH AN APPLICATION

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In this paper, we enlarge the class of rectangular b-metric spaces by considering the class of extended rectangular b-metric spaces and utilize the same to prove our fixed point results. Our main result extends and improves many results of the existing literature. We adopt an example to highlight the utility of our main result. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

Keywords: Extended rectangular b-metric; orbit; fixed point.

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1. Introduction

In 1922, Banach proved his classical contraction principle. The investigation of existence and uniqueness of fixed point for a self-mapping and common fixed points for two or more mappings has become a very active and natural subject of interest. Many researchers proved Banach contraction principle in multitude of generalized metric spaces. In 1993, Stefan Czerwik [5] introduced the concept of b-metric space by replacing triangular inequality with a relatively more general condition which is also utilize to improve generalizing Banach contraction mapping theorem. In recent years, Imdad [8], Mustafa [10], Suzuki [14], Wong [15], Piri-Afshari [11] and others proved some fixed point results in b-metric spaces (see [1,7,13]). Very recently, Kamran et al. [9] introduced a new type of generalized b-metric space and termed it as extended b-metric space. Thereafter, Samreen et al. [12] also proved some fixed point results in extended b-metric space via contraction condition involved a new class of comparison functions.

In 2000, Branciari [2] generalized the idea of metric space by replacing the triangular inequality with more general inequality, namely, quadrilateral inequality (namely, involving four points instead of three) for introducing the notion of rectangular metric spaces and generalized Banach contraction theorem. After eight years, George et al. [6] introduced rectangular b-metric spaces in order to generalized rectangular metric spaces. Finally, authors proved the analogue of Banach contraction mapping principle in the framework of rectangular b-metric space.

Inspired by the concepts of extended b-metric space and rectangular b-metric space, we introduce extended rectangular b-metric space and utilize the same to prove fixed point result. We, also furnish an example to establish the genuineness of our newly proved result.

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2. **Preliminaries**

In what follows, we collect relevant definitions needed in our subsequent discussions.

Definition 2.1. [5] Let X be a non-empty set. A mapping $\sigma : X \times X \to \mathbb{R}_+$ is said to be a b-metric with coefficient $s \geq 1$, if σ satisfies the following (for all $x, y, z \in X$):

(1) $\sigma(x, y) = 0$ if and only if $x = y$,
(2) $\sigma(x, y) = \sigma(y, x)$,
(3) $\sigma(x, y) \leq s[\sigma(x, z) + \sigma(z, y)]$.

Then the pair (X, σ) is said to be a b-metric space.

Definition 2.2. [9] Let X be a non-empty set and $\xi : X \times X \to [1, \infty)$. A mapping $\sigma_\xi : X \times X \to \mathbb{R}_+$ is said to be an extended b-metric space, if σ_ξ satisfies the following (for all $x, y, z \in X$):

(1) $\sigma_\xi(x, y) = 0$ if and only if $x = y$,
(2) $\sigma_\xi(x, y) = \sigma_\xi(y, x)$,
(3) $\sigma_\xi(x, y) \leq \xi(x, y)[\sigma_\xi(x, z) + \sigma_\xi(z, y)]$.

Then the pair (X, σ_ξ) is said to be an extended b-metric space.

Definition 2.3. [2] Let X be a non-empty set. A mapping $r : X \times X \to \mathbb{R}^+$ is said to be a rectangular metric on X if, r satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\},$):

(1) $r(x, y) = 0$ if and only if $x = y$,
(2) $r(x, y) = r(y, x)$,
(3) $r(x, y) \leq r(x, u) + r(u, v) + r(v, y)$.

Then the pair (X, r) is said to be a rectangular metric space.

Definition 2.4. [6] Let X be a non-empty set with the coefficient $s \geq 1$. A mapping $r_b : X \times X \to \mathbb{R}^+$ is said to be a rectangular b-metric on X if, r_b satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$):

(1) $r_b(x, y) = 0$ if and only if $x = y$,
(2) $r_b(x, y) = r_b(y, x)$,
(3) $r_b(x, y) \leq s[r_b(x, u) + r_b(u, v) + r_b(v, y)]$.

Then the pair (X, r_b) is said to be a rectangular b-metric space.

3. **Results**

In this section, we introduce yet another type of generalized metric space, which we refer as extended rectangular b-metric space. We also establish a fixed point theorem besides deducing natural corollaries.

Definition 3.1. Let X be a non-empty set and $\xi : X \times X \to [1, \infty)$. A mapping $r_\xi : X \times X \to \mathbb{R}^+$ is said to be an extended rectangular b-metric on X if, r_ξ satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$):

$(1r_\xi)$ $r_\xi(x, y) = 0$, if and only if $x = y$,
$(2r_\xi)$ $r_\xi(x, y) = r_\xi(y, x)$,
$(3r_\xi)$ $r_\xi(x, y) \leq \xi(x, y)\big[r_b(x, u) + r_\xi(u, v) + r_\xi(v, y)\big]$.

Then the pair (X, r_ξ) is said to be an extended rectangular b-metric space.

The following implications amongst several generalized metric spaces defined earlier are natural. However, the inverse implications need not be true.

$$\begin{tikzcd} \text{m.s.} \arrow[r] \arrow[d] & b\text{-m.s.} \arrow[r] \arrow[d] & \text{extended } b\text{-m.s.} \arrow[d] \\ \text{rectangular m.s.} \arrow[r] & \text{rectangular } b\text{-m.s.} \arrow[r] & \text{extended rectangular } b\text{-m.s.} \end{tikzcd}$$

The following example demonstrates that the results proved in this paper are genuinely new.

Example 3.1. *Consider $X = \{1, 2, 3, 4, 5\}$. Define $\xi : X \times X \to [1, \infty)$ by:*

$$\xi(x, y) = x + y + 1, \text{ for all } x, y \in X.$$

Define $r_\xi : X \times X \to \mathbb{R}^+$ by:

$$\begin{aligned} r_\xi(x, x) &= 0, \text{ for all } x \in X; \\ r_\xi(x, y) &= r_\xi(y, x), \text{ for all } x, y \in X; \\ r_\xi(1, 3) &= r_\xi(2, 5) = 70, \; r_\xi(1, 4) = 1000 \text{ and } r_\xi(1, 5) = 1200; \\ r_\xi(1, 2) &= r_\xi(2, 3) = r_\xi(3, 4) = 60, \; r_\xi(3, 5) = r_\xi(4, 5) = r_\xi(2, 4) = 400. \end{aligned}$$

Now, we show that r_ξ is an extended rectangular b-metric space. Here, ($1r_\xi$) and ($2r_\xi$) are trivial. Now, for ($3r_\xi$), we have

$$r_\xi(1, 5) = 1200, \; \xi(1, 5)\big[r_\xi(1, 3) + r_\xi(3, 2) + r_\xi(2, 5)\big] = 7(70 + 60 + 70) = 1400$$

and

$$r_\xi(1, 4) = 1000, \; \xi(1, 4)\big[r_\xi(1, 2) + r_\xi(2, 3) + r_\xi(3, 4)\big] = 6(60 + 60 + 60) = 1080.$$

Similarly, other cases can also be argued. Thus, for all $x, y \in X$ with distinct $u, v \in X \backslash \{x, y\}$, we get

$$r_\xi(x, y) \leq \xi(x, y)\big[r_b(x, u) + r_\xi(u, v) + r_\xi(v, y)\big].$$

Hence, (X, r_ξ) is extended rectangular b-metric space with ξ.

In an extended rectangular b-metric space, the concepts of basic topological notions, such as: Cauchy sequence, convergent sequence and complete extended rectangular b-metric space can be easily adopted as under.

Definition 3.2. A sequence $\{x_n\}$ in (X, r_ξ) is said to be Cauchy if

$$\lim_{n,m \to \infty} r_\xi(x_n, x_m) = 0.$$

Definition 3.3. A sequence $\{x_n\}$ in (X, r_ξ) is said to be convergent to $x \in X$ if

$$\lim_{n \to \infty} r_\xi(x_n, x) = 0.$$

Definition 3.4. An extended rectangular b-metric space (X, r_ξ) is said to be a complete if every Cauchy in X is convergent to some point in X.

The following lemma is needed in the proof of our main result.

Lemma 3.1. *Let (X, r_ξ) be an extended rectangular b-metric space and $\{x_n\}$ a Cauchy sequence in X such that $x_n \neq x_m$ whenever $m \neq n$. Then $\{x_n\}$ converges at most one point.*

Proof. Let a sequence $\{x_n\}$ in X has two limit point $x, y \in X$, that is, $\lim_{n\to\infty} r_\xi(x_n, x) = 0$ and $\lim_{n\to\infty} r_\xi(x_n, y) = 0$. Since, $\{x_n\}$ is Cauchy then for $x_n \neq x_m$ whenever $m \neq n$, so from ($3r_\xi$), we have

$$\begin{aligned} r_\xi(x, y) &\leq \xi(x, y)[r_\xi(x, x_n) + r_\xi(x_n, x_m) + r_\xi(x_m, y)] \\ &\to 0 \text{ as } n, m \to \infty. \end{aligned}$$

which implies that

$$r_\xi(x,y)=0,$$

Therefore, $\{x_n\}$ converges at unique limit point. $\square$

In 1974, *Ćirić* considered the concept of orbit and proved some fixed point results (see [3]).

Definition 3.5. [3] Let (X,r_ξ) be an extended rectangular b-metric space. For a self-mapping $f:X\to X$, we define (for $x\in X$ and $n\in\mathbb{N}$)

$$\mathcal{O}(x;n)=\{x,fx,...,f^nx\} \text{ and } \mathcal{O}(x;\infty)=\{x,fx,...,f^nx,..\}.$$

The set $\mathcal{O}(x;\infty)$ or simply $\mathcal{O}(x)$ is called an orbit of f.

Our main theorem is an analogue of Banach contraction principle in the setting of extended rectangular b-metric space. All through this section, for a mapping $f:X\to X$ and $x\in X$, we consider an orbit $\mathcal{O}(x)=\{x,fx,...,f^nx,..\}$.

Definition 3.6. Let (X,r_ξ) be an extended rectangular b-metric space. A self-mapping $f:X\to X$ is called orbitally continuous if $\lim\limits_{k\to\infty}f^{n_k}x=x$ for some $x\in X$ implies $\lim\limits_{k\to\infty}f(f^{n_k}x)=fx$. Besides, (X,r_ξ) is called f-orbitally complete if every Cauchy sequence which is obtained in $\{x,fx,...,f^nx,..\}$ for some $x\in X$ converges to X.

Now, we state and prove our main result as follows:

Theorem 3.1. *Let (X,r_ξ) be an extended rectangular b-metric space and $f:X\to X$. Suppose that the following conditions hold:*

(i) for all $x,y\in X$, we have

$$r_\xi(fx,fy)\le\lambda r_\xi(x,y)$$

where $\lambda\in[0,1)$,

(ii) $\lim\limits_{n,m\to\infty}\xi(x_n,x_m)<\frac{1}{\lambda}$,

(iii) (X,r_ξ) is f-orbitally complete,

(iv) f is orbitally continuous.

Then f has a unique fixed point.

Proof. With initial point $x_0\in X$, construct an iterative sequence $\{x_n\}$ by:

$$x_1=fx_0,\ x_2=f^2x_0,\ x_3=f^3x_0,...,x_n=f^nx_0,...$$

Now, we assert that $\lim\limits_{n\to\infty}r_\xi(x_n,x_{n+1})=0$. On setting $x=x_n$ and $y=x_{n+1}$ in condition (i), we get

$$\begin{aligned} r_\xi(f^nx_0,f^{n+1}x_0) &= r_\xi(fx_n,fx_{n+1})\\ &\le \lambda r_\xi(x_n,x_{n+1})\\ &\le \lambda^n r_\xi(x_0,x_1), \end{aligned}$$

which on making $n\to\infty$, gives rise

$$\lim_{n\to\infty}r_\xi(f^nx_0,f^{n+1}x_0)=0$$

$$\text{and } \lim_{n\to\infty}r_\xi(f^nx_0,f^{n+2}x_0)=0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X,r_ξ). In doing so, we distinguish two cases as under:

Case 1. Firstly let p is odd, that is $p = 2m + 1$ for any $m \geq 1$. Now using $(3r_\xi)$ for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
r_\xi(x_n, x_{n+2m+1}) &\leq \xi(x_n, x_{n+2m+1})\big[r_\xi(x_n, x_{n+1}) + r_\xi(x_{n+1}, x_{n+2}) + r_\xi(x_{n+2}, x_{n+2m+1})\big] \\
&\leq \xi(x_n, x_{n+2m+1})\big[\lambda^n r_\xi(x_0, x_1) + \lambda^{n+1} r_\xi(x_0, x_1)\big] + \xi(x_n, x_{n+2m+1}) \times \\
&\quad r_\xi(x_{n+2}, x_{n+2m+1}) \\
&= \xi(x_n, x_{n+2m+1})(\lambda^n + \lambda^{n+1}) r_\xi(x_0, x_1) + \xi(x_n, x_{n+2m+1}) \times \\
&\quad r_\xi(x_{n+2}, x_{n+2m+1}) \\
&\leq \xi(x_n, x_{n+2m+1})(\lambda^n + \lambda^{n+1}) r_\xi(x_0, x_1) + \xi(x_n, x_{n+2m+1}) \times \\
&\quad \xi(x_{n+2}, x_{n+2m+1})(\lambda^{n+2} + \lambda^{n+3}) r_\xi(x_0, x_1) + ... + \\
&\quad \xi(x_n, x_{n+2m+1})....\xi(x_{n+2m-2}, x_{n+2m+1})(\lambda^{n+2m-2} + \lambda^{n+2m-1}) \times \\
&\quad r_\xi(x_0, x_1) + \xi(x_n, x_{n+2m+1})...\xi(x_{n+2m-2}, x_{n+2m+1})\lambda^{n+2m} r_\xi(x_0, x_1) \\
&= \lambda^n(1+\lambda) r_\xi(x_0, x_1) \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{n+2j}, x_{n+2m+1}) + \\
&\quad \lambda^{n+2m} \prod_{j=0}^{m-1} \xi(x_{n+2j}, x_{n+2m+1}) r_\xi(x_0, x_1),
\end{aligned}$$

yielding thereby

$$\sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{n+2j}, x_{n+2m+1}) \leq \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{2j}, x_{n+2m+1}).$$

As, in view of condition (ii), we have $\lim\limits_{n,m\to\infty} \xi(x_n, x_m)\lambda < 1$, therefore utilizing the ratio test, we conclude that the series $\sum_{i=0}^{\infty} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{2j}, x_{n+2m+1})$ is convergent for each $m \in \mathbb{N}$. Assume that

$$S = \sum_{i=0}^{\infty} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{2j}, x_{n+2m+1}), \;\; S_n = \sum_{i=0}^{n} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{2j}, x_{n+2m+1}).$$

Therefore, from the above inequality, we have

$$\begin{aligned}
r_\xi(x_n, x_{n+2m+1}) &\leq \lambda^n(1+\lambda) r_\xi(x_0, x_1)[S_{m-1} - S_{n-1}] + \\
&\quad \lambda^{n+2m} \prod_{j=0}^{m-1} \xi(x_{n+2j}, x_{n+2m+1}) r_\xi(x_0, x_1). \qquad (3.1)
\end{aligned}$$

Letting $n \to \infty$ in equation (3.1), we conclude that $r_\xi(x_n, x_{n+2m+1}) \to 0$.

Case 2. Secondly, assume that p is even, that is $p = 2m$ for any $m \geq 1$. Then

$$\begin{aligned}
r_\xi(x_n, x_{n+2m}) &\leq \xi(x_n, x_{n+2m})\big[r_\xi(x_n, x_{n+1}) + r_\xi(x_{n+1}, x_{n+2}) + r_\xi(x_{n+2}, x_{n+2m})\big] \\
&\leq \xi(x_n, x_{n+2m})\big[\lambda^n r_\xi(x_0, x_1) + \lambda^{n+1} r_\xi(x_0, x_1)\big] + \xi(x_n, x_{n+2m}) \times \\
&\quad r_\xi(x_{n+2}, x_{n+2m}) \\
&= \xi(x_n, x_{n+2m})(\lambda^n + \lambda^{n+1}) r_\xi(x_0, x_1) + \xi(x_n, x_{n+2m}) r_\xi(x_{n+2}, x_{n+2m}) \\
&\leq \xi(x_n, x_{n+2m})(\lambda^n + \lambda^{n+1}) r_\xi(x_0, x_1) + \xi(x_n, x_{n+2m}) \times \\
&\quad \xi(x_{n+2}, x_{n+2m})(\lambda^{n+2} + \lambda^{n+3}) r_\xi(x_0, x_1) + ... + \\
&\quad \xi(x_n, x_{n+2m})....\xi(x_{n+2m-2}, x_{n+2m})(\lambda^{n+2m-2} + \lambda^{n+2m-1}) r_\xi(x_0, x_1) + \\
&\quad \xi(x_n, x_{n+2m+1})...\xi(x_{n+2m-2}, x_{n+2m+1})\lambda^{n+2m} r_\xi(x_0, x_1) \\
&= \lambda^n(1+\lambda) r_\xi(x_0, x_1) \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(x_{n+2j}, x_{n+2m}) + \\
&\quad \lambda^{n+2m-2} \prod_{j=0}^{m-1} \xi(x_{n+2j}, x_{n+2m}) r_\xi(x_0, x_2),
\end{aligned}$$

so that

$$\begin{aligned}
r_\xi(x_n, x_{n+2m}) &\leq \lambda^n(1+\lambda) r_\xi(x_0, x_1)[S_{m-1} - S_{n-1}] + \\
&\quad \lambda^{n+2m-2} \prod_{j=0}^{m-1} \xi(x_{n+2j}, x_{n+2m}) r_\xi(x_0, x_2). \qquad (3.2)
\end{aligned}$$

Taking limit $n \to \infty$, in (3.2), we get $r_\xi(x_n, x_{n+2m}) \to 0$. Therefore, in both the cases, we have

$$\lim_{n\to\infty} r_\xi(x_n, x_{n+p}) = 0,$$

which shows that the sequence $\{x_n\}$ is Cauchy in X. Since X is f-orbitally complete then there exists $x \in X$ such that $x_n \to x$. Since, f is orbitally continuous so, we have

$$\begin{aligned}
r_\xi(fx, x) &\leq \xi(fx, x)\big[r_b(fx, x_n) + r_\xi(x_n, x_{n+1}) + r_\xi(x_{n+1}, x)\big] \\
&\leq \xi(fx, x)\big[r_b(fx, x_n) + r_\xi(fx_{n-1}, fx_n) + r_\xi(x_{n+1}, x)\big] \\
&= \xi(fx, x)\big[r_b(fx, x_n) + \lambda r_\xi(x_{n-1}, x_n) + r_\xi(x_{n+1}, x)\big]
\end{aligned}$$

which are making as $n \to \infty$ gives rise

$$r_\xi(fx, x) \to 0,$$

so that, $r_\xi(fx, x) = 0$. Therefore, $fx = x$. Hence, x is a fixed point of f. Observe that, in view of Lemma 3.1, a sequence $\{x_n\}$ converges uniquely at point $x \in X$. □

Now, we present an example which illustrates the utility of our newly proved result:

Example 3.2. *Let $X = [0,1]$. Define, $r_\xi(x,y) = |x-y|^2$ and $\xi(x,y) = x + y + 3$, for all $x, y \in X$. Then, (X, r_ξ) is a complete extended rectangular b-metric space. Define a mapping $f : X \to X$ by $fx = \frac{x}{2}$.*

Observe that, all the conditions of Theorem 3.1 are satisfied and $x = 0$ is a unique fixed point of the involved map f.

The following corollary deduce form Theorem 3.1 remains a new result (due to improvement in orbital consideration).

Corollary 3.1. *Let (X, r_ξ) be an extended rectangular b-metric space and $f : X \to X$. Suppose that the following conditions hold:*

(i) for all $x, y \in X$, we have $r_\xi(fx, fy) \leq \lambda r_\xi(x, y)$ where $\lambda \in [0, 1)$,
(ii) $\lim\limits_{n,m\to\infty} \xi(x_n, x_m) < \frac{1}{\lambda}$,
(iii) (X, r_ξ) is complete,
(iv) f is continuous.

Then f has a unique fixed point.

By setting $\xi(x, y) = s \geq 1$ (for all $x, y \in X$) in Theorem 3.1, we deduce a sharpened version of Theorem 2.1 due to George et al. [6]

Corollary 3.2. *Let (X, r_b) be a rectangular b-metric space with $s \geq 1$ and $f : X \to X$. Suppose that the following conditions hold:*

(i) for all $x, y \in X$, we have $r_b(fx, fy) \leq \lambda r_b(x, y)$, where $\lambda \in [0, \frac{1}{s})$,
(ii) (X, r) is f-orbitally complete,
(iii) f is orbitally continuous.

Then f has a unique fixed point.

On setting $\xi(x, y) = 1$ for all $x, y \in X$ in Theorem 3.1, we get the following corollary due to Das and Lakshmi [4], in 2007.

Corollary 3.3. *Let (X, r) be a rectangular metric spacef $: X \to X$. Suppose that the following conditions hold:*

(i) for all $x, y \in X$, we have $r(fx, fy) \leq \lambda r(x, y)$, where $\lambda \in [0, 1)$,
(ii) (X, r) is f-orbitally complete,
(iii) f is orbitally continuous.

Then f has a unique fixed point.

4. Application

In this section, we endeavor to apply Theorem 3.1 to prove the existence and uniqueness of solution of the following integral equation of Fredholm type:

$$x(t) = \int_a^b G(t, s, x(s))ds + h(t) \text{ for } t, s \in [a, b] \tag{4.1}$$

where, $G, h \in C([a, b], \mathbb{R})$ (say $X = C([a, b], \mathbb{R})$). Define $r_\xi : X \times X \to \mathbb{R}^+$ by

$$r_\xi(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2 \text{ and } \xi(x, y) = x + y + 3, \text{ for all } x, y \in X,$$

where $\xi : X \times X \to [1, \infty)$. Then, (X, r_ξ) is a complete extended rectangular b-metric space. Now, we are equipped to state and prove our result as follows:

Theorem 4.1. *Assume that (for all $x, y \in C([a, b], \mathbb{R})$)*

$$|G(t, s, x(s)) - G(t, s, y(s))| \leq \frac{1}{3(b - a)}|x(s) - y(s)|, \tag{4.2}$$

for all $t, s \in [a, b]$. Then the integral equation (4.1) has a unique solution.

Proof. Define $f : X \to X$ by $fx(t) = \int_a^b G(t, s, x(s))ds + h(t)$ for all $t, s \in [a, b]$. It is clear that, x is a fixed point of the operator f if and only if it is a solution of the integral equation

(4.1). Now, for all $x, y \in X$, we have

$$\begin{aligned} |fx(t)-fy(t)|^2 &\leq \Big(\int_a^b |G(t,s,x(s))-G(t,s,y(s))|ds\Big)^2 \\ &\leq \Big(\int_a^b \frac{1}{3(b-a)}|x(s)-y(s)|ds\Big)^2 \\ &\leq \frac{1}{9(b-a)^2}\sup_{t\in[a,b]}|x(t)-y(t)|^2\Big(\int_a^b ds\Big)^2 \\ &\leq \frac{1}{9}r_\xi(x,y). \end{aligned}$$

Thus, the condition (4.2) is satisfied with $\lambda = \frac{1}{3(b-a)} \in [0,1)$. Hence, the operator f has a unique fixed point, that is, the Fredholm integral Equation (4.1) has a unique solution. □

5. **Conclusion**

As the rectangular b-metric space is relatively new addition to the existing literature, therefore, in this note, we endeavor to further enrich this notion by introducing the idea of extended rectangular b-metric spaces wherein we generalized the constant $s \geq 1$ by a function $\xi(x,y)$ in quadrilateral inequality. Our main result (i.e., Theorem 3.1) is an analogue of Banach contraction principle wherein we have also exploited the idea of orbit. An example is also adopted to highlight the realized improvements in our newly proved result. Finally, we apply Theorem 3.1 to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

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