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$\mathcal{A}_S$ −SCALAR OPERATORS

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În această lucrare studiem o clasă nouă de operatori numiți $\mathcal{A}_S$ -scalari. *Aceştia apar în mod natural, ca o generalizare a operatorilor* $\mathcal{A}$ *-scalari introduşi în* [4] *şi se definesc cu ajutorul homomorfismelor* S *–spectrale* (*funcții* $\mathcal{A}_S$ *-spectrale*), *care, la rândul lor, sunt generalizări ale homomorfismelor spectrale* (*funcții* $\mathcal{A}$ *-spectrale*) *din* [4]. *Pe parcursul lucrării sunt prezentate unele proprietăți ale acestor operatori, dintre care, cea mai semnificativă este aceea cum că sunt* S *-decompozabili*, *în sensul din* [1].

This paper is devoted to the study of a new class of operators, called $\mathcal{A}_S$ *-scalar operators, naturally appearing as a generalization of the* $\mathcal{A}$ *-scalar operators* [4]. *This study uses the concept of* $\mathcal{A}_S$ *-spectral homomorphism which is also the generalization of the spectral homomorphism* ($\mathcal{A}$ *-spectral function*) *studied in* [4]. *Furthermore, we prove some properties concerning the* $\mathcal{A}_S$ *-scalar* operators; *their main quality is that of being* S *– decomposable* [1].

Keywords: scalar ($\mathcal{A}$ -scalar); spectral ($\mathcal{A}$ -spectral); $\mathcal{A}_S$ -scalar operator; $\mathcal{A}_S$ -spectral function; restriction and quotient of an operator.

1. Introduction

Let X be a Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X and let $\mathbb{C}$ be the complex plane. If $T \in \mathbf{B}(X)$ and $Y \subset X$ is a (closed) subspace invariant to T, let us denote by $T \mid Y$ the restriction of T to Y, respectively by $\dot{T}$ the operator induced by T in the quotient space $\dot{X} = X / Y$. In what follows, by subspace of X we understand a closed linear manifold of X. Recall that Y is a *spectral maximal space* of T if it is an invariant subspace to T such that for any other subspace $Z \subset X$, also invariant to T, the inclusion

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$\sigma(T\,|\,Z)\subset\sigma(T\,|\,Y)$ implies $Z\subset Y$ ([6]). A family of open sets $G_S\cup\{G_i\}_{i=1}^{n}$ is said to be an $S-$*covering* of the closed set $\sigma\subset\mathbb{C}$ if $G_S\cup\left(\bigcup_{i=1}^{n}G_i\right)\supset\sigma\cup S$ and $\overline{G}_i\cap S=\varnothing$ $(i=1,2,...,n)$ (where $S\subset\mathbb{C}$ is also closed) ([11]).

The operator $T\in\mathbf{B}(X)$ is $S-$*decomposable* (where $S\subset\sigma(T)$ compact) if for any finite open $S-$covering $G_S\cup\{G_i\}_{i=1}^{n}$ of $\sigma(T)$, there is a system $Y_S\cup\{Y_i\}_{i=1}^{n}$ of spectral maximal spaces of T such that $\sigma(T\,|\,Y_S)\subset G_S$, $\sigma(T\,|\,Y_i)\subset G_i$ $(i=1,2,...,n)$ and $X=Y_S+\sum_{i=1}^{n}Y_i$ ([1]). If $S=\varnothing$, then T is decomposable ([6]). An open set $\Omega\subset\mathbb{C}$ is said to be a *set of analytic uniqueness* for $T\in\mathbf{B}(X)$ if for any open set $\omega\subset\Omega$ and any analytic function $f_0:\omega\to X$ satisfying the equation $(\lambda I-T)f_0(\lambda)\equiv 0$, it follows that $f_0(\lambda)\equiv 0$ in ω ([10]). For $T\in\mathbf{B}(X)$ there is a unique maximal open set Ω_T of analytic uniqueness (2.1., [10]). We denote by $S_T=\complement\Omega_T=\mathbb{C}\setminus\Omega_T$ the *analytic spectral residuum of* T. For $x\in X$, a point λ is in $\delta_T(x)$ if in a neighborhood V_λ of λ there is at least an analytic X-valued function f_x (called T-*associated to* x) such that $(\mu I-T)f_x(\mu)\equiv x$, for all $\mu\in V_\lambda$. We shall put

$$\gamma_T(x)=\complement\delta_T(x),\ \rho_T(x)=\delta_T(x)\cap\Omega_T,\ \sigma_T(x)=\complement\rho_T(x)=\gamma_T(x)\cup S_T \text{ and}$$
$$X_T(F)=\{x\in X;\ \sigma_T(x)\subset F\},\text{ where } S_T\subset F\subset\mathbb{C}\ \text{([10], [11]).}$$

An operator $T\in\mathbf{B}(X)$ is said to have the *single-valued extension property* if for any analytic function $f:\omega\to X$ (where $\omega\subset\mathbb{C}$ open), with $(\lambda I-T)f(\lambda)\equiv 0$, it results that $f(\lambda)\equiv 0$ ([5]). T has the single-valued extension property if and only if $S_T=\varnothing$; then we have $\sigma_T(x)=\gamma_T(x)$ and there is in $\rho_T(x)=\delta_T(x)$ an unique analytic function $x(\lambda)$, $T-$associated to x, for any $x\in X$ ([10]). We recall that if $T\in\mathbf{B}(X)$, $S_T\neq\varnothing$, $S_T\subset F$ and $X_T(F)$ is closed, for $F\subset\mathbb{C}$ closed, then $X_T(F)$ is a spectral maximal space of T and $\sigma(T\,|\,X_T(F))\subset F$ ([10], Propositions 2.4. and 3.4.).

We remind that a set $A \subset \mathbb{C}$ is *of dimension* 0 (*totally disconnected*) if any subset of it is both open and closed in the relative topology of A, or, equivalently, if any connected component of A is reduced to a single point ([9]).

2. Preliminaries

Definition 2.1. Let Ω be a set of the complex plane $\mathbb{C}$ and let $S \subset \overline{\Omega}$ be a compact subset. An algebra $\mathcal{A}_S$ of $\mathbb{C}$-valued functions defined on Ω is called S-*normal* if for any finite open S-covering $G_S \cup \{G_i\}_{i=1}^{n}$ of $\overline{\Omega}$, there are the functions f_S, $f_i \in \mathcal{A}_S$ $(1 \le i \le n)$ such that:

1) $f_S(\Omega) \subset [0,1]$, $f_i(\Omega) \subset [0,1]$ $(1 \le i \le n)$;
2) $\operatorname{supp}(f_S) \subset G_S$, $\operatorname{supp}(f_i) \subset G_i$ $(1 \le i \le n)$;
3) $f_S + \sum_{i=1}^{n} f_i = 1$ on Ω,

where *the support* of $f \in \mathcal{A}_S$ is defined as: $\operatorname{supp}(f) = \overline{\{\mu \in \Omega;\ f(\mu) \ne 0\}}$.

Definition 2.2. An algebra $\mathcal{A}_S$ of $\mathbb{C}$-valued functions defined on Ω is called S-*admissible* if:

1) $\lambda, 1 \in \mathcal{A}_S$ (where $\lambda, 1$ denote the functions $f(\lambda) \equiv \lambda,\ f(\lambda) \equiv 1$);
2) $\mathcal{A}_S$ is S-normal;
3) for any $f \in \mathcal{A}_S$ and any $\xi \notin \operatorname{supp}(f)$, the function $f_\xi : \Omega \to \mathbb{C}$

$$f_\xi(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi - \lambda}, & \text{for } \lambda \in \Omega \setminus \{\xi\} \\ 0, & \text{for } \lambda \in \Omega \cap \{\xi\} \end{cases}$$

belongs to $\mathcal{A}_S$.

Definition 2.3. An operator $T \in \mathbf{B}(X)$ is said to be $\mathcal{A}_S$-*scalar* if there are an S-admissible algebra $\mathcal{A}_S$ and an algebraic homomorphism $U : \mathcal{A}_S \to \mathbf{B}(X)$ such that $U_1 = I$ and $U_\lambda = T$ (where 1 is the function $f(\lambda) \equiv 1$ on $\mathbb{C}$, respectively λ is the identical function $f(\lambda) \equiv \lambda$ on $\mathbb{C}$). The application U is called $\mathcal{A}_S$-*spectral homomorphism* ($\mathcal{A}_S$-*spectral function* or $\mathcal{A}_S$-*functional calculus*) for T.

If $S=\varnothing$, then we put $\mathcal{A}=\mathcal{A}_{\varnothing}$ and we obtain an $\mathcal{A}$-spectral function and an $\mathcal{A}$-scalar operator ([4]).

Definition 2.4. A subspace Y of X is said to be *invariant with respect to* an $\mathcal{A}_S$-spectral function $U:\mathcal{A}_S\to\mathbf{B}(X)$ if $U_f Y\subseteq Y$, for any $f\in\mathcal{A}_S$.

Definition 2.5. *The support* of an $\mathcal{A}_S$-spectral function U is denoted by $\operatorname{supp}(U)$ and it is defined as the smallest closed subset of $\overline{\Omega}$ such that $U_f=0$ for any $f\in\mathcal{A}_S$ with $\operatorname{supp}(f)\cap\operatorname{supp}(U)=\varnothing$.

We recall several important proprieties of an $\mathcal{A}$-spectral function U (see [4]), because we want to obtain similar properties for an $\mathcal{A}_S$-spectral function:

1) $\operatorname{supp}(U)=\sigma(U_\lambda)$, where λ is the identical function $f(\lambda)\equiv\lambda$;
2) U_λ has the single-valued extension property;
3) $\sigma_{U_\lambda}(U_f x)\subset\operatorname{supp}(f)$, for any $f\in\mathcal{A}$ and $x\in X$;
4) $\sigma_{U_\lambda}(x)\cap\operatorname{supp}(f)=\varnothing\Rightarrow U_f(x)=0$;
5) $x\in X_{U_\lambda}(F)\Leftrightarrow U_f(x)=0$, for any $f\in\mathcal{A}$ with property:
$$\operatorname{supp}(f)\cap F=\varnothing,\ F\subset\Omega \text{ closed;}$$
6) U_λ is decomposable.

3. $\mathcal{A}_S$-spectral functions and $\mathcal{A}_S$-scalar operators

Theorem 3.1. *Let $T\in\mathbf{B}(X)$ be an $\mathcal{A}_S$-scalar operator and let U be an $\mathcal{A}_S$-spectral function for T. Then we have the relations*:
$$\operatorname{supp}(U)\subset\sigma(T)\cup S \text{ and } \sigma(T)\subset\operatorname{supp}(U)\cup S.$$

Proof. Let us consider $f\in\mathcal{A}_S$ such that $\operatorname{supp}(f)\cap(\sigma(T)\cup S)=\varnothing$. If $\xi\notin\operatorname{supp}(f)$ and λ is the identical function $f(\lambda)\equiv\lambda$, then we have
$$(\xi I-T)U_{f_\xi}=(\xi I-U_\lambda)U_{f_\xi}=U_{(\xi-\lambda)f_\xi}=U_f$$
whence
$$U_{f_\xi}=\Re(\xi,T)U_f,\text{ for } \xi\in\rho(T)\cap\complement\operatorname{supp}(f).$$

The function $F:\mathbb{C}\to\mathbf{B}(X)$

$$F(\xi)=\begin{cases}\Re(\xi,T)U_f, & \text{for } \xi\in\rho(T)\\ U_{f_\xi}, & \text{for } \xi\in\complement\operatorname{supp}(f)\end{cases}$$

is entire and

$$\lim_{|\xi|\to\infty}\|F(\xi)\|=0,$$

therefore $F\equiv 0$. It follows that $U_{f_\xi}=0$ on $\complement\operatorname{supp}(f)$ and $U_f=0$, accordingly $\operatorname{supp}(U)\subset\sigma(T)\bigcup S$.

Let now $\xi_0\notin\operatorname{supp}(U)\bigcup S$, let V_{ξ_0} be an open neighborhood of ξ_0 and let W be an open neighborhood of $\operatorname{supp}(U)\bigcup S$ such that $V_{\xi_0}\bigcap W=\varnothing$. The algebra $\mathcal{A}_S$ being S-normal, there is a function $f\in\mathcal{A}_S$ with $f(\mu)=1$ for $\mu\in W$ and $f(\mu)=0$ for $\mu\in V_{\xi_0}$. Therefore

$$\operatorname{supp}(1-f)\bigcap\left(\operatorname{supp}(U)\bigcup S\right)=\varnothing,$$

whence

$$U_{1-f}=0,\text{ hence } U_f=I.$$

It follows that

$$U_{f_{\xi_0}}(\xi_0I-T)=U_{f_{\xi_0}}(\xi_0I-U_\lambda)=$$
$$=(\xi_0I-U_\lambda)U_{f_{\xi_0}}==U_{(\xi_0-\lambda)f_{\xi_0}}=U_f=I$$

therefore we finally have $\xi_0\notin\sigma(U_\lambda)=\sigma(T)$ and hence $\sigma(T)\subset\operatorname{supp}(U)\bigcup S$.

Lemma 3.1. *If* $(\lambda_0I-U_\lambda)x_0=0$, *with* $x_0\neq 0$ *and* $f\in\mathcal{A}_S$ *with* $f(\lambda)=$ $=c$, *for* $\lambda\in G\bigcap\Omega$, *where* G *is an open neighborhood of* λ_0, *then*

$$U_fx_0=cx_0.$$

Proof. From the equality $U_\lambda x_0=\lambda_0x_0$, *with* $x_0\neq 0$, it results that λ_0 is the eigenvalue of U_λ corresponding to the eigenvector x_0, hence

$$\lambda_0\in\sigma_p(U_\lambda)\subset\sigma(U_\lambda)\subset\operatorname{supp}(U)\bigcup S\subset\Omega$$

whence $G\bigcap\Omega\neq\varnothing$ (where $\sigma_p(U_\lambda)$ is the point spectrum of U_λ, i.e. the set of all eigevalues of U_λ). If we denote $g=f-c$, then we can write

$$U_fx_0-cx_0=U_gx_0=U_{(\lambda_0-\lambda)g_{\lambda_0}}x_0=U_{g_{\lambda_0}}(\lambda_0I-U_\lambda)x_0=0$$

and consequently $U_fx_0=cx_0$.

Theorem 3.2. *If* U *is an* $\mathcal{A}_S$*-spectral function for* $T \in \mathbf{B}(X)$, *then* $S_T \subset S$. *Moreover, if* $\dim(S) \leq 1$, *we have* $S_T = \varnothing$ (i.e. T *has the single-valued extension property*).

Proof. Let $f : G_f \to X$ ($G_f \subset \mathbb{C}$ open, $G_f \cap S = \varnothing$) be an analytic function such that $(\xi I - T) f(\xi) \equiv 0$.

Let us suppose that there is $\xi_0 \in G_0 \subset G_f$ (G_0 is a connected component of G_f) with $f(\xi_0) \neq 0$. Then $f(\xi) \neq 0$, for $\xi \in D_0 \subset G_0$, where D_0 is a disk with center in ξ_0. If $D_0 = D(\xi_0, r_0)$ and $D = D(\xi_0, r)$, $0 < r < r_0$, the algebra $\mathcal{A}_S$ being S-normal, it results that there is a function $g \in \mathcal{A}_S$ such that

$$g(\xi) = \begin{cases} 1, \text{ for } \xi \in \Omega \cap \overline{D} \\ 0, \text{ for } \xi \in \Omega \setminus (\Omega \cap D_0) \end{cases}.$$

According to Lemma 3.1, we have

$$U_g f(\xi) = \begin{cases} f(\xi), \text{ for } \xi \in G_0 \cap D \\ 0, \quad\quad \text{for } \xi \in G_0 \setminus D_0 \end{cases}.$$

By analytic extension, it results that $U_g f(\xi) = 0$, for $\xi \in G_0$, hence $f(\xi) = U_g f(\xi) = 0$, for $\xi \in G_0$ and thus we have obtained a contradiction. Consequently, $f(\xi) = 0$ on G_f, therefore $\complement S \subset \Omega_T$, i.e. $S_T \subset S$.

Proposition 3.1. *If* U *is an* $\mathcal{A}_S$*-spectral function for* $T \in \mathbf{B}(X)$, *then* $\gamma_T(U_f x) \subset \operatorname{supp}(f)$, *for any* $f \in \mathcal{A}_S$ *and* $x \in X$.

Moreover, if $\operatorname{supp}(f) \supset S$, *then* $\sigma_T(U_f x) \subset \operatorname{supp}(f)$.

Proof. For any $\xi \notin \operatorname{supp}(f)$, we have $f_\xi \in \mathcal{A}_S$ and the X-valued function $\xi \to U_{f_\xi} x$ is analytic. Consequently,

$$(\xi I - T) U_{f_\xi} x = (\xi I - U_\lambda) U_{f_\xi} x = U_f x,$$

therefore $\xi \in \delta_T(U_f x)$, hence $\gamma_T(U_f x) \subset \operatorname{supp}(f)$.

Moreover, for $f \in \mathcal{A}_S$ with $\operatorname{supp}(f) \supset S$, it follows that

$$\sigma_T(U_f x) = S_T \cup \gamma_T(U_f x) \subset S \cup \gamma_T(U_f x) \subset \operatorname{supp}(f).$$

Proposition 3.2. *Let* $T \in \mathbf{B}(X)$ *be an* $\mathcal{A}_S$*-scalar operator having an* $\mathcal{A}_S$*-spectral function* U *and let* Y *be a spectral maximal space of* T *such that* $Y = X_T(F)$*, for* $F \subset \mathbb{C}$ closed, $F \supset S$. *Then* $T \mid Y$ *is an* $\mathcal{A}_S$*-scalar operator.*

Proof. A spectral maximal space Y of T is also an ultrainvariant subspace to T, therefore Y is invariant to U_f, for any $f \in \mathcal{A}_S$. It is easy to prove that $U \mid Y : \mathcal{A}_S \to \mathbf{B}(Y)$ is an $\mathcal{A}_S$-spectral function for $T \mid Y \in \mathbf{B}(Y)$, hence $T \mid Y$ is $\mathcal{A}_S$-scalar.

Theorem 3.3. *Let* $T \in \mathbf{B}(X)$ *be an* $\mathcal{A}_S$*-scalar operator and let* U *be an* $\mathcal{A}_S$*-spectral function for* T. *Then* T *is an* S*-decomposable operator.*

Poof. According to some results studied in [1] and [2], it is enough to show that T is $(1,S)$-decomposable, i.e. for any open $(1,S)$-covering $\{G_S, G_1\}$ of the complex plane $\mathbb{C}$, there is a system $\{Y_S, Y_1\}$ of invariant subspaces to T such that

$\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_1) \subset G_1$ and $X = Y_S + Y_1$.

$(1,S)$-covering $\{G_S, G_1\}$ of the complex plane $\mathbb{C}$ is also an S-covering of $\overline{\Omega}$, thus there are the functions $f_S, f_1 \in \mathcal{A}_S$ such that

$0 \le f_S(\lambda), f_1(\lambda) \le 1$, $\operatorname{supp}(f_S) \subset G_S$, $\operatorname{supp}(f_1) \subset G_1$, $f_S + f_1 = 1$ on Ω.

For every $x \in X$ we have the relation

$$x = U_1 x = U_{f_S} x + U_{f_1} x \overset{\text{not}}{=} y_S + y_1 .$$

Let $F \subset \mathbb{C}$ be closed such that $F \cap S = \varnothing$ or $F \supset S$ and

$$\mathcal{E}(F) = \bigcap \left\{ \ker U_f \,;\, f \in \mathcal{A}_S, \operatorname{supp}(f) \cap F = \varnothing \right\} =$$
$$= \left\{ x \in X \,;\, f \in \mathcal{A}_S, \operatorname{supp}(f) \cap F = \varnothing \Rightarrow U_f x = 0 \right\}.$$

Obviously, $\mathcal{E}(F)$ are closed subspaces of X invariant to T.

Let us show that $\sigma(T \mid \mathcal{E}(F)) \subset F$. For $\xi \in \mathbb{C} \setminus F$, there is a function $f \in \mathcal{A}_S$ such that $f = 1$ on $F \cap \Omega$ and $f = 0$ on $V \cap \Omega$, for an open suitable neighborhood V of ξ (when $F \cap S = \varnothing$ and $\xi \in S$, we must take $V \supset S$). Therefore $\operatorname{supp}(1-f) \cap F \cap \Omega = \varnothing$, $U_{1-f} x = (I - U_f) x = 0$, for any $x \in \mathcal{E}(F)$; consequently $U_f x = x$ or $U_f \mid \mathcal{E}(F) = I$.

But $(\xi - \lambda) f_\xi(\lambda) = f(\lambda)$ $(\lambda \in \Omega)$, hence

$$U_{(\xi-\lambda)f_{\xi}} = U_f = (\xi I - T) U_{f_{\xi}}.$$

In the last equality, if we consider the restriction to $\mathcal{E}(F)$, it results that $\xi \in \rho(T \mid \mathcal{E}(F))$ and thus $\sigma(T \mid \mathcal{E}(F)) \subset F$. Then, for any $x \in X$, it follows that

$y_S = U_{f_S} x \in \mathcal{E}(\text{supp}(f_S)) \subset \mathcal{E}(\overline{G}_S)$

$y_1 = U_{f_1} x \in \mathcal{E}(\text{supp}(f_1)) \subset \mathcal{E}(\overline{G}_1)$ and

$X = \mathcal{E}(\text{supp}(f_S)) + \mathcal{E}(\text{supp}(f_1))$

and on account of the above results we deduce that T is S-decomposable.

Remark 3.1. If $T \in \mathbf{B}(X)$ is an $\mathcal{A}_S$-scalar operator and U is an $\mathcal{A}_S$-spectral function for T, then T is S-decomposable, hence $S \subset \sigma(T)$ and on account of Theorem 3.1, it results that $\text{supp}(U) \subset \sigma(T)$.

Example 3.1. Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}$-scalar operator and let $U : \mathcal{A} \to \mathbf{B}(X)$ be its $\mathcal{A}$-spectral function; let also Y be a closed invariant subspace to T, which is not invariant with respect to U (i.e. there is a function $f \in \mathcal{A}$ such that Y is not invariant to U_f).

Then the quotient operator $\dot{T}$ induced by T in the quotient space $\dot{X} = X / Y$ is an $\mathcal{A}_S$-scalar operator, where $S = \sigma(T \mid Y)$ and $\mathcal{A}_S$ is the subalgebra of $\mathcal{A}$ composed by all functions $f \in \mathcal{A}$ which have one of the following properties:

(1) $\text{supp}(f) \cap S = \varnothing$;

(2) $\text{supp}(f) \supset S$ and $U_f Y \subset Y$.

If the functions $f \in \mathcal{A}$ satisfy the condition (1), then for $y \in Y$, we have the relation $\text{supp}(f) \cap \sigma_T(y) = \varnothing$. Accordingly $U_f y = 0$ and $\dot{U}_f$ makes sense, where $\dot{U}_f$ is the operator induced by $U_f \in \mathbf{B}(X)$ in the quotient space $\dot{X} = X / Y$.

If the functions $f \in \mathcal{A}$ satisfy the condition (2), then the functions $g = 1 - f$ verify the relation $\text{supp}(g) \cap S = \varnothing$. Therefore for $y \in Y$ we have $U_g y = 0$, i.e. $U_f y = y$. Since for $\xi \notin \text{supp}(f) \supset \sigma(T \mid Y)$, Y is invariant to the resolvent function $R(\xi,T)$ of T, it results that $f_\xi \in \mathcal{A}_S$ and hence the operators $\dot{U}_f$, $\dot{U}_{f_\xi}$ make sense also in this case.

It is easy to verify that the application $\dot{U}:\mathcal{A}_S \to \mathbf{B}(\dot{X})$ defined by $\dot{U}(f)=\dot{U}_f$, $f \in \mathcal{A}_S$, is an $\mathcal{A}_S$-spectral function for $\dot{T} \in \mathbf{B}(\dot{X})$, therefore $\dot{T}$ is $\mathcal{A}_S$-scalar.

Example 3.2. Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}$-scalar operator, let $U:\mathcal{A} \to \mathbf{B}(X)$ be its $\mathcal{A}$-spectral function and let also Y be a closed invariant subspace to T, which is not invariant to U.

Then the restriction $T\,|\,Y$ is an $\mathcal{A}_S$-scalar operator, where $S=\sigma(\dot{T})$ and $\mathcal{A}_S$ is an S-admissible subalgebra of $\mathcal{A}$ composed by all functions f which have one of the following properties:

(1) $\operatorname{supp}(f) \cap S = \varnothing$;

(2) $\operatorname{supp}(f) \supset S$.

The functions $f \in \mathcal{A}$ with the properties $\operatorname{supp}(f) \cap S \neq \varnothing$ and $S \not\subset \operatorname{supp}(f)$, do not belong to the algebra $\mathcal{A}_S$.

It can be easily to verify that the restriction $U\,|\,\mathcal{A}_S:\mathcal{A}_S \to \mathbf{B}(Y)$ defined by $U\,|\,\mathcal{A}_S(f)=U_f$, $f \in \mathcal{A}_S$, is an $\mathcal{A}_S$-spectral function for $T\,|\,Y \in \mathbf{B}(Y)$, therefore $T\,|\,Y$ is $\mathcal{A}_S$-scalar.

Remark 3.2. If $T \in \mathbf{B}(X)$ is an $\mathcal{A}$-scalar operator, $U:\mathcal{A} \to \mathbf{B}(X)$ is an $\mathcal{A}$-spectral function for T and Y is a closed subspace invariant to both T and U, then the restriction $T\,|\,Y$ and the quotient $\dot{T}$ induced by T in the quotient space $\dot{X}=X/Y$ are $\mathcal{A}$-scalar operators.

If $S \subset \sigma(T)$ is totally disconnected (i.e. $\dim S = 0$), then the topology τ of $\mathbb{C}$ induces a topology τ_0 of countable metric space on S. It is shown that τ_0 has a basis composed by spectral sets for T. According to the Lindelöf theorem, we obtain for τ_0 a countable basis $(\delta_n)_{n\in\mathbb{N}}$ of spectral sets for T (a set $\delta \subset \mathbb{C}$ is said to be a *spectral set* for T if $\delta \cap \sigma(T)$ is both open and closed in $\sigma(T)$) and we define the projector $E(\delta)$ by the relation

$$E(\delta) = \frac{1}{2\pi i}\int_C \Re(\lambda, T)\,\mathrm{d}\lambda$$

for any δ in the field Σ of all spectral sets, where C is a Jordan curve (admissible contour) containing $\delta \cap \sigma(T)$ and does not contain "inside" another points of $\sigma(T)$.

Example 3.3. If an operator $T \in \mathbf{B}(X)$ has the spectrum $\sigma(T) = S \cup \delta$, with $\dim S \geq 1$ and $\delta = \sigma(T) \setminus S$, with $\dim \delta = 0$, S and δ being spectral sets for T, then, according to Example 1.20, Chapter 3, [4], the operator $T \mid E(\delta)X$ is $\mathcal{A}$-scalar, with $\sigma(T \mid E(\delta)X) = \delta$ and the operator $T \mid E(S)X$ has the spectrum equal to S. It is easy to verify that T is $\mathcal{A}_S$-scalar.

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