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n-ARY H_v-MODULES WITH EXTERNAL n-ARY P-HYPEROPERATION

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The class of (m, n)-ary H_v-modules is larger than the well known class H_v-modules. A wide subclass of (m, n)-ary H_v-modules is n-ary P-H_v-modules. In this paper, we consider and study a module over a ring and we define three kinds of external n-ary P-hyperoperations. By using external n-ary P-hyperoperations and certain conditions, we construct several (m, n)-ary H_v-modules.

Keywords: hyperoperation, H_v-group, H_v-ring, H_v-module, P-hyperoperation.
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1. Introduction and basic definitions

Hyperstructure theory was born in 1934 when Marty [19] defined hypergroups as a generalization of groups. Let H be a non-empty set and let $\wp^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we denote $A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. Under certain conditions, we obtain the so-called semihypergroups and hypergroups . Basic definitions and results about the hyperstructures are found in [2, 3]. Hyperrings are essentially rings with approximately modified axioms. There are several kinds of hyperrings that can be defined on a non-empty set . In 2007, Davvaz and Leoreanu-Fotea [9] published a book titled Hyperring Theory and Applications. Sometimes, external hyperoperation is considered. An example of a hyperstructure, endowed both with an internal hyperoperation and an external hyperoperation is the so-called hypermodule.

The theory of H_v-structures has been introduced by Vougiouklis [25]. The concept of H_v-structures constitutes a generalization of the well-known algebraic hyperstructures (hypergroups, hyperrings, hypermodules). Actually, some axioms concerning the above hyperstructures are replaced by their corresponding weak axioms. Basic definitions and results about the H_v-structures are found in [6, 24]. A hypergroupoid $(H, \circ)$ is called an *H_v-semigroup* if for all x, y, z of H we have $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$, which means that $\bigcup_{u\in x\circ y} u \circ z \cap \bigcup_{v\in y\circ z} x \circ v \neq \emptyset$.

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We say that an H_v-semigroup $(H, \circ)$ is an H_v*-group* [25] if for all $x \in H$, we have $x \circ H = H \circ x = H$.

A multivalued system $(R, +, \cdot)$ is an H_v*-ring* [24] if

(1) $(R, +)$ is an H_v-group,
(2) $(R, \cdot)$ is an H_v-semigroup,
(3) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x \cdot (y + z) \cap x \cdot y + x \cdot z \neq \emptyset$ and $(x + y) \cdot z \cap x \cdot z + y \cdot z \neq \emptyset$.

An H_v-ring may be commutative with respect either to $(+)$ or $(\cdot)$. If H is commutative with respect to both $(+)$ and $(\cdot)$, then we call it a *commutative* H_v*-ring*. If there exists $u \in R$ such that $x \cdot u = u \cdot x = \{x\}$ for all $x \in R$, then u is called the scalar unit of R, which obviously is unique, and is denoted by 1.

A non-empty set M is an H_v-module over an H_v-ring R, if $(M, +)$ is a (commutative) H_v-group and there exists a map $\cdot : R \times M \longrightarrow \wp^*(M)$, $(r, x) \mapsto rx$, such that

(1) $r(x + y) \cap (rx + ry) \neq \emptyset$,
(2) $(r + s)x \cap (rx + sx) \neq \emptyset$,
(3) $(rs)x \cap r(sx) \neq \emptyset$,

for all $r, s \in R$ and $x, y \in M$.

The notion of P-hyperoperations introduced for hypergroups in [26] and generalized in [23], also see [27]. A wide class of H_v-rings is the class of H_v-rings with P-hyperoperations [22]. A nice application of P-hyperstructures appeared in [10]. Let $(M, +)$ be a module over the ring R. According to [28], three kinds of external P-hyperoperations, for all $(\lambda, v) \in R \times M$, can be defined as follows:

(1) If P is a non-empty subset of R, then $\lambda P^* v = (\lambda P)v$.
(2) If P is a non-empty subset of M, then $\lambda P_* v = \lambda(P + v)$.
(3) If P_1 is a non-empty subset of R and P_2 is a non-empty subset of M, then $\lambda P^*_{1,2} v = (\lambda P_1)(P_2 + v)$.

Note that P^* is a special case of $P^*_{1,2}$ because it is obtained by setting $P_2 = \{0\}$ and $P_1 = P$.

Our aim in this paper is to give generalizations of following theorems:

Theorem 1.1. [28] *Let M be a module over the ring R. Let P be a non-empty subset of R and $a \in P \cap Z(R)$ such that $a^2 \in P$. Then, $(M, +, P^*)$ is an H_v-module over R.*

Theorem 1.2. [28] *Let M be a module over the ring R. Let P be a non-empty subset of M such that $0 \in P$. Then, $(M, +, P_*)$ is an H_v-module over R.*

Theorem 1.3. [28] *Let M be a module over the ring R, P_1 be a non-empty subset of R and P_2 be a non-empty subset of M. If there exists $a \in P_1 \cap Z(R)$ such that $a^2 = a$ and there exists $b \in P_2$ such that $a \cdot b = 0$, then $(M, +, P^*_{1,2})$ is an H_v-module over R.*

2. n-ary H_v-modules

The notion of an n-ary group was introduced by Dörnte [12], which is a natural generalization of group. One can find the basic results on n-ary groups in Post [21]. The notion of n-ary hypergroup was first introduced by Davvaz and Vougiouklis as a generalization of n-ary group [7], and studied mainly in [8, 13, 14, 16, 17, 18].

In general, a mapping $f : H^n \longrightarrow \wp^*(H)$ is called an *n-ary hyperoperation* [7]. Let f be an n-ary hyperoperation on H and $A_1, \cdots, A_n$ be non-empty subsets of H. We define $f(A_1, \cdots, A_n) = \bigcup_{x_i \in A_i} f(x_1, \cdots, x_n)$. The sequence $x_i, x_{i+1}, \cdots, x_j$ is denoted by x_i^j. For $j < i$, x_i^j is the empty set. In this convention $f(x_1, \cdots, x_i, y_{i+1}, \cdots, y_j, z_{j+1}, \cdots, z_n)$ is written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. Also, for $y \in H$ and $1 \leq i \leq n$ we denote the $f(x_1^i, \underbrace{y, \cdots, y}_{n-i})$ by $f(x_1^i, y_.)$. A non-empty set H with an n-ary hyperoperation $f : H^n \longrightarrow \wp^*(H)$ is called an *n-ary H_v-semigroup* [13] if the weak associativity is valid, i.e., for every $x_1, \cdots, x_{2n-1} \in H$,

$$\bigcap_{1 \leq i \leq n} f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) \neq \phi.$$

If for every $x_1, \cdots, x_{2n-1} \in H$ and $i, j \in \{1, \cdots, n\}$

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n+i-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

then (H, f) is called an *n-ary semihypergroup* [7]. An n-ary H_v-semigroup (H, f) in which, for every $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, b \in H$ and $1 \leq i \leq n$, there exists $x_i \in H$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ is called an n-ary H_v-group. This condition can be formulated by $f(a_1^{i-1}, H, a_{i+1}^n) = H$. Moreover, if for all $(x_1, \cdots, x_n) \in H^n$, the set $f(x_1^n)$ is singleton, then f is an n-ary operation and (H, f) is an *n-ary semigroup* or *n-ary group.*

The notion of (m, n)-rings introduced by Crombez [4], Crombez and Timm [5], and Dudeck [11]. Recently, the notation of (m, n)-hyperrings studied by Mirvakili and Davvaz [20] and obtained (m, n)-rings from (m, n)-hyperrings by fundamental relations. The following definition is a general form of the concept that investigated in [15].

Definition 2.1. *An (m, n)-ary H_v-ring is an algebraic structure (R, f, g) which satisfies the following axioms:*

1. *(R, f) is an m-ary H_v-group,*
2. *(R, g) is an n-ary H_v-semigroup,*
3. *the n-ary hyperoperation g is weak distributive with respect to the hyperoperation f, i.e.,*

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) \cap f(g(a_1^{i-1}, x_1, a_{i+1}^n), \cdots, g(a_1^{i-1}, x_m, a_{i+1}^n)) \neq \phi,$$

for every sequence a_1^{i-1}, a_{i+1}^n, x_1^m in R and $1 \leq i \leq n$.

When (R, f) is an m-ary hypergroup, (R, g) is an n-ary semihypergroup and g is distributive with respect to f, then (R, f, g) is called an *(m, n)-ary hyperring.*

Also, if f and g are m-ary and n-ary operations, respectively, then (R, f) is an n-ary group, (R, g) is an n-ary semigroup, the n-ary operation g is distributive with respect to the m-ary operation f and in this case (R, f, g) is an (m, n)*-ary ring.*

In [1], Anvariyeh et al. studied the class of (m, n)-ary hypermodules and they gave several properties and examples of them. Now, we introduce the concept of (m, n)-ary H_v-module over an (m, n)-ary H_v-ring R as follows:

Definition 2.2. *Let M be a non-empty set. Then, $M = (M, h, k)$ is an (m, n)-ary H_v-module over an (m, n)-ary H_v-ring R, if (M, h) is a (commutative) m-ary H_v-group and the map*

$$k : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$

satisfies in the following conditions:

(1) $k(r_1^{n-1}, h(x_1^m)) \cap h(k(r_1^{n-1}, x_1), \ldots, k(r_1^{n-1}, x_m)) \neq \emptyset$,
(2) $k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x) \cap h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)) \neq \emptyset$,
(3) $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) \cap k(r_1^{n-1}, k(r_n^{2n-2}, x)) \neq \emptyset$.

If k is a scalar n-ary hyperoperation, $S_1, \ldots, S_{n-1}$ are non-empty subsets of R and $M_1 \subseteq M$, we set

$$k(S_1, \ldots, S_{n-1}, M_1) = \cup\{k(r_1, \ldots, r_{n-1}, x) |\ r_i \in S_i, i = 1, \ldots, n-1, x \in M_1\}.$$

An (m, n)-ary H_v-module M is an H_v*-module*, if $n = 2$.

3. n-ary P-hyperoperations

Definition 3.1. *Let $(M, +)$ be a module over the ring R. Then, three kinds of external n-ary P-hyperoperations can be defined as follows:*

(1) *If P is a non-empty subset of R, then*

$$P_R : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$
$$(r_1^{n-1}, x) \mapsto (r_1 \ldots r_{n-1} P)x.$$

(2) *If P is a non-empty subset of M, then*

$$P_M : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$
$$(r_1^{n-1}, x) \mapsto r_1 \ldots r_{n-1}(P + x).$$

(3) *If P_1 is a non-empty subset of R and P_2 is a non-empty subset of M, then*

$$P_{RM} : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$
$$(r_1^{n-1}, x) \mapsto (r_1 \ldots r_{n-1} P_1)(P_2 + x).$$

Remark 3.1. *Let M be a module over the ring R. We define*

$$\begin{array}{l} f : R \times \ldots \times R \longrightarrow R,\ f(r_1^m) = r_1 + \ldots + r_m, \\ g : R \times \ldots \times R \longrightarrow R,\ g(r_1^n) = r_1 \ldots r_n, \\ h : M \times \ldots \times M \longrightarrow M,\ h(x_1^m) = x_1 + \ldots + x_m, \\ k : R \times \ldots \times R \times M \longrightarrow M,\ k(r_1^{n-1}, x) = (r_1 \ldots r_{n-1})x. \end{array}$$

Then, (M, h, k) is an (m, n)-ary module over the (m, n)-ary ring (R, f, g).

Note that every (m, n)-ary module can consider as an (m, n)-ary H_v-module. By consideration the above remark, we have the following results:

Theorem 3.1. *Let M be a module over the ring R. Let P be a non-empty subset of R and $a \in P \cap Z(R)$ such that $a^2 \in P$. Then, (M, h, P_R) is an (m, n)-ary H_v-module over (R, f, g).*

Proof. Indeed, we have

$$\begin{array}{l} P_R\left(r_1^{n-1}, h(x_1^m)\right) = (r_1 \ldots r_{n-1}P)(x_1 + \ldots + x_m) \\ = \{(r_1 \ldots r_{n-1}p)(x_1 + \ldots + x_m) \mid p \in P\} \\ = \{(r_1 \ldots r_{n-1}p)x_1 + \ldots + (r_1 \ldots r_{n-1}p)x_m \mid p \in P\} \\ \subseteq \{(r_1 \ldots r_{n-1}p_1)x_1 + \ldots + (r_1 \ldots r_{n-1}p_m)x_m \mid p_1, \ldots, p_m \in P\} \\ = (r_1 \ldots r_{n-1}P)x_1 + \ldots + (r_1 \ldots r_{n-1}P)x_m \\ = h\left((r_1 \ldots r_{n-1}P)x_1, \ldots, (r_1 \ldots r_{n-1}P)x_m\right) \\ = h\left(P_R(r_1^{n-1}, x_1), \ldots, P_R(r_1^{n-1}, x_m)\right); \end{array}$$

$$\begin{array}{l} P_R\left(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x\right) = (r_1 \ldots r_{i-1} f(s_1^m) r_{i+1} \ldots r_{n-1}P)x \\ = (r_1 \ldots r_{i-1}(s_1 + \ldots + s_m)r_{i+1} \ldots r_{n-1}P)x \\ = (r_1 \ldots r_{i-1}s_1r_{i+1} \ldots r_{n-1}P)x + \ldots + (r_1 \ldots r_{i-1}s_mr_{i+1} \ldots r_{n-1}P)x \\ = h\left(r_1 \ldots r_{i-1}s_1r_{i+1} \ldots r_{n-1}P)x, \ldots, r_1 \ldots r_{i-1}s_mr_{i+1} \ldots r_{n-1}P)x\right) \\ = h\left(P_R(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, P_R(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)\right); \end{array}$$

also, we have

$$\begin{array}{rl} (r_1 \ldots r_{2n-2}a^2)x \in & (r_1 \ldots r_{2n-2}P)x = P_R(r_1^{i-1}, (r_i \ldots r_{i+n-1}), r_{i+n}^{2n-2}, x) \\ & = P_R(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) \\ (r_1 \ldots r_{2n-2}a^2)x \in & (r_1 \ldots r_{2n-2}PP)x = (r_1 \ldots r_{n-1}r_n \ldots r_{2n-2}PP)x \\ & = (r_1 \ldots r_{n-1}Pr_n \ldots r_{2n-2}P)x \\ & = P_R(r_1^{n-1}, P_R(r_n^{2n-2}, x)). \end{array}$$

Therefore,

$$P_R(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) \cap P_R(r_1^{n-1}, P_R(r_n^{2n-2}, x)) \neq \emptyset.$$

□

Theorem 3.2. *Let M be a module over the ring R. Let P be a non-empty subset of M such that $0 \in P$. Then, (M, h, P_R) is an (m, n)-ary H_v-module over (R, f, g).*

Proof. Indeed, we have

$$
\begin{aligned}
&P_M\left(r_1^{n-1}, h(x_1^m)\right) = (r_1 \dots r_{n-1})(P + x_1 + \dots + x_m)\\
&= \{(r_1 \dots r_{n-1})(p + x_1 + \dots + x_m) \mid p \in P\}\\
&= \{(r_1 \dots r_{n-1})(p + x_1) + (r_1 \dots r_{n-1})(0 + x_2) + \dots\\
&\quad +(r_1 \dots r_{n-1})(0 + x_m) \mid p \in P\}\\
&\subseteq \{(r_1 \dots r_{n-1})(p_1 + x_1) + \dots + (r_1 \dots r_{n-1})(p_m + x_m) \mid p_1, \dots, p_m \in P\}\\
&= (r_1 \dots r_{n-1})(P + x_1) + \dots + (r_1 \dots r_{n-1})(P + x_m)\\
&= h\left((r_1 \dots r_{n-1})(P + x_1), \dots, (r_1 \dots r_{n-1})(P + x_m)\right)\\
&= h\left(P_M(r_1^{n-1}, x_1), \dots, P_M(r_1^{n-1}, x_m)\right);
\end{aligned}
$$

$$
\begin{aligned}
&P_M\left(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x\right) = (r_1 \dots r_{i-1} f(s_1^m) r_{i+1} \dots r_{n-1})(P + x)\\
&= (r_1 \dots r_{i-1}(s_1 + \dots + s_m) r_{i+1} \dots r_{n-1})(P + x)\\
&= (r_1 \dots r_{i-1} s_1 r_{i+1} \dots r_{n-1})(P + x) + \dots + (r_1 \dots r_{i-1} s_m r_{i+1} \dots r_{n-1})(P + x)\\
&= h\left((r_1 \dots r_{i-1} s_1 r_{i+1} \dots r_{n-1})(P + x), \dots, (r_1 \dots r_{i-1} s_m r_{i+1} \dots r_{n-1})(P + x)\right)\\
&= h\left(P_M(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \dots, P_M(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)\right);
\end{aligned}
$$

and finally

$$
\begin{aligned}
&P_M(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) = P_M(r_1^{i-1}, (r_i \dots r_{i+n-1}), r_{i+n}^{2n-2}, x)\\
&= (r_1 \dots r_{2n-2})(P + x) = (r_1 \dots r_{n-1})(r_n \dots r_{2n-2})(P + x)\\
&= (r_1 \dots r_{n-1})(0 + r_n \dots r_{2n-2}(P + x))\\
&\subseteq (r_1 \dots r_{n-1})(P + r_n \dots r_{2n-2}(P + x))\\
&= P_M\left(r_1^{n-1}, r_n \dots r_{2n-2}(P + x)\right) = P_M\left(r_1^{n-1}, P_M(r_n^{2n-2}, x)\right).
\end{aligned}
$$

□

Theorem 3.3. *Let M be a module over the commutative Boolean ring R, P_1 be a non-empty subset of R and P_2 be a non-empty subset of M such that there exist $a \in R$ and $b \in M$ such that $ab = 0$. Then, (M, h, P_{RM}) is an (m, n)-ary H_v-module over (R, f, g).*

Proof. Indeed, we have

$$
\begin{aligned}
&(r_1 \dots r_{n-1} a) x_1 + \dots + (r_1 \dots r_{n-1} a) x_m\\
&= (r_1 \dots r_{n-1} a) b + (r_1 \dots r_{n-1} a) x_1 + \dots + (r_1 \dots r_{n-1} a) x_m\\
&\in \{(r_1 \dots r_{n-1} s) u + (r_1 \dots r_{n-1} s) x_1 + \dots + (r_1 \dots r_{n-1} s) x_m \mid s \in P_1, u \in P_2\}\\
&= \{(r_1 \dots r_{n-1} s)(u + x_1 + \dots + x_m) \mid s \in P_1, u \in P_2\}\\
&= (r_1 \dots r_{n-1} P_1)(P_2 + x_1 + \dots + x_m)\\
&= P_{RM}(r_1^{n-1}, h(x_1^m)),
\end{aligned}
$$

$$
\begin{aligned}
&(r_1 \dots r_{n-1} a) x_1 + \dots + (r_1 \dots r_{n-1} a) x_m\\
&(r_1 \dots r_{n-1} a)(b + x_1) + \dots + (r_1 \dots r_{n-1} a)(b + x_m)\\
&\in \{(r_1 \dots r_{n-1} s_1)(u_1 + x_1) + \dots + (r_1 \dots r_{n-1} s_m)(u_m + x_m) \mid s_i \in P_1, u_i \in P_2\}\\
&= (r_1 \dots r_{n-1} P_1)(P_2 + x_1) + \dots + (r_1 \dots r_{n-1} P_1)(P_2 + x_m)\\
&= h\left((r_1 \dots r_{n-1} P_1)(P_2 + x_1), \dots, (r_1 \dots r_{n-1} P_1)(P_2 + x_m)\right)\\
&= h(P_{RM}(r_1^{n-1}, x_1), \dots, P_{RM}(r_1^{n-1}, x_m)).
\end{aligned}
$$

Therefore,

$$P_{RM}(r_1^{n-1}, h(x_1^m)) \cap h(P_{RM}(r_1^{n-1}, x_1), \ldots, P_{RM}(r_1^{n-1}, x_m)) \neq \emptyset.$$

$$\begin{aligned}
&P_{RM}\left(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x\right) = (r_1 \ldots r_{i-1} f(s_1^m) r_{i+1} \ldots r_{n-1} P_1)(P_2 + x) \\
&= (r_1 \ldots r_{i-1}(s_1 + \ldots + s_m) r_{i+1} \ldots r_{n-1} P_1)(P_2 + x) \\
&= (r_1 \ldots r_{i-1} s_1 r_{i+1} \ldots r_{n-1} P_1)(P_2 + x) + \ldots \\
&\quad + (r_1 \ldots r_{i-1} s_m r_{i+1} \ldots r_{n-1} P_1)(P_2 + x) \\
&= h(r_1 \ldots r_{i-1} s_1 r_{i+1} \ldots r_{n-1} P_1)(P_2 + x), \ldots, \\
&\quad r_1 \ldots r_{i-1} s_m r_{i+1} \ldots r_{n-1} P_1)(P_2 + x)) \\
&= h\left(P_{RM}(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, P_{RM}(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)\right);
\end{aligned}$$

$$\begin{aligned}
&(r_1 \ldots r_{2n-2} a)(b + x) \\
&\in (r_1 \ldots r_{2n-2} P_1)(P_2 + x) \\
&= P_{RM}(r_1^{i-1}, (r_i \ldots r_{i+n-1}), r_{i+n}^{2n-2}, x) \\
&= P_{RM}(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x),
\end{aligned}$$

$$\begin{aligned}
&(r_1 \ldots r_{2n-2} a)(b + x) = (r_1 \ldots r_{n-1} a) b + (r_1 \ldots r_{2n-2} a^2)(b + x) \\
&\in (r_1 \ldots r_{n-1} P_1) P_2 + (r_1 \ldots r_{2n-2} P_1 P_1)(P_2 + x) \\
&= (r_1 \ldots r_{n-1} P_1)(P_2 + (r_n \ldots r_{2n-2} P_1)(P_2 + x)) \\
&= P_{RM}\left(r_1^{n-1}, (r_n \ldots r_{2n-2} P_1)(P_2 + x)\right) \\
&= P_{RM}\left(r_1^{n-1}, P_{RM}(r_n^{2n-2}, x)\right).
\end{aligned}$$

Therefore,

$$P_{RM}(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) \cap P_{RM}(r_1^{n-1}, P_{RM}(r_n^{2n-2}, x)) \neq \emptyset.$$

□

Corollary 3.1. *Let N be an H_v-submodule of the (m,n)-ary P_R-H_v-module (M, h, P_R). We define*

$$h^* : \underbrace{M/N \times \ldots \times M/N}_{m} \longrightarrow M/N$$
$$(x_1 + N, \ldots, x_m + N) \mapsto x_1 + \ldots + x_m + N,$$
$$P_R^* : \underbrace{R \times \ldots \times R}_{n-1} \times M/N \longrightarrow \wp^*(M/N)$$
$$(r_1^{n-1}, x + N) \mapsto \{y + N \mid y \in (r_1 \ldots r_{n-1} P)x\}.$$

Then, $(M/N, h^*, P_R^*)$ is an (m,n)-ary H_v-module over (R, f, g).

Corollary 3.2. *Let N be an H_v-submodule of the (m,n)-ary P_M-H_v-module (M, h, P_M). We define*

$$h^* : \underbrace{M/N \times \ldots \times M/N}_{m} \longrightarrow M/N$$
$$(x_1 + N, \ldots, x_m + N) \mapsto x_1 + \ldots + x_m + N,$$
$$P_M^* : \underbrace{R \times \ldots \times R}_{n-1} \times M/N \longrightarrow \wp^*(M/N)$$
$$(r_1^{n-1}, x + N) \mapsto \{y + N \mid y \in r_1 \ldots r_{n-1}(P + x)\}.$$

Then, $(M/N, h^*, P_M^*)$ is an (m,n)-ary H_v-module over (R, f, g).

Corollary 3.3. *Let N be an H_v-submodule of the (m,n)-ary P_{RM}-H_v-module (M, h, P_{RM}). We define*

$$h^* : \underbrace{M/N \times \ldots \times M/N}_{m} \longrightarrow M/N$$
$$(x_1 + N, \ldots, x_m + N) \mapsto x_1 + \ldots + x_m + N,$$
$$P^*_{RM} : \underbrace{R \times \ldots \times R}_{n-1} \times M/N \longrightarrow \wp^*(M/N)$$
$$(r_1^{n-1}, x + N) \mapsto \{y + N \mid y \in (r_1 \ldots r_{n-1}P_1)(P_2 + x)\}.$$

Then, $(M/N, h^*, P^*_{RM})$ is an (m,n)-ary H_v-module over (R, f, g).

Remark 3.2. *Let (M_1, h_1, k) and (M_2, h_2, k) be two (m,n)-ary H_v-modules over an (m,n)-ary H_v-ring R. A strong homomorphism from M_1 to M_2 is a mapping $\varphi : M_1 \longrightarrow M_2$ such that*

(1) $\varphi(h_1(a_1, \ldots, a_m)) = h_2(\varphi(a_1), \ldots, \varphi(a_m))$,
(2) $\varphi(k(r_1, \ldots, r_{n-1}, a)) = k(r_1, \ldots, r_{n-1}, \varphi(a))$,

for all $a_1, \ldots, a_m, a \in M$ and $r_1, \ldots, r_{n-1} \in R$.

φ is called a weak homomorphism if in (1) and (2) we have empty intersection instead of equality.

Theorem 3.4. *Let (M, h, P_M) and $(M', h', P'_{M'})$ be two (m,n)-ary H_v-modules over (R, f, g) and $1 \in R$. Let $\varphi : M \longrightarrow M'$ be an ordinary homomorphism of modules. Then, $\varphi : (M, h, P_M) \longrightarrow (M', h', P'_{M'})$ is a weak homomorphism if and only if $\varphi(P) \cap P' \neq \emptyset$, and it is strong homomorphism if and only if $\varphi(P) = P'$.*

Proof. Assume that φ is a weak homomorphism. Then, $\varphi(P_M(r_1^{n-1}, x)) \cap P'_{M'}(r_1^{n-1}, \varphi(x)) \neq \emptyset$, for all $r_1, \ldots r_{n-1} \in R$ and $x \in M$. So, $\varphi(P_M(1, \ldots, 1, 0)) \cap P'_{M'}(1, \ldots 1, \varphi(0)) \neq \emptyset$ which implies that $\varphi((1 \ldots 1)(P + 0)) \cap (1 \ldots 1)(P' + 0) \neq \emptyset$. Thus, $\varphi(P) \cap P' \neq \emptyset$.

Conversely, suppose that $y \in \varphi(P) \cap P'$. Then,

$$(r_1 \ldots r_{n-1})(\varphi(x) + y) \in (r_1 \ldots r_{n-1})(\varphi(x) + \varphi(P)) \cap (r_1 \ldots r_{n-1})(\varphi(x) + P'),$$

for all $r_1, \ldots, r_{n-1} \in R$ and $x \in M$. Hence,

$$\varphi((r_1 \ldots r_{n-1})(x + P)) \cap (r_1 \ldots r_{n-1})(\varphi(x) + P') \neq \emptyset.$$

Therefore,

$$\varphi(P_M(r_1^{n-1}, x)) \cap P'_{M'}(r_1^{n-1}, \varphi(x)) \neq \emptyset.$$

Now, suppose that φ is a strong homomorphism. Then, $\varphi(P_M(r_1^{n-1}, x)) = P'_{M'}(r_1^{n-1}, \varphi(x))$, for all $r_1, \ldots, r_{n-1} \in R$ and $x \in M$. So, $\varphi(P_M(1, \ldots, 1, 0)) = P'_{M'}(1, \ldots, 1, \varphi(0))$ which implies that $\varphi((1 \ldots 1)(P + 0)) = (1 \ldots 1)(P' + 0)$. Thus, $\varphi(P) = P'$.

Conversely, suppose that $\varphi(P) = P'$. Then,

$$(r_1 \ldots r_{n-1})(\varphi(x) + \varphi(P)) = (r_1 \ldots r_{n-1})(\varphi(x) + P'),$$

for all $r_1, \ldots, r_{n-1} \in R$ and $x \in M$. Hence,

$$\varphi((r_1 \ldots r_{n-1})(x + P)) = (r_1 \ldots r_{n-1})(\varphi(x) + P').$$

Therefore, $\varphi(P_M(r_1^{n-1}, x)) = P'_{M'}(r_1^{n-1}, \varphi(x))$. □

REFERENCES

[1] *S. M. Anvariyeh, S. Mirvakili and B. Davvaz*, Fundamental relation on (m, n)-ary hypermodules on $(m, n-$-ary hyperring, ARS Combinatoria, **94** (2010), 273-288.

[2] *P. Corsini*, Prolegomena of hypergroup theory, Second edition, Aviani editore, 1993.

[3] *P. Corsini and V. Leoreanu*, Applications of hyperstructure theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[4] *G. Crombez*, On $(n; m)$-rings, Abh. Math. Semin. Univ. Hamburg, **37** (1972), 180-199.

[5] *G. Crombez and J. Timm*, On $(n; m)$-quotient rings, Abh. Math. Semin. Univ. Hamburg, **37** (1972), 200-203.

[6] *B. Davvaz*, A brief survey of the theory of H_v-structures, Proc. 8^{th} International Congress on Algebraic Hyperstructures and Applications, 1-9 Sep., 2002, Samothraki, Greece, Spanidis Press, (2003) 39-70.

[7] *B. Davvaz and T. Vougiouklis*, n-Ary hypergroups, Iranian Journal of Science and Technology, Transaction A, **30** (2006), 165-174.

[8] *B. Davvaz, W.A. Dudek and T. Vougiouklis*, A generalization of n-ary algebraic systems, Communications in Algebra, **37** (2009), 1248-1263.

[9] *B. Davvaz, V. Leoreanu-Fotea*, Hyperring Theory and Applications, International Academic Press, USA, 2007.

[10] *B. Davvaz, R. M. Santilli and T. Vougiouklis*, Studies of multivalued hyperstructures for the characterization of matter-antimatter systems and their extension, Algebras Groups and Geometries, **28** (2011), 105-116.

[11] *W.A. Dudek*, On the divisibility theory in (m, n)-rings, Demonstratio Math., **14** (1981), 19-32.

[12] *W. Dörnte*, Unterschungen uber einen uerallgemeinerten gruppenbe griff, Math. Z., **29** (1929), 1-19.

[13] *M. Ghadiri, B. N. Waphare, B. Davvaz*, n-ary H_v-Structures, Southeast Asian Bull. Math., **34** (2010), 243-255.

[14] *M. Ghadiri, B. N. Waphare*, n-ary P- H_v-Groups, J. Appl. Math. and Informatics, **26** (2008), 945-959.

[15] *M. Ghadiri, B. N. Waphare and B. Davvaz*, Quotient n-Ary P-H_v-Rings, Carpathian J. Math., **28** (2012), 239-246.

[16] *V. Leoreanu-Fotea and B. Davvaz*, n-hypergroups and binary relations, European Journal of Combinatorics, **29** (2008), 1027-1218.

[17] *V. Leoreanu-Fotea and B. Davvaz*, Join n-spaces and lattices, Journal Multiple-valued Logic and Soft Computing, **15** (2009), 421-432.

[18] *V. Leoreanu-Fotea and B. Davvaz*, Roughness in n-ary hypergroups, Information Sciences, 178 (2008) 4114-4124.

[19] *F. Marty*, Sur une generalization de la notion de group, 8th Congres Math. Scandinaves, Stockholm, (1934), 45-49.

[20] *S. Mirvakili and B. Davvaz*, Relations on Krasner (m, n)-hyperrings, European Journal of Combinatorics, **31** (2010), 790-802.

[21] *E.L. Post*, Polyadic groups, Trans. Amer. Math. Soc., **48** (1940), 208-350.

[22] *S. Spartalis*, Quotients of P-H_v-rings, New frontiers in hyperstructures (Molise, 1995), 167-177, Ser. New Front. Adv. Math. Ist. Ric. Base, Hadronic Press, Palm Harbor, FL, 1996.

[23] *T. Vougiouklis*, Generalization of P-hypergroups, Rent. de Circolo Math, di Palermo, S. II, **36** (1987), 114-121.

[24] *T. Vougiouklis*, Hyperstructures and their Representations, Hadronic Press, Inc., 115, Palm Harber, USA, 1994.

[25] *T. Vougiouklis*, The fundamental relation in hyperrings. The general hyperfield, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Appl. (AHA 1990), World Scientific, (1991), 203-211.

[26] *T. Vougiouklis*, Cyclicity in a special class of hypergroups, Acta Un. Car.-Math. Ph., **22** (1981), 3-6.

[27] *T. Vougiouklis*, Isomorphisms on P-hypergroups and cyclicity, Ars Combinatoria., **29** (1990), 241-245.

[28] *T. Vougiouklis and A. Dramalidis*, H_v-modulus with external P-hyperoperations, Algebraic hyperstructures and applications (Iasi, 1993), 191-197, Hadronic Press, Palm Harbor, FL, 1994.