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(IMPLICATIVE) PSEUDO-VALUATIONS ON R_0-ALGEBRAS

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The concepts of (implicative) pseudo-valuations on R_0-algebras are introduced and some related characterizations are investigated. The relationship between a pseudo-valuation and an implicative pseudo-valuation is provided. In particular, we show that a pseudo-valuation on R_0-algebras is Boolean if and only if it is implicative. Finally, we prove that the binary operation in R_0-algebras is uniformly continuous based on the notion of pseudo-valuations.

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1. **Introduction**

The concept of R_0-algebras was first introduced by Wang in [7] by providing an algebraic proof of the completeness theorem of a formal deductive system. Further, Pei [6] proved NM-algebras are categorically isomorphic to R_0-algebras. From [6, 7], we can find some concrete applications of R_0-algebras. In 2008, Iorgulescu published a book *Algebras of logic as BCK-algebras*. In this book, she introduced some logical algebras and obtained some important results.

Busneag [2] defined a pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Also, Busneag [3] provided several theorems on extensions of pseudo-valuations. Busneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras[3]).

In this paper, we introduce the concepts of (implicative) pseudo-valuations on R_0-algebras and investigate some related characterizations. The relationship between a pseudo-valuation and an implicative pseudo-valuation is provided. In particular, we show that a pseudo-valuation on R_0-algebras is Boolean if and only if it is implicative. Finally, we prove that the binary operation in R_0-algebras is uniformly continuous based on the notion of pseudo-valuations.

This paper is an application of the concept of (implicative) pseudo-valuations on R_0-algebras, that is, we discuss a theoretical approach of the algebraic system in

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R_0-algebras by using the notion of (implicative) pseudo-valuations. Many interesting applications in information and engineering one can find (for example) in [4, 7].

2. Preliminaries

By an R_0-algebra [7], we mean a bounded lattice $L = (L, \leq, \wedge, \vee, ', \rightarrow, 0, 1)$ which $'$ is an order-reversing involution and with a binary operation $\rightarrow$ such that the following conditions hold:

(R1) $x \rightarrow y = y' \rightarrow x'$;
(R2) $1 \rightarrow x = x$;
(R3) $(y \rightarrow z) \wedge ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y \rightarrow z$;
(R4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(R5) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$;
(R6) $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (x' \vee y)) = 1$.

In any R_0-algebra L, the following statements are true (see [6, 7]):

(a1) $x \leq y \Leftrightarrow x \rightarrow y = 1$,
(a2) $x \leq y \rightarrow x$,
(a3) $x' = x \rightarrow 0$,
(a4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,
(a5) $x \rightarrow y \Rightarrow x \rightarrow z \geq y \rightarrow z$,
(a6) $x \rightarrow y \Rightarrow z \rightarrow x \geq z \rightarrow y$,
(a7) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,
(a8) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x) \rightarrow x)$,
(a9) $x \odot x' = 0, x \oplus x' = 1$,
(a10) $x \odot y \leq x \wedge y, x \odot (x \rightarrow y) \leq x \wedge y$,
(a11) $x \odot y \rightarrow z = x \rightarrow (y \rightarrow z)$,
(a12) $x \leq y \rightarrow (x \odot y)$,
(a13) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,
(a14) $x \rightarrow y \leq x \odot z \leq y \rightarrow z$,
(a15) $x \rightarrow y \leq (y \rightarrow z) \leq (x \rightarrow z)$,
(a16) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

In what follows, L is an R_0-algebra unless otherwise specified.

A non-empty subset A of L is called a filter of L if it satisfies the following conditions: (A1) $1 \in A$; (A2) $\forall x \in A, y \in L, x \rightarrow y \in A \Rightarrow y \in A$.

Now, we call a non-empty subset A of L an implicative filter if it satisfies (A1) and (A3) $x \rightarrow (y \rightarrow z) \in A, x \rightarrow y \in A \Rightarrow x \rightarrow z \in A$. Equivalently, a non-empty subset A of L is an implicative filter of L if and only if it satisfies (A1) and (A4) $x \rightarrow ((y \rightarrow z) \rightarrow y) \in A, x \in A \Rightarrow y \in A$, for all $x, y, z \in L$.

3. Pseudo-valuations

In this section, we introduce the notion of pseudo-valuations on an R_0-algebra.

Definition 3.1. *A real-valued function $\varphi : L \rightarrow \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, is called an pseudo-valuation on L if for all $x, y \in L$,*

(pv1) $\varphi(1) = 0$,
(pv2) $\varphi(y) \leq \varphi(x \to y) + \varphi(x)$.
A pseudo-valuation φ *on* L *satisfies the following:*
(pv3) $\forall x \in L, x \neq 1 \to \varphi(x) \neq 0$ *is called a valuation on* L.

Example 3.1. *Let* $L = \{0, a, b, c, 1\}$, *where* $0 < a < b < c < 1$. *Define* $'$ *and* $\to$ *as follows:*

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $(L, \wedge, \vee, ', \to)$ *is an* R_0*-algebra. Define two real-valued functions* φ_1 *and* φ_2 *on* L *by*

$$\varphi_1 = \begin{pmatrix} 0 & a & b & c & 1 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } \varphi_2 = \begin{pmatrix} 0 & a & b & c & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then φ_1 *and* φ_2 *are two pseudo-valuations on* L.

Proposition 3.1. *For any pseudo-valuation* φ *on* L*, then*
(1) φ *is order reversing.*
(2) $\forall x \in L, \varphi(x) \geq 0$.
(3) $\forall x, y \in L, \varphi(x \to y) \leq \varphi(y)$.

Proof. (1) Let $x, y \in L$ be such that $x \leq y$, then $x \to y = 1$, and so

$$\varphi(y) \leq \varphi(x \to y) + \varphi(x) = \varphi(1) + \varphi(x) = 0 + \varphi(x) = \varphi(x).$$

(2) Putting $y = 1$ in (pv2), we have

$$0 = \varphi(1) \leq \varphi(x \to 1) + \varphi(x) = \varphi(x).$$

(3) By (a2), $y \leq x \to y$. Thus, from (1), we have $\varphi(x \to y) \leq \varphi(y)$. □

Theorem 3.1. *If* φ *is a pseudo-valuation on* L*, then for all* $x, y, z \in L$*, we have*
(pv4) $x \to (y \to z) = 1 \Rightarrow \varphi(z) \leq \varphi(x) + \varphi(y)$.

Proof. Let φ be a pseudo-valuation on L, then by (pv2), we have

$$\varphi(z) \leq \varphi(y) + \varphi(y \to z)$$

and

$$\varphi(y \to z) \leq \varphi(x \to (y \to z)) + \varphi(x).$$

If $x \to (y \to z) = 1$, then

$$\varphi(y \to z) \leq \varphi(1) + \varphi(x) = \varphi(x).$$

Hence, $\varphi(z) \leq \varphi(y) + \varphi(y \to z) \leq \varphi(y) + \varphi(x)$. □

Theorem 3.2. *Let φ be a real-valued function. Then φ is pseudo-valuation on L if and only if it satisfies the conditions (pv1) and (pv4).*

Proof. Since $(x \to y) \to (x \to y) = 1$, then by (pv4), we have $\varphi(y) \leq \varphi(x \to y) + \varphi(x)$. This proves that (pv2) holds. Hence φ is a pseudo-valuation on L.

Conversely, if φ is a pseudo-valuation on L, then from Definition 3.1 and Theorem 3.1, we know (pv1) and (pv4) hold. □

Since $(x \odot y) \to z = x \to (y \to z)$, the following are consequences of Theorems 3.1 and 3.1.

Theorem 3.3. *Let φ be a real-valued function on L, then φ is pseudo-valuation on L if and only if it satisfies the conditions (pv1) and*

(pv5) $\forall x, y, z \in L, x \odot y \leq z \Rightarrow \varphi(z) \leq \varphi(x) + \varphi(y)$.

Corollary 3.1. *If φ is a pseudo-valuation on L, then for all $x, y \in L$, we have*

(pv6) $\varphi(x \odot y) \leq \varphi(x) + \varphi(y)$.

(pv7) $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$.

Theorem 3.4. *Every pseudo-valuation on L satisfies:*

(pv8) $\forall x, y, z \in L, \varphi(x \to (y \to z)) \leq \varphi((x \to y) \to z)$.

(pv9) $\forall x, y, z \in L, \varphi(x \to z) \leq \varphi((x \to y) + \varphi(y \to z)$.

Proof. For any $x, y, z \in L$, then by (a2) and (a15),

$$1 = y \to (x \to y) \leq ((x \to y) \to z) \to (y \to z),$$

and so

$$(x \to y) \to z \leq y \to z.$$

Thus

$$\varphi(y \to z) \leq \varphi((x \to y) \to z).$$

Since $y \to z \leq x \to (y \to z)$, we have

$$\varphi(x \to (y \to z)) \leq \varphi(y \to z) \leq \varphi((y \to z) \to z).$$

Thus, (pv8) holds.

Since $(x \to y) \odot (y \to z) \leq x \to z$ by (a16), it follows from (pv6) that

$$\varphi(x \to z) \leq \varphi((x \to y) \odot ((y \to z)) \leq \varphi(x \to y) + \varphi(y \to z).$$

Thus (pv9) holds. □

Theorem 3.5. *If φ is a pseudo-valuation on L, then the set $F = \{x \in L | \varphi(x) = 0\}$ is a filter of L.*

Proof. Since $\varphi(1) = 0$, we have $1 \in F$. Let $x, y \in L$ be such that $x \to y \in L$ and $x \in L$, then $\varphi(x \to y) = 0$ and $\varphi(x) = 0$. Then $\varphi(y) \leq \varphi(x \to y) + \varphi(x) = 0$, and so $\varphi(y) = 0$, that is, $y \in F$. Hence, F is a filter of L. □

The following example shows that the converse of Theorem 3.5 may not be true.

Example 3.2. *Consider an R_0-algebra L as in Example 3.1. Define a real-valued function φ on L by*

$$\varphi = \begin{pmatrix} 0 & a & b & c & 1 \\ 2 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then $F = \{x \in L | \varphi(x) = 0\} = \{1, c\}$ is a filter of L, but φ is not a pseudo-valuation on L since $\varphi(0) = 2 \nleq \varphi(a \to 0) + \varphi(a) = \varphi(c) + \varphi(a) = 1$.

4. Implicative pseudo-valuations

Definition 4.1. *A real-valued function φ on L is called an implicative pseudo-valuation on L if it satisfies (pv1) and*

(pv10) $\forall x, y, z \in L, \varphi(x \to z) \leq \varphi(x \to (y \to z)) + \varphi(x \to y)$.

Example 4.1. Let $L = \{0, a, b, c, d, 1\}$, where $0 < a < b < c < d < 1$. Define $'$ and $\to$ as follows:

x	x'
0	1
a	d
b	c
c	b
d	a
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Define a real-valued function φ on L by

$$\varphi = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then φ is a pseudo-valuation on L.

The following proposition is obvious and we omit the proof.

Proposition 4.1. Every implicative pseudo-valuation on L is a pseudo-valuation on L.

The converse of Proposition 4.1 may not be true. In fact, let φ_1 be a pseudo-valuation on L in Example 3.2. We know that φ_1 is a pseudo-valuation on L, but it is not an implicative pseudo-valuation on L since $2 = \varphi_1(1 \to 0) \nleq \varphi_1(1 \to (a \to 0)) + \varphi_1(1 \to a) = 0$.

Now, we mainly investigate the characterizations of pseudo-valuations of R_0-algebras.

Theorem 4.1. *Let φ be a pseudo-valuation on L, then φ is an implicative pseudo-valuation on L if and only if for all $x, y, z \in L$, it satisfies:*

(pv11) $\varphi(x \to z) \leq \varphi(x \to (z' \to y)) + \varphi(y \to z)$.

Proof. Assume that φ is an implicative pseudo-valuation on L. For any $x, y, z \in L$, we have

$$\begin{aligned}(\varphi(x \to z) &= \varphi(z' \to x') && (by \;\; R1)\\ &\le \varphi(z' \to (y' \to x')) + \varphi(z' \to y') && (by \;\; pv10)\\ &= \varphi(x \to (z' \to y)) + \varphi(y \to z) && (by\ R4\ and\ R1).\end{aligned}$$

Thus, (pv11) holds.

Conversely, assume that φ is an implicative pseudo-valuation on L satisfies (pv11). Then

$$\begin{aligned}\varphi(x \to z) &= \varphi(z' \to x')\\ &\le \varphi(z' \to (x'' \to y')) + \varphi(y' \to x')\\ &= \varphi(x \to (y \to z)) + \varphi(x \to y).\end{aligned}$$

Thus, (pv10) holds, and so φ is an implicative pseudo-valuation on L. □

Theorem 4.2. *Let φ be a pseudo-valuation on L. Then the following are equivalent:*

(1) φ is an implicative pseudo-valuation on L;

(2) $\forall x, z \in L, \varphi(x \to z) \le \varphi(x \to (z' \to z))$;

(3) $\forall x, y, z \in L, \varphi(x \to z) \le \varphi(y \to (x \to (z' \to z))) + \varphi(y)$.

Proof. (1) ⇒ (2). Assume that φ is an implicative pseudo-valuation on L. Putting $y = z$ in (pv11), we have

$$\begin{aligned}\varphi(x \to z) &= \varphi(x \to (z' + z)) + \varphi(z \to z)\\ &= \varphi(x \to (z' \to z)) + \varphi(1)\\ &= \varphi(x \to (z' \to z)).\end{aligned}$$

(2) ⇒ (3). For any $x, y, z \in L$, we have

$$\varphi(x \to (z' \to z)) \le \varphi(y \to (x \to (z' \to z))) + \varphi(y).$$

Using (2), we obtain

$$\begin{aligned}\varphi(x \to z) &= \varphi(x \to (z' \to z))\\ &= \varphi(y \to (x \to (z' \to z))) + \varphi(y).\end{aligned}$$

(3) ⇒ (1). Let φ be a pseudo-valuation on L satisfies the condition (3). Then by Theorem 3.8 (pv9), we have

$$\varphi(x \odot z' \to z) \le \varphi(x \odot z' \to y) + \varphi(y \to z).$$

By (a11), we have

$$\varphi(x \to (z' \to z)) \le \varphi(x \odot z' \to y) + \varphi(y \to z).$$

Putting $y = 1$ in (3), we have

$$\begin{aligned}\varphi(x \to z) &\le \varphi(1 \to (x \to (z' \to z))) + \varphi(1)\\ &= \varphi(x \to (z' \to z))\\ &\le \varphi(x \to (z' \to y)) + \varphi(y \to z).\end{aligned}$$

Thus, (pv11) holds. It follows from Theorem 4.1 that φ is an implicative pseudo-valuation on L. □

Theorem 4.3. *Let φ be a pseudo-valuation on L. Then the following are equivalent:*

(1) φ is an implicative pseudo-valuation on L;
(2) $\forall x \in L, \varphi(x) \le \varphi(x' \to x)$;
(3) $\forall x, y, z \in L, \varphi(x) \le \varphi((x \to y) \to x)$;
(4) $\forall x, y, z \in L, \varphi(x) \le \varphi(z \to ((x \to y) \to x)) + \varphi(z)$.

Proof. (1) ⇒ (2). From Theorem 4.2(2), we have

$$\varphi(x) = \varphi(1 \to x) \le \varphi(1 \to (x' \to x)) = \varphi(x' \to x).$$

(2) ⇒ (3). Since $x' \le x \to y$ by (a6), we have

$$(x \to y) \to x \le x' \to x$$

from (a5). Since φ is a pseudo-valuation on L, we have

$$\varphi(x' \to x) \le \varphi((x \to y) \to x)$$

by Proposition 3.1(1). Thus, from (2), we deduce that

$$\varphi(x) \le \varphi(x' \to x) \le \varphi((x \to y) \to x).$$

Hence (3) holds.

(3) ⇒ (4). Since φ is a pseudo-valuation on L, we have

$$\varphi((x \to y) \to x) \le \varphi(z \to ((x \to y) \to x)) + \varphi(z).$$

Thus (4) holds.

(4) ⇒ (1). Since $z \le x \to z$ by (a2), we have $(x \to z)' \le z'$ and $z' \to (x \to z) \le (x \to z)' \to (x \to z)$. Thus, $\varphi((x \to z)' \to (x \to z)) \le \varphi(z' \to (x \to z))$. It follows from (4) that

$$\begin{aligned}\varphi(x \to z) &\le \varphi(1 \to (((x \to z) \to 0) \to (x \to z))) + \varphi(1)\\ &= \varphi((x \to z)' \to (x \to z))\\ &\le \varphi(z' \to (x \to z)).\end{aligned}$$

Thus, from Theorem 4.2(2), φ is an implicative pseudo-valuation on L. □

Definition 4.2. *A pseudo-valuation φ on L is called Boolean if it satisfies:*
(pv12) $\forall x \in L, \varphi(x \vee x') = 0$.

Theorem 4.4. *A pseudo-valuation φ on L is Boolean if and only if it implicative.*

Proof. Assume that φ is an implicative pseudo-valuation on L. Since

$$x' \to (((x' \to x) \to x) \to (x' \to x)') = ((x' \to x) \to x) \to ((x \to x) \to x) = 1$$

and

$$x' \to ((x' \to x) \to x) = 1,$$

we have

$$\begin{aligned}\varphi((x' \to x) \to x) &= \varphi(x' \to (x' \to x)') \\ &\leq \varphi(x' \to (((x' \to x) \to x) \to (x' - x)')) + \varphi(x' \to ((x' \to x) \to x)) \\ &= \varphi(1) + \varphi(1) = 0,\end{aligned}$$

that is, $\varphi((x' \to x) \to x) = 0$ by Proposition 3.3(2).

Similarly, we can obtain $\varphi((x \to x') \to x') = 0$. Thus, by (a8) and (pv7), we have

$$\begin{aligned}\varphi(x \vee x') &= \varphi(((x' \to x) \to x) \wedge ((x \to x') \to x')) \\ &\leq \varphi((x' \to x) \to x) + \varphi((x \to x') \to x') \\ &= 0,\end{aligned}$$

and so $\varphi(x \vee x') = 0$. Thus, φ is a Boolean pseudo-valuation on L.

Conversely, let φ be a Boolean pseudo-valuation on L. Then

$$\begin{aligned}\varphi(x \to y) &\leq \varphi((y \vee y') \to (x \to y)) + \varphi(y \vee y') \\ &= \varphi((y \vee y') \to (x \to y)) \\ &= \varphi((y \to (x \to y)) \wedge (y' \to (x \to y))) \\ &\leq \varphi(y \to (x \to y)) + \varphi(y' \to (x \to y)) \\ &= \varphi(1) + \varphi(y' \to (x \to y)) \\ &= \varphi(y' \to (x \to y)).\end{aligned}$$

Thus, from Theorem 4.2(2), φ is an implicative pseudo-valuation on L. □

Similar to Theorem 3.5, we can obtain:

Theorem 4.5. If φ is an implicative pseudo-valuation on L, then the set $F = \{x \in L|\varphi(x) = 0\}$ is an implicative filter of L.

The following example shows that the converse of Theorem 4.5 may not be true.

Example 4.2. *Consider an R_0-algebra L as in Example 4.1. Define a real-valued function φ on L by*

$$\varphi = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 2 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $F = \{x \in L|\varphi(x) = 0\} = \{1, c, d\}$ is an implicative filter of L, but φ is not an implicative pseudo-valuation on L since $2 = \varphi(1 \to 0) \not\leq 1 = \varphi(1 \to (a \to 0)) + \varphi(1 \to a)$.

5. Pseudo-metric spaces

By a pseudo-metric space we mean an ordered pair (M, d), where M is a non-empty set and $d : M \times M \to \mathbb{R}$ is a positive function such that the following hold:

(1) $d(x, x) = 0$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in M$.

If in $(M, d), d(x, y) = 0 \Rightarrow x = y$, then (M, d) is called a metric space.

For a real-valued function φ on L, define a mapping $d_\varphi : L \times L \to R$ by

$$d_\varphi(x, y) = \varphi(x \to y) + \varphi(y \to x)$$

for all $(x, y) \in L \times L$.

We say d_φ is the pseudo-metric introduced by pseudo-valuation φ.

Theorem 5.1. *If a real-valued function φ on L is a pseudo-valuation on L, then d_φ is a pseudo-metric on L, and so (L, d_φ) is a pseudo-metric space.*

Proof. For any $x \in L, d_\varphi(x, x) = \varphi(x \to x) + \varphi(x \to x) = 0$. It is clear that $d_\varphi(x, y) = d_\varphi(y, x)$. Let $x, y, z \in L$. By Theorem 3.8 (pv9), we have

$$\begin{aligned} d_\varphi(x, y) + d_\varphi(y, z) &= (\varphi(x \to y) + \varphi(y \to x)) + (\varphi(y \to z) + \varphi(z \to y)) \\ &= (\varphi(x \to y) + \varphi(y \to z)) + (\varphi(z \to y) + \varphi(y \to x)) \\ &\geq \varphi(x \to z) + \varphi(z \to x) \\ &= d_\varphi(x, z). \end{aligned}$$

Thus, (L, d_φ) is a pseudo-metric space. □

Theorem 5.2. If $\varphi : L \to \mathbb{R}$ is a valuation on L, then (L, d_φ) is a metric space.

Proof. By Theorem 5.1, (L, d_φ) is a pseudo-metric space. Let $x, y \in L$ be such that $d_\varphi(x, y) = 0$. Then $0 = d_\varphi(x, y) = \varphi(x \to y) + \varphi(y \to x)$, and so $\varphi(x \to y) = \varphi(y \to x) = 0$. Since φ is a valuation on L, then $x \to y = y \to x = 1$. By (a1), we have $x = y$. Hence (L, d_φ) is a metric space. □

Proposition 5.1. *Every pseudo-metric d_φ induced by pseudo-valuation φ on L satisfies:*

(1) $d_\varphi(x, y) \geq d_\varphi(x \to a, y \to a)$,
(2) $d_\varphi(x, y) \geq d_\varphi(a \to x, a \to y)$,
(3) $d_\varphi(x \to y, a \to b) \leq d_\varphi(x \to y, a \to y) + d_\varphi(a \to y, a \to b)$

for all $x, y, a, b \in L$.

Proof. (1) Let $x, y, a \in L$. By (a15), $(x \to y) \to ((y \to a) \to (x \to a)) = 1$ and $(y \to x) \to ((x \to a) \to (y \to a)) = 1$. Hence, by Proposition 3.1(1), we have

$\varphi(x \to y) \geq \varphi((y \to a) \to (x \to a))$ and $\varphi(y \to x) \geq \varphi((x \to a) \to (y \to a))$. Thus,

$$\begin{aligned} d_\varphi(x,y) &= \varphi(x \to y) + \varphi(y \to x) \\ &\geq \varphi((y \to a) \to (x \to a)) + \varphi((x \to a) \to (y \to a)) \\ &= d_\varphi(x \to a, y \to a). \end{aligned}$$

(2) is similar to (1).

(3) By Theorem 3.4(pv9), we have
$\varphi((x \to y) \to (a \to b)) \leq \varphi((x \to y) \to (a \to y)) + \varphi((a \to y) \to (a \to b))$
and $\varphi((a \to b) \to (x \to y)) \leq \varphi((a \to b) \to (a \to y)) + \varphi((a \to y) \to (x \to y))$.
Thus, we have

$$\begin{aligned} d_\varphi(x \to y, a \to b) &= \varphi((x \to y) \to (a \to b)) + \varphi((a \to b) \to (x \to y)) \\ &\leq (\varphi((x \to y) \to (a \to y)) + \varphi((a \to y) \to (a \to b))) \\ &+ (\varphi((a \to b) \to (a \to y)) + \varphi((a \to y) \to (x \to y))) \\ &= (\varphi((x \to y) \to (a \to y)) + \varphi((a \to y) \to (x \to y))) \\ &+ (\varphi((a \to y) \to (a \to b)) + \varphi((a \to b) \to (a \to y))) \\ &= d_\varphi(x \to y, a \to y) + d_\varphi(a \to y, a \to b). \end{aligned}$$

□

Theorem 5.3. *For a real-valued function φ on L, if d_φ is a pseudo-metric on L, then $(L\times L, d^*_\varphi)$ is a pseudo-metric space, where $d^*_\varphi((x,y),(a,b)) = \max\{d_\varphi(x,a), d_\varphi(y,b)\}$ for all $(x,y),(a,b) \in L \times L$.*

Proof. Let d_φ be a pseudo-metric on L. For any $(x,y),(a,b) \in L \times L$, we have

(1) $d^*_\varphi((x,y),(x,y)) = \max\{d_\varphi(x,x), d_\varphi(y,y)\} = 0$.

(2) $d^*_\varphi((x,y),(a,b)) = \max\{d_\varphi(x,a), d_\varphi(y,b)\} = \max\{d_\varphi(a,x), d_\varphi(b,y)\} = d^*_\varphi((a,b),(x,y))$.

(3) Let $(x,y),(a,b),(u,v) \in L \times L$.

Then we have

$$\begin{aligned} d^*_\varphi((x,y),(u,v)) + d^*_\varphi((u,v),(a,b)) &= \max\{d_\varphi(x,u), d_\varphi(y,v)\} + \max\{d_\varphi(u,a), d_\varphi(v,b)\} \\ &\geq \max\{d_\varphi(x,u) + d_\varphi(u,a), d_\varphi(y,v), d_\varphi(v,b)\} \\ &\geq \max\{d_\varphi(x,a), d_\varphi(y,b)\} \\ &= d^*_\varphi((x,y),(a,b)). \end{aligned}$$

Hence $(L \times L, d^*_\varphi)$ is a pseudo-metric space. □

Corollary 5.1. *If $\varphi : L \to \mathbb{R}$ is a pseudo-valuation on L, then $(L \times L, d^*_\varphi)$ is a pseudo-metric space.*

Theorem 5.4. *If $\varphi : L \to \mathbb{R}$ is a valuation on L, then $(L \times L, d^*_\varphi)$ is a metric space.*

Proof. By Corollary 5.1, $(L\times L, d^*_\varphi)$ is a pseudo-metric space. For any $(x,y),(a,b)\in L\times L$ such that $d^*_\varphi((x,y),(a,b))=0$. Then $0=d^*_\varphi((x,y),(a,b))=\max\{d_\varphi(x,a),d_\varphi(y,b)\}$, and so $d_\varphi(x,a)=d_\varphi(y,b)=0$. Hence

$$0=d_\varphi(x,a)=\varphi(x\to a)+\varphi(a\to x)$$

and

$$0=d_\varphi(y,b)=\varphi(y\to b)+\varphi(b\to y).$$

Thus, $\varphi(x\to a)=\varphi(a\to x)=0$ and $\varphi(y\to b)=\varphi(b\to y)=0$. Thus, $x\to a=a\to x=1$ and $y\to b=b\to y=1$, that is, $x=a$ and $y=b$, and so $(x,y)=(a,b)$. Hence $(L\times L, d^*_\varphi)$ is a metric space. □

Theorem 5.5. *If φ is a valuation on L, then the operation $*$ in L is uniformly continuous.*

Proof. For any $\varepsilon>0$, if $d^*_\varphi(x,y),(a,b))<\frac{\varepsilon}{2}$, then $d^*_\varphi(x,a)<\frac{\varepsilon}{2}$ and $d^*_\varphi(y,b)<\frac{\varepsilon}{2}$. By Proposition 5.1, we have

$$\begin{aligned}d_\varphi(x\to y,a\to b)&\le d_\varphi(x\to y,a\to y)+d_\varphi(a\to y,a\to b)\\&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Then $*:L\times L\to L$ is uniformly continuous. □

6. Conclusions

In this paper, we introduce the concepts of (implicative) pseudo-valuations on R_0-algebras and investigate some related characterizations. Finally, we prove that the binary operation in R_0-algebras is uniformly continuous based on the notion of pseudo-valuations.

Based on these results, we will consider some its applications to knowledge-based information systems in the future.

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